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ON THE REGESSION DESIGN PROBLEM OF SACKS AND YLVISAKER[1]

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0. Summary. We consider the experimental design problem of Sacks and Ylvisaker. We consider only the case of the (noise) stochastic process X satisfying a stochastic differential equation of the form

$$L_m X(t) = dW(t)/dt \qquad 0 \leqq t \leqq 1 \tag{0.1}$$

where L_m is an mth order differential operator whose null space is spanned by an ECT system and $W(t)$ is a Wiener process. We show that the non-degeneracy of the covariance matrix of $\{X^{(v)}(t_i), v = 0, 1, 2, \cdots, m-1, t_i \in [0, 1], i = 1, 2, \cdots, n\}$ is equivalent to the total positivity properties of the Green's function for $L_m{}^* L_m$ with appropriate boundary conditions. An asymptotically optimal sequence of designs is found for this case and its dependence on the characteristic discontinuity of the above mentioned Green's function is exhibited. Finally we show that a special case of the problem is equivalent to the problem of the optimal approximation of a monomial by a Spline function in the L_2 norm. Some recent results are available on this latter problem which provide some information concerning existence and uniqueness of optimal designs with distinct points.

1. Introduction. Consider the linear regression model in which one may observe a stochastic process Y having the form

$$Y(t) = \theta f(t) + X(t) \qquad 0 \leqq t \leqq 1. \tag{1.1}$$

θ is an unknown constant, $f(t)$ is a known function and X is assumed to have mean value function zero and known continuous covariance kernel $Q(t, t') = EX(t)X(t')$. Let T be a subset of [0, 1] and let $\hat{\theta}_T$ be the best linear estimate (if it exists) of θ based on observing $\{Y(t), t \in T\}$. Let $\sigma_T{}^2$ be $E(\theta - \hat{\theta}_T)^2$. Let $\mathscr{D}_n = \{T_n \mid T_n = t_0, t_1, \cdots, t_n, 0 \leqq t_0 < \cdots < t_n \leqq 1\}$. Sacks and Ylvisaker, in a series of papers [8], [9], [10], consider the problem of finding a member $T_n{}^*$ of $\mathscr{D}_n$ for which

$$\sigma^2_{T_n^*} = \inf_{T_n \in \mathscr{D}_n} \sigma^2_{T_n}. \tag{1.2}$$

In [8], [9] they consider processes $X(t)$ which are assumed to have no quadratic mean derivatives and satisfy a number of other conditions. [10] considered situations where $X(t)$ has exactly $m-1$ quadratic mean derivatives. It is assumed there that $X(t)$ has a representation

$$X(t) = \int_0^t dt_{m-1} \int_0^{t_{m-1}} dt_{m-2} \cdots \int_0^{t_2} X_0(t_1)\, dt_1 \tag{1.3}$$

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where

(1.4) $$EX_0(t) = 0$$

$$EX_0(s)X_0(t) = K(s, t)$$

with

(1.5) $$\lim_{s\downarrow t}\frac{\partial}{\partial s}K(s, t) - \lim_{s\uparrow t}\frac{\partial}{\partial s}K(s, t) = \alpha(t) = \text{const} > 0$$

and $K(s, t)$ satisfies some other conditions.

Throughout [8], [9], [10] it is assumed that $f(t)$ is of the form

(1.6) $$f(t) = \int_0^1 Q(t, t')\rho(t')\, dt', \qquad \rho \text{ continuous}$$

where $Q(t, t') = EX(t)X(t')$, and f satisfies some other conditions.

A sequence T_n^*, $n = 1, 2, \cdots$ of designs $T_n^* \in \mathscr{D}_n$, is said by Sacks and Ylvisaker to be asymptotically optimum if

(1.7) $$\lim_{n\to\infty} \frac{\sigma_{T_n^*}^2 - \sigma^2}{\inf_{T_n \in \mathscr{D}_n} \sigma_{T_n}^2 - \sigma^2} = 1$$

where $\sigma^2 = \sigma_T^2$ with $T = [0, 1]$. It is well known that $\sigma^2 > 0$ if $f \in \mathscr{H}_Q$, where $\mathscr{H}_Q$ is the reproducing kernel Hilbert space associated with the kernel Q, and that (1.6) insures that $f \in \mathscr{H}_Q$. (See [7] for details).

$\mathscr{H}_Q$, for any Q positive definite on $[0, 1] \times [0, 1]$, has the following properties (see [1]):

(i) $Q_t(\cdot) \in \mathscr{H}_Q$, $\forall t \in [0, 1]$ where $Q_t(\cdot) = Q(t, \cdot)$

(ii) $\langle Q_t, h\rangle_Q = h(t)$, $\forall h \in \mathscr{H}_Q$, $t \in [0, 1)$.

We are using the symbol $\langle \cdot, \cdot \rangle_Q$ for the inner product in $\mathscr{H}_Q$.

Let $\mathscr{H}_X$ be the Hilbert space spanned by the random variables $\{X(t), t \in [0, 1]\}$, with inner product

$$\langle Z_1, Z_2\rangle = EZ_1Z_2 \qquad Z_1, Z_2 \in \mathscr{H}_X.$$

There is an isometric isomorphism between $\mathscr{H}_Q$ and $\mathscr{H}_X$ generated by the correspondence $X(t) \sim Q_t(\cdot)$. $\forall t \in [0, 1]$ which follows from the fact that

$$EX(t)X(t') = Q(t, t') = \langle Q_t, Q_{t'}\rangle_Q, \qquad t, t' \in [0, 1].$$

It is well known that if $Z \in \mathscr{H}_X$ and $f(\cdot) \in \mathscr{H}_Q$, then

$$Z \sim f \Leftrightarrow EZX(t) = f(t)$$

and it is easy to check that f of (1.6) satisfies $\int_0^1 X(t')\rho(t')\, dt' \sim f(\cdot)$.

If $Z \in \mathscr{H}_Q$ and $Z \sim f$, it will be convenient to use the symbol $\langle f, X\rangle_\sim$ to represent the random variable Z, which corresponds to the element f of $\mathscr{H}_Q$ under this congruence. It is well known that if $\hat{\theta}_T$ is a best linear estimate for θ given $\{Y(t), t \in T\}$ it satisfies

(1.8) $$\hat{\theta}_T - \theta = \langle P_T f, X\rangle_\sim / \langle P_T f, P_T f\rangle_Q$$

where P_T is the projection operator on the subspace of $\mathscr{H}_Q$ spanned by $\{Q_t(\cdot), t \in T\}$. Hence, using the fact that $E\langle \rho_1, X\rangle_\sim \langle \rho_2, X\rangle_\sim = \langle \rho_1, \rho_2\rangle_Q$ we have

$$\text{(1.9)} \qquad \operatorname{Var} \hat{\theta}_T = [\|P_T f\|_Q{}^2]^{-1}$$

where $\|\cdot\|_Q$ denotes the norm in $\mathscr{H}_Q$. Thus (1.7) is equivalent to

$$\text{(1.10)} \qquad \lim_{n\to\infty} \frac{\|f\|_Q{}^2 - \|P_{T_n*}f\|_Q{}^2}{\|f\|_Q{}^2 - \sup_{T_n \in \mathscr{D}_n} \|P_{T_n}f\|_Q{}^2} = 1.$$

Suppose that $X(t)$ has $m-1$ quadratic mean derivatives (which entails that f of the form (1.6) has $2m$ continuous derivatives). Let $\hat{\theta}_{m,T_n}$ be the best linear estimate, if it exists, of θ, based on observing $\{Y^{(\nu)}(t), \nu = 0, 1, 2, \cdots m-1, t \in T_n\}$. Allowing $m-1$ (quadratic mean) derivatives to be observable at the design points T_n, the definition of asymptotically optimal may be revised to read: $T_n{}^* \in \mathscr{D}_n$ is asymptotically optimal if

$$\text{(1.11)} \qquad \lim_{n\to\infty} \frac{\sigma^2_{m,T_n*} - \sigma^2}{\inf_{T_n \in \mathscr{D}_n} \sigma^2_{m,T_n} - \sigma^2} = 1$$

where

$$\text{(1.12)} \qquad \sigma^2_{m,T_n} = E(\hat{\theta}_{m,T_n} - \theta)^2.$$

In this case we have

$$\text{(1.13)} \qquad \hat{\theta}_{m,T_n} - \theta = \langle P_{m,T_n}f, X\rangle_\sim / \langle P_{m,T_n}f, P_{m,T_n}f\rangle_Q$$

where P_{m,T_n} is the projection operator in $\mathscr{H}_Q$ onto the subspace of $\mathscr{H}_Q$ spanned by

$$\text{(1.14)} \qquad \{Q_t{}^{(\nu)}(\cdot), t \in T_n, \nu = 0, 1, 2, \cdots, m-1\}$$

where $Q_t{}^{(\nu)}(\cdot) = (\partial^\nu/\partial s^\nu)Q(s, \cdot)|_{s=t}$ since $Q_t{}^{(\nu)}(\cdot) \sim X^{(\nu)}(t)$.

Hence,

$$\operatorname{Var} \hat{\theta}_{m,T_n} = [\|P_{m,T_n}f\|_Q{}^2]^{-1}.$$

If $X(t)$ and its first $m-1$ derivatives are continuous in quadratic mean, then $X^{(\nu)}(t)$, $\nu \leqq m-1$ may be approximated arbitrarily closely by $\{X(t+\delta_i(t))\}_{i=1}^{\nu+1}$ if we are allowed to choose $\{\delta_i(t)\}_{i=1}^{\nu+1}$ arbitrarily close to t, and

$$\text{(1.15)} \qquad \inf_{T_{nm} \in \mathscr{D}_{nm}} \|f - P_{T_{nm}}f\|_Q \leqq \inf_{T_n \in \mathscr{D}_n} \|f - P_{m,T_n}\|_Q \leqq \inf_{T_n \in \mathscr{D}_n} \|f - P_{T_n}f\|_Q.$$

Suppose that the mn elements in brackets in (1.14) are linearly independent for every T_n in $\mathscr{D}_n$ and every finite n. Then it is easy to see that if f has a representation of the form (1.6) then f cannot be in the range of P_{m,T_n}, for any $T_n \in \mathscr{D}_n$, $n < \infty$, that is f cannot have a representation of the form

$$f(\cdot) = \sum_{\nu=0}^{m-1} \sum_{i=0}^{n} c_{\nu_i} Q_{t_i}^{(\nu)}(\cdot), \qquad t_i \in T_n,$$

and conversely. Thus it becomes apparent that different analyses are required according as some condition like (1.6) holds or not.

In this note we consider only f of the form (1.6) and primarily the situation where derivatives are allowed. Sacks and Ylvisaker prove the following

THEOREM (Sacks and Ylvisaker). *Under some assumptions on Q and f stated in* [10] *and including* (1.3), (1.5) *and* (1.6), $T_n^* = \{t_{in}^*\}_{i=0}^n$ *given by*

$$(1.16) \qquad \int_0^{t^*_{in}} \rho^{2/(2m+1)}(u)\,du = \frac{i}{n}\int_0^1 \rho^{2/(2m+1)}(u)\,du, \qquad i = 1, 2, \cdots, n$$

is an asymptotically optimal sequence, and

$$(1.17) \qquad n^{2m}\|f - P_{m,T_n^*}f\|_Q^2 = \frac{m!^2}{(2m)!(2m+1)!}\left[\int_0^1 \rho^{2/(2m+1)}(u)\,du\right]^{2m+1} + o(1).$$

In this note we consider a special class of stochastic processes. $X(t)$ is assumed to (formally) satisfy the stochastic differential equation

$$(1.18) \qquad L_m X(t) = \frac{dW(t)}{dt}$$

with random (left) boundary conditions where $W(t)$ is the Wiener process and L_m is an mth order linear differential operator whose null space is spanned by an extended complete Tchebychev (ECT) system, of continuity class C^{2m}. For these processes we will have

$$(1.19) \qquad \lim_{s\downarrow t}\frac{\partial^{2m-1}}{\partial s^{2m-1}}Q(s, t) - \lim_{s\uparrow t}\frac{\partial^{2m-1}}{\partial s^{2m-1}}Q(s, t) = (-1)^m\alpha(t)$$

where $\alpha(t) > 0$ but may not be a constant. Thus this class is not covered by [10].

Processes of the form of (1.18) have a number of interesting properties. Q is a Green's function for $L_m^* L_m$, with appropriate self adjoint boundary conditions, where L_m^* is the adjoint operator to L_m and $(-1)^m\alpha(t)$ is the characteristic discontinuity of the Green's function. These processes are m-ple Markov processes in the sense of Hida [3]. In Section 2, we define the class of processes under consideration and point out that it is an immediate consequence of the total positivity properties of Green's functions for certain self-adjoint differential operators that the dimension of the subspace spanned by the set (1.14) is nm. In Section 3, by writing down an appropriate representation of the Green's function for L_m we obtain

THEOREM 2. *Let* $EX(s)X(t) = Q(s, t)$, $s, t \in [0, 1]$, *where* $X(t)$ *satisfies*

$$L_m X(t) = dW(t)/dt$$

$$X^{(\nu)}(0) = \xi_{\nu+1}, \qquad \nu = 0, 1, 2, \cdots, m-1$$

where $W(t)$ is a Wiener process, $\{\xi_v\}_{v=1}^m$ are m linearly independent normal, zero mean random variables independent of $W(t)$, and L_m is an mth order differential operator with null space spanned by an ECT *system of continuity class C^{2m}.*

Let

$$f(s) = \int_0^1 Q(s,t)\rho(t)\,dt \tag{1.20}$$

and

$$\lim_{s\downarrow t}\frac{\partial^{2m-1}}{\partial s^{2m-1}}Q(s,t) - \lim_{s\uparrow t}\frac{\partial^{2m-1}}{\partial s^{2m-1}}Q(s,t) = (-1)^m\alpha(t).$$

Suppose ρ is strictly positive and has a bounded first derivative on $[0, 1]$. Then $T_n^* = \{t_{in}\}_{i=0}^n$ with t_{in}^* given by

$$\int_0^{t^*_{in}} [\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}\,du = \frac{i}{n}\int_0^1 [\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}\,du, \quad i = 1, 2, \cdots, n \tag{1.21}$$

$$t_{0n}^* = 0$$

is an asymptotically optimal sequence, and

$$\|f - P_{m,T_n^*}f\|_Q^{\,2} = \frac{1}{n^{2m}}\frac{(m!)^2}{(2m)!(2m+1)!}\left[\int_0^1 [\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}\,du\right]^{2m+1} + o\left(\frac{1}{n^{2m}}\right). \tag{1.22}$$

2. Extended complete Tchebychev systems and associated stochastic processes. In this section we quote some basic definitions and Theorems which will be used in the sequel. They may be found in [4].

Let $\{\Phi_i(t)\}_{i=1}^m$ be a set of m functions. The set is said to be a Tchebychev system if the determinant

$$\begin{vmatrix} \Phi_1(t_1) & \cdots & \Phi_2(t_m) \\ \vdots & & \vdots \\ \Phi_m(t_1) & \cdots & \Phi_m(t_m) \end{vmatrix}$$

is strictly positive whenever $0 < t_1 < t_2 < \cdots < t_m < 1$, and a complete Tchebychev system if $\{\Phi_i\}_{i=1}^v$ is a Tchebychev system for each $v = 1, 2, \cdots m$. Suppose $\Phi_i(t)$ has $m-1$ continuous derivatives on $(0, 1)$. The domain of definition of the determinant may be extended to $0 < t_{1'} \leqq t_2 \leqq \cdots t_{m-1} \leqq t_m < 1$, where, whenever we have an r tuple coincidence $t_v = t_{v+1} = \cdots = t_{v+r-1}$, the $v+j$th column of the determinant is replaced by

$$\begin{pmatrix} \Phi_1^{(j)}(t_v) \\ \vdots \\ \Phi_m^{(j)}(t_v) \end{pmatrix}$$

for $j = 1, 2, \cdots, r-1$.

(See [4] page 48 for details). If the determinant is always strictly positive, with this interpretation then $\{\Phi_\nu\}_{\nu=1}^m$ is called an extended Tchebychev (ET) system, and if $\{\Phi_\nu\}_{\nu=1}^m$ is an ET system for each $\nu = 1, 2, \cdots, m$ then it is called an extended complete Tchebychev (ECT) system. The following theorem will be useful to motivate our requirement that L_m have a null space spanned by an ECT system.

Theorem ([4] page 276). Let $\{\Phi_i\}_{i=1}^m$ be of class C^{m-1} on [0, 1] obeying the initial conditions

$$\Phi_k^{(p)}(0) = 0 \qquad p = 0, \qquad 1, 2, \cdots, k-2, \; k = 2, 3, \cdots, m. \tag{2.1}$$

Then the following three assertions are equivalent:

(a) $\{\Phi_i\}_{i=1}^m$ has a representation of the form

$$\begin{aligned} \Phi_1(t) &= \omega_1(t) \\ \Phi_2(t) &= \omega_1(t) \textstyle\int_0^t \omega_2(\xi_1)\, d\xi_1 \\ &\vdots \\ \Phi_m(t) &= \omega_1(t) \textstyle\int_0^t \omega_2(\xi_1)\, d\xi_1 \int_0^{\xi_1} \omega_3(\xi_2)\, d\xi_2 \cdots \int_0^{\xi_{m-2}} \omega_m(\xi_{m-1})\, d\xi_{m-1} \end{aligned} \tag{2.2}$$

where $\{\omega_i\}_{i=1}^m$ are m strictly positive functions with ω_k of continuity class $C^{m-k}[0, 1]$;

(b) $\{\Phi_i\}_{i=1}^m$ is an ECT system;

(c) The Wronskian of $\{\Phi_i\}_{i=1}^\nu$ is strictly positive on [0, 1], for $\nu = 1, 2, \cdots, m$.

Now let the first order differential operator D_i be defined by

$$(D_i\Phi)(t) = \frac{d}{dt}\frac{1}{\omega_i(t)}\Phi(t) \qquad i = 1, 2, \cdots, m \tag{2.3}$$

and the mth order differential operator L_m be defined by

$$L_m\Phi = D_m D_{m-1} \cdots D_1 \Phi. \tag{2.4}$$

It may be verified that $\{\Phi_\nu\}_{\nu=1}^m$ given by (2.2) are the solutions of $L_m\Phi = 0$ satisfying the initial conditions $M_\nu\Phi_k(0) = \delta_{k,\nu+1}\,\omega_k(0)$, $\nu = 0, 1, 2, \cdots, m-1$, where

$$\begin{aligned} M_\nu &= D_\nu D_{\nu-1} \cdots D_1, \qquad \nu = 1, 2, \cdots, m-1 \\ M_0 &= I. \end{aligned} \tag{2.5}$$

Let

$$\begin{aligned} G_m(t, s) &= \omega_1(t) \textstyle\int_s^t \omega_2(\xi_1)\, d\xi_1 \int_s^{\xi_1} \omega_3(\xi_2)\, d\xi_2 \cdots \int_s^{\xi_{m-2}} \omega_m(\xi_{m-1})\, d\xi_{m-1} \qquad t \geqq s \\ &= 0 \qquad t \leqq s. \end{aligned} \tag{2.6}$$

$G_m(t, s)$ is well known to be the Green's function for the differential operator L_m with boundary conditions $\mathscr{B}$:

$$\mathscr{B}: \{(M_\nu f)(0) = 0, \qquad \nu = 0, 1, 2, \cdots, m-1\}. \tag{2.7}$$

That is, the solution to the equation

$$L_m f = g, \qquad f \in \mathscr{B} \tag{2.8}$$

is given by

$$f(t) = \int_0^1 G_m(t, u) g(u)\, du. \tag{2.9}$$

Let now

$$X(t) = \sum_{i=1}^m \xi_1 \Phi_i(t) + \int_0^t G_m(t, u)\, dW(u) \tag{2.10}$$

where $W(t)$ is a Wiener process and $\{\xi_i\}_{i=1}^m$ are m zero mean normal random variables with non-degenerate covariance matrix $S = \{s_{ij}\}$, independent of $W(t)$. We say that a stochastic process $X(t)$ constructed as in (2.10) formally satisfies the stochastic differential equation $L_m X = dW/dt$ with (random) boundary conditions $M_\nu X(0) = \xi_{\nu+1}$, $\nu = 0, 1, 2, \cdots, m-1$.

We have

$$EX(s)X(t) = Q_0(s, t) + Q(s, t) = \tilde{Q}(s, t) \tag{2.11}$$

where

$$Q_0(s, t) = \sum_{\mu=1}^m \sum_{\nu=1}^m s_{\mu\nu} \Phi_\mu(s) \Phi_\nu(t) \tag{2.12}$$

and

$$Q(s, t) = \int_0^{\min(s,t)} G_m(s, u) G_m(t, u)\, du. \tag{2.13}$$

To insure that $Q(s, t)$ has the usual continuity properties for Green's functions (see e.g. [6] page 29) we now further assume that Φ_ν is of continuity class C^{2m}, $\nu = 1, 2, \cdots, m$.

It can be shown that $Q(s, t)$ is the Green's function for the differential operator $L_m^* L_m$ with boundary conditions $\mathscr{B} \cap \mathscr{B}^*$, where

$$\begin{aligned} \mathscr{B}^* : L_m \Phi(1) &= 0 \\ D_m^* L_m \Phi(1) &= 0 \\ &\vdots \\ D_2^* \cdots D_m^* L_m \Phi(1) &= 0 \end{aligned} \tag{2.14}$$

where

$$D_i^* \Phi(t) = -\frac{1}{\omega_i(t)} \frac{d\Phi(t)}{dt}. \tag{2.15}$$

We will later on use the properties of the characteristic discontinuity of Green's functions for differential equations, (see [6]) namely

$$\lim_{s \downarrow t} \frac{\partial^{2m-1}}{\partial s^{2m-1}} Q(s, t) - \lim_{s \uparrow t} \frac{\partial^{2m-1}}{\partial s^{2m-1}} Q(s, t) = (-1)^m \alpha(t) \tag{2.16}$$

where $[(-1)^m\alpha(t)]^{-1}$ is the coefficient of $\partial^{2m}/\partial t^{2m}$ in the expansion of $L_m{}^* L_m$. Here we have

(2.17) $$\alpha(t) = \textstyle\prod_{i=1}^m \omega_i{}^2(t).$$

Let

(2.18) $$\tilde{X}(t) = \textstyle\int_0^t G_m(t, u)\, dW(u).$$

We have

(2.19) $$E\tilde{X}^{(\mu)}(t_i)\tilde{X}^{(\nu)}(t_j) = \frac{\partial^{\mu+\nu}}{\partial r^\mu ds^\nu} Q(r, s)\bigg|_{r=t_i,\, s=t_j}.$$

Let $\tilde{\Sigma}$ be the $mn \times mn$ covariance matrix of the mn random variables $\{\tilde{X}^{(\mu)}(t_i),\ \mu = 0, 1, 2, \cdots, m-1, i = 1, 2, \cdots, n\}$ with entries given by (2.19). We have the following

THEOREM 1. $\det \tilde{\Sigma} > 0$.

PROOF. The remarkable fact that $\tilde{\Sigma} > 0$ is a direct consequence of Theorem (8.1) page 547, [4] concerning the strict total positivity of Green's functions for differential operators of the form $L_m{}^* L_m$ with (self-adjoint) boundary conditions $\mathscr{B} \cap \mathscr{B}^*$.

COROLLARY. *Let* Σ *be the* $(n+1)m \times (n+1)m$ *covariance matrix of the* $(n+1)m$ *random variables*

$$X^{(\mu)}(t_i), \begin{Bmatrix} \mu = 0, 1, 2, \cdots, m-1 \\ i = 0, 1, 2, \cdots, n,\ t_0 = 0 \end{Bmatrix}$$

then $\det \Sigma > 0$.

The reproducing kernel Hilbert space $\mathscr{H}_{\tilde{Q}}$ with $\tilde{Q}$ given by (2.11), corresponding to the stochastic process X, consists of all functions f for which $M_{m-1}f$ is absolutely continuous and $L_m f \in L_2[0, 1]$, with inner product

(2.20) $$\langle f_1, f_2\rangle = \textstyle\sum_{\mu=1}^m \sum_{\nu=1}^m s^{\mu\nu}(M_{\mu-1}f_1)(0)(M_{\nu-1}f_2)(0) + \int_0^1 (L_m f_1)(u)(L_m f_2)(u)\, du$$

where $S^{-1} = \{s^{\mu\nu}\}$.

If $X(t)$, $0 \leqq t \leqq 1$ is a segment of a stationary stochastic process with spectral density

$$f(\lambda) = \left|\textstyle\sum_{\nu=0}^m \alpha_\nu (i\lambda)^\nu\right|^{-2}$$

where the polynomial $\sum_{\nu=0}^m \alpha_\nu z^\nu$ has no real zeroes, then $X(t)$, $0 \leqq t \leqq 1$ is an example of (2.10) with $L_m\Phi = \sum_{\nu=0}^m \alpha_\nu \Phi^{(\nu)}$ (compare (2.20) and equation (5.17) of [7]). The simplest example is the unpinned, integrated Wiener process (see [12]), $L_m\Phi = (d^m/dt^m)\Phi$,

(2.21) $$G_m(t, u) = \frac{(t-u)_+{}^{m-1}}{(m-1)!}, \qquad \begin{aligned}(x)_+ &= x, \ x > 0 \\ &= 0 \quad \text{otherwise}\end{aligned}$$

and $\Phi_i(t) = t^{i-1}/(i-1)!$. (In both these examples, $\alpha(t)$ is a constant).

We may always add a fixed finite number of points to each member of an asymptotically optimum sequence of designs without modifying the asymptotic optimality. Thus we may without loss of generality restrict ourselves to processes of the form

$$X(t) = \int_0^t G_m(t, u)\, dW(u), \tag{2.22}$$

since the random variables $\{\xi_i\}_{i=1}^m$ are known arbitrarily accurately if we may observe $X(s_i)$, $i = 1, 2, \cdots, m$ for s_i arbitrarily near zero, or exactly if we observe $X^{(\nu)}(0)$, $\nu = 0, 1, 2, \cdots, m-1$.

3. An asymptotically optimal sequence of designs. The goal of this section is to prove Theorem 2. This is done via several lemmas which study the behavior of $\|f - P_{m,T_n} f\|_Q^2$:

LEMMA 1. *Let* X(*t*) *by given by* (2.10) *and let*

$$f(t) = \int_0^1 Q(t, u)\rho(u)\, du, \qquad \textit{where} \quad \rho \in L_2[0, 1]. \tag{3.1}$$

and let $t_0 = 0$, $t_n = 1$.

Then

$$\|f - P_{m,T_n} f\|_Q^2 = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \rho(s) B_i(s, t) \rho(t)\, ds\, dt \tag{3.2a}$$

where

$$\begin{aligned} B_i(s, t) &= \int_{t_i}^{\min(s,t)} G_m(s, u) G_m(t, u)\, du \\ &\quad - \sum_{\mu,\nu=0}^{m-1} \int_{t_i}^{s} G_m(s, u) G_{m,\mu}(t_{i+1}, u)\, du\, s_i^{\mu\nu} \\ &\quad \cdot \int_{t_i}^{t} G_m(t, v) G_{m,\nu}(t_{i+1}, v)\, dv, \qquad s, t \in [t_i, t_{i+1}] \\ &= 0 \qquad \textit{otherwise}, \end{aligned} \tag{3.2b}$$

with

$$\begin{aligned} G_{m,\mu}(s, u) &= M_{\mu(s)} G_m(s, u), \qquad \mu = 1, 2, \cdots, m-1 \\ G_{m,0}(s, u) &= G_m(s, u), \end{aligned} \tag{3.2c}$$

$M_{\mu(s)}$ *is the operator* M_μ *defined by* (2.5), *applied to the variable s, and* $\{s_i^{\mu\nu}\}_{\mu,\nu=0}^{m-1}$ *are defined by*

$$S_i^{-1} = \{s_i^{\mu\nu}\}_{\mu,\nu=0}^{m-1}, \qquad S_i = \{s_{i,\mu\nu}\}_{\mu,\nu=0}^{m-1} \tag{3.2d}$$

with

$$s_{i,\mu\nu} = \int_{t_i}^{t_{i+1}} G_{m,\mu}(t_{i+1}, u) G_{m,\nu}(t_{i+1}, u)\, du, \tag{3.2e}$$

PROOF. Let

$$P_{m,T_n} X(u) = E\{X(u) \mid X^{(\nu)}(t_i), \nu = 0, 1, 2, \cdots, m-1, t_i \in T_n\}. \tag{3.3}$$

Then, since

$$(t) = EX(t) \int_0^1 X(u)\rho(u)\, du \tag{3.4a}$$

and

(3.4b) $$P_{m,T_n}f(t) = EX(t)\int_0^1 P_{m,T_n}X(u)\,du$$

we have

(3.4c) $$f(\cdot) \sim \int_0^1 X(u)\rho(u)\,du$$

(3.4d) $$P_{m,T_n}f(\cdot) \sim \int_0^1 P_{m,T_n}X(u)\rho(u)\,du$$

and

(3.5) $$\|f-P_{m,T_n}f\|_Q{}^2 = \int_0^1\int_0^1 \rho(s)\rho(t)E(X(s)-P_{m,T_n}X(s))(X(t)-P_{m,T_n}X(t))\,ds\,dt.$$

We will evaluate the right-hand side of (3.5).

Since $t_0 = 0 \in T_n$, it is only necessary to carry out the proof for $X(t)$ of the form (2.21), that is, $X \in \mathscr{B}$. This follows, since, in calculating $X(t)-P_{m,T_n}X(t)$, it makes no difference whether $X^{(v)}(0)$, $v = 0, 1, 2, \cdots, m-1$ are observed, or known to be zero. Now, for

(3.6) $$L_mX(t) = dW(t)/dt,$$
$$X^{(v)}(0) = 0,\ v = 0, 1, 2, \cdots, m-1$$

$X(t)$ has the representation.

(3.7) $$X(t) = \omega_1(t)\int_0^t \omega_2(t_1)\,dt_1 \int_0^{t_1} \omega_3(t_2)\cdots\int_0^{t_{m-2}} \omega_m(t_{m-1})W(t_{m-1})\,dt_{m-1}.$$

It will be convenient to work with so-called generalized derivatives, $M_vX(t)$, $v = 0, 1, 2, \cdots, m-1$. We have the representations

(3.8) $$M_vX(t) = \omega_{v+1}(t)\int_0^t \omega_{v+2}(t_{v+1})\,dt_{v+1}\cdots\int_0^{t_{m-2}} \omega_m(t_{m-1})W(t_{m-1})\,dt_{m-1} \qquad v = 0, 1, 2, \cdots, m-2$$
$$M_{m-1}X(t) = \omega_m(t)W(t),$$

and

(3.9a) $$M_vX(t) = M_{v(t)}\int_0^t G_m(t, s)\,dW(s) \qquad v = 0, 1, 2, \cdots, m-1$$

(3.9b) $$= \int_0^t G_{m,v}(t, s)\,dW(s).$$

$G_{m,v}(t, s)$ is the Green's function for the operator $L_{m,v}$ given by

(3.10) $$L_{m,v}\Phi = D_mD_{m+1}\cdots D_{v+1}\Phi, \qquad v = 1, 2, \cdots, m-1$$

with boundary conditions $\mathscr{B}_{m,v}$: $\{\Phi^{(\mu)}(0) = 0,\ \mu = 0, 1, 2, \cdots, m-v\}$.
We will use another representation for $G_{m,v}(t, s)$, $v = 1, 2, \cdots, m-1$.

Let

(3.11)
$$\Phi_{1,v}(s) = \omega_{m-v+1}(s)$$
$$\Phi_{2,v}(s) = \omega_{m-v+1}(s)\int_0^s \omega_{m-v+2}(\xi_1)\,d\xi_1$$
$$\vdots$$
$$\Phi_{v,v}(s) = \omega_{m-v+1}(s)\int_0^s \omega_{m-v+2}(\xi_1)\,d\xi\int_0^{\xi_1} \omega_{m-v+3}(\xi_2)\,d\xi_2\cdots \int_0^{\xi_{v-2}} \omega_m(\xi_{v-1})\,d\xi_{v-1}, \qquad v = 1, 2, \cdots, m.$$

and let

$$\begin{aligned}(3.12)\qquad &\Phi_1(s) = \Phi_{1.m}(s) = \omega_1(s)\\ &\Phi_2(s) = \Phi_{2,m}(s) = \omega_1(s)\int_0^s \omega_2(\xi_1)\,d\xi_1\\ &\vdots\\ &\Phi_m(s) = \Phi_{m,m}(s) = \omega_1(s)\int_0^s \omega_2(\xi_1)\,d\xi_1 \int_0^{\xi_1} \omega_3(\xi_2)\,d\xi_2 \cdots \int_0^{\xi_{m-2}} \omega_m(\xi_{m-1})\,d\xi_{m-1}\end{aligned}$$

as before.

Also let

$$\begin{aligned}(3.13)\qquad &\Phi_1{}^*(s) = (-1)^2\\ &\Phi_2{}^*(s) = (-1)^3 \int_0^s \omega_m(\xi_{m-1})\,d\xi_{m-1}\\ &\vdots\\ &\Phi_m{}^*(s) = (-1)^{m+1} \int_0^s \omega_m(\xi_{m-1})\,d\xi_{m-1} \int_0^{\xi_{m-1}} \omega_{m-1}(\xi_{m-2})\,d\xi_{m-2} \cdots \int_0^{\xi_2} \omega_2(\xi_1)\,d\xi_1.\end{aligned}$$

Algebraic manipulations on the representation of the Green's function in the form of (2.6) give the Green's function, in another, familiar form:

$$\begin{aligned}(3.14)\qquad G_{m,\nu}(t, s) &= \textstyle\sum_{\mu=1}^{m-\nu} \Phi_{m-\nu-\mu+1,\, m-\nu}(t)\Phi_\mu{}^*(s) \qquad && t \geqq s,\ \nu = 0, 1, 2, \cdots, m-1\\ &= 0 && t < s.\end{aligned}$$

Substituting (3.14) into (3.9b), we have that the random variables $M_\nu X(t_i)$, have the representation

$$(3.15)\qquad M_\nu X(t_i) = \textstyle\sum_{\mu=1}^{m-\nu} \Phi_{m-\nu-\mu+1,\, m-\nu}(t_i) \int_0^{t_i} \Phi_\mu{}^*(u)\,dW(u),\ \nu = 0, 1, 2, \cdots, m-1$$

and that the m-dimensional space $\mathscr{H}_{t_i}$ spanned by $\{M_\nu X(t_i)\}_{\nu=0}^{m-1}$ is also spanned by $\{\int_0^{t_i} \Phi_\mu{}^*(u)dW(u)\}_{\mu=1}^m$. (We are using the fact that the three systems of (3.11), (3.12), and (3.13) are each ECT).

Now we have, for $t \geqq t_i$

$$(3.16)\qquad X(t) - E\{X(t) \mid M_\nu X(t_i),\ \nu = 0, 1, 2, \cdots, m-1\} = \int_{t_i}^t G_m(t, u)\,dW(u).$$

This (well-known) result follows by writing

$$\begin{aligned}(3.17)\qquad X(t) &= \int_0^{t_i} G_m(t, u)\,dW(u) + \int_{t_i}^t G_m(t, u)\,dW(u)\\ &= \{\textstyle\sum_{\mu=1}^m \Phi_{m-\mu+1,m}(t) \int_0^{t_i} \Phi_\mu{}^*(u)\,dW(u)\} + \{\int_{t_i}^t G_m(t, u)\,dW(u)\}\end{aligned}$$

and the first term in brackets is in $\mathscr{H}_{t_i}$, while the second term is perpendicular to it. Using (3.15) and the remarks following, it follows that $\mathscr{H}_{t_i} \cup \mathscr{H}_{t_{i+1}}$ is also spanned by $\{M_\nu X(t_i), \int_{t_i}^{t_{i+1}} G_{m,\nu}(t_{i+1}, u)\,dW(u),\ \nu = 0, 1, 2, \cdots, m-1\}$.

It may then be calculated, for $t_i \leqq t \leqq t_{i+1}$, that

$$\begin{aligned}(3.18\text{a})\qquad X(t) - E\{X(t) \mid M_\nu X(t_i),\ M_\nu X(t_{i+1}),\ \nu = 0, 1, 2, \cdots, m-1\}&\\ = \int_{t_i}^{t_{i+1}} H_i(t, u)\,dW(u),&\end{aligned}$$

where

(3.18b) $$H_i(t, u) = G_m(t, u) - \sum_{\mu,\nu=0}^{m-1} G_{m,\mu}(t_{i+1}, u) s_i^{\mu\nu} \cdot \int_{t_i}^{t_{i+1}} G_m(t, v) G_{m,\nu}(t_{i+1}, v)\, dv,$$

where $S_i^{-1} = \{s_i^{\mu\nu}\}$ is given by

(3.19) $$S_i = \{s_{i,\mu\nu}\}, \quad s_{i,\mu\nu} = \int_{t_i}^{t_{i+1}} G_{m,\mu}(t_{i+1}, u) G_{m,\nu}(t_{i+1}, u)\, du \qquad \mu, \nu = 0, 1, 2, \cdots, m-1$$

Finally, we have that, for $t_i \leqq t \leqq t_{i+1}$

(3.20) $$\begin{aligned} X(t) - P_{m,T_n}X(t) &= X(t) - E\{X(t) \mid M_\nu X(t_j), \nu = 0, 1, 2, \cdots, m-1, t_j \in T_n\} \\ &= X(t) - E\{X(t) \mid M_\nu X(t_i), M_\nu X(t_{i+1}), \\ &\qquad \nu = 0, 1, 2, \cdots, m-1\} \end{aligned}$$

since a direct check shows that this last random variable, as given by (3.18) is already orthogonal to each random variable of the form

$$\int_{t_j}^{t_{j+1}} G_{m,\mu}(t_{j+1}, u)\, dW(u), \qquad \mu = 0, 1, 2, \cdots, m-1, j = 0, 1, 2, \cdots, n-1.$$

Finally, it also follows that

(3.21) $$E(X(s) - P_{m,T_n}X(s))(X(t) - P_{m,T_n}X(t)) = 0, \qquad s \in [t_j, t_{j+1}],\ t \in [t_i, t_{i+1}],\ i \neq j.$$

A quick calculation from (3.18) shows

(3.22) $$\begin{aligned} E(X(s) - P_{m,T_n}X(s))(X(t) - P_{m,T_n}X(t) &= \int_{t_i}^{t_{i+1}} H_i(s, u) H_i(t, u)\, du \\ &= B_i(s, t), \qquad s, t \in [t_i, t_{i+1}], \end{aligned}$$

and the lemma is proved.

LEMMA 2. *Let*

(3.23) $$Q(s, t) = \int_0^1 \frac{(s-u)_+^{m-1}}{(m-1)!} \frac{(t-u)_+^{m-1}}{(m-1)!}\, du.$$

Then

(3.24) $$\int_{t_i}^{t_{i+1}} ds \int_{t_i}^{t_{i+1}} B_i(s, t)\, dt = \frac{(m!)^2}{(2m)!(2m+1)!} (t_{i+1} - t_i)^{2m+1}.$$

PROOF. A proof of this lemma appears in [10]. An alternative proof using the classical Hermite remainder formula appears in [13].
The next step is to evaluate

(3.25) $$\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \rho(s) B_i(s, t) \rho(t)\, ds\, dt = \int_{t_i}^{t_{i+1}} du \left[\int_{t_i}^{t_{i+1}} H_i(t, u) \rho(t)\, dt\right]^2$$

for the general case.

We remark that $B_i(s, t)$, $s, t \in [t_i, t_{i+1}]$ is the Green's function for the operator $L_m^* L_m$ with boundary conditions $\mathscr{B}_i \cap \mathscr{B}_{i+1}$, where

$$\mathscr{B}_j : \{(M_\nu f)(t_j) = 0,\ \nu = 0, 1, 2, \cdots, m-1\}, \qquad j = 0, 1, 2, \cdots, n. \tag{3.26}$$

LEMMA 3. *Suppose* $\rho(t)$ *has a bounded derivative on* $[0, 1]$,[2] *and* $t_0 = 0$, $t_n = 1$. *Then*

$$\|f - P_{m,T_n} f\|_Q^2 = \frac{(m!)^2}{(2m)!(2m+1)!} \sum_{i=0}^{n-1} \rho^2(t_i)\alpha(t_i)[(t_{i+1} - t_i)^{2m+1}(1 + O(\Delta_i))] \tag{3.27}$$

where

$$\Delta_i = |t_{i+1} - t_i|. \tag{3.28}$$

PROOF. By the assumptions on ρ, the mean value theorem, and (3.25)

$$\int_{t_i}^{t_{i+1}} du \left[\int_{t_i}^{t_{i+1}} H_i(t, u)\rho(t)\, dt\right]^2 = \rho^2(\theta_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} B_i(s, t)\, ds\, dt \tag{3.29}$$

where θ_i is some number in $[t_i, t_{i+1}]$, and so, by Lemma 1,

$$\|f - P_{m,T_n} f\|_Q^2 = \sum_{i=0}^{n-1} \rho^2(t_i)(1 + O(\Delta_i)) \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} B_i(s, t)\, ds\, dt. \tag{3.30}$$

Hence it remains to show that

$$\int_{t_i}^{t_{i+1}} ds \int_{t_i}^{t_{i+1}} B_i(s, t)\, dt = \frac{(m!)^2}{(2m)!(2m+1)!} \alpha(t_i)[(t_{i+1} - t_i)^{2m+1}(1 + O(\Delta_i))]. \tag{3.31}$$

For the case of Lemma 2, we have

$$G_{m,\nu}(t, s) = \frac{(t-s)_+^{m-\nu-1}}{(m-\nu-1)!}, \qquad \nu = 0, 1, 2, \cdots, m-1. \tag{3.32}$$

In general, we have, by analogy with (2.6), and the mean value theorem

$$\begin{aligned} G_{m,\nu}(t, s) &= \omega_{\nu+1}(t) \int_s^t \omega_{\nu+2}(\xi_{\nu+1})\, d\xi_{\nu+1} \cdots \int_s^{\xi_{m-2}} \omega_m(\xi_{m-1})\, d\xi_{m-1}, & t > s \\ &= 0 & t \leqq s \\ &= \prod_{i=\nu+1}^{m} \omega_i(\theta_{i\nu}) \frac{(t-s)_+^{m-\nu-1}}{(m-\nu-1)!}, & \nu = 0, 1, 2, \cdots, m-1 \end{aligned} \tag{3.33}$$

where $\theta_{i\nu} \in [s, t]$.

Thus, for $s, t \in [t_i, t_{i+1}]$ we may write

$$\int_{t_i}^{\min(s,t)} G_m(s, u)G_m(t, u) - \sum_{\mu,\nu=0}^{m-1} \sum \int_{t_i}^{s} G_m(s, u)G_{m,\mu}(t_{i+1}, u)\, du\ s_i^{\mu\nu}$$
$$\times \int_{t_i}^{t} G_m(t, v)G_{m,\nu}(t_{i+1}, v)\, dv$$

[2] Recall that since $\Phi_\nu \in C^{2m}$, $\omega_\nu \in C^{2m-(\nu-1)}$ and hence $\alpha(t)$ has at least a bounded first derivative, $m \geqq 1$.

$$(3.34)\quad = \prod_{j=1}^{m}\prod_{k=1}^{m}\omega_j(\theta_{j1})\omega_k(\theta_{k2})\int_{t_i}^{\min(s,t)}\frac{(s-u)_+^{m-1}(t-u)_+^{m-1}}{(m-1)!(m-1)!}du$$

$$-\left[\sum_{\mu,\nu=0}^{m-1}\prod_{j=1}^{m}\prod_{k=\mu+1}^{m}\omega_j(\theta_{j3})\omega_k(\theta_{k4})\int_{t_i}^{s}\frac{(s-u)_+^{m-1}}{(m-1)!}\frac{(t_{i+1}-u)_+^{m-\mu-1}}{(m-\mu-1)!}du\right.$$

$$\left.\cdot s_i^{\mu\nu}\times\prod_{j=1}^{m}\prod_{k=\nu+1}^{m}\omega_j(\theta_{j5})\omega_k(\theta_{k6})\int_{t_i}^{t}\frac{(t-v)_+^{m-1}}{(m-1)!}\frac{(t_{i+1}-v)_+^{m-\nu-1}}{(m-\nu-1)!}dv\right]$$

where $S_i^{-1} = \{s_i^{\mu\nu}\}$, $S_i = \{s_{i,\mu\nu}\}$, and

$$(3.35)\quad s_{i,\mu\nu} = \prod_{j=\mu+1}^{m}\prod_{k=\nu+1}^{m}\omega_j(\theta_{j7})\omega_k(\theta_{k8})\int_{t_i}^{t_{i+1}}\frac{(t_{i+1}-u)_+^{2m-\mu-\nu-2}}{(m-\mu-1)!(m-\nu-1)!}du$$

and where $\{\theta_{jl}, \theta_{kl}, j, k = 1, 2, \cdots, m,\ l = 1, 2, \cdots, 8\}$ are all in the interval $[t_i, t_{i+1}]$.

Then, by the continuity and strict positivity properties of the $\{\omega_i\}_{i=1}^m$,

$$(3.36)\quad \textstyle\prod_{j=\nu+1}^{m}\omega_j(\theta_{jl}) = \prod_{j=\nu+1}^{m}\omega_j(t_i)(1+O(\Delta_i))$$

$$\nu = 0, 1, 2, \cdots, m-1, \quad l = 1, 2, \cdots, 8.$$

In particular

$$(3.37)\quad s_{i,\mu\nu} = \textstyle\prod_{j=\mu+1}^{m}\prod_{k=\nu+1}^{m}\omega_j(t_i)\omega_k(t_i)(1+O(\Delta_i))\tilde{s}_{i,\mu\nu}$$

where

$$(3.38)\quad \tilde{s}_{i,\mu\nu} = \int_{t_i}^{t_{i+1}}\frac{(t_{i+1}-u)_+^{2m-\mu-\nu-2}}{(m-\mu-1)!(m-\nu-1)!}du.$$

The matrix $\tilde{S}_i$, $\tilde{S}_i = \{\tilde{s}_{i,\mu\nu}\}$, is strictly positive definite. We have by (3.35) and the continuity properties of the matrix inverse transformation

$$(3.39)\quad s_i^{\mu\nu} = [\textstyle\prod_{j=\mu+1}^{m}\prod_{k=\nu+1}^{m}\omega_j(t_i)\omega_k(t_i)]^{-1}\tilde{s}_i^{\mu\nu}(1+O(\Delta_i))$$

where $\tilde{s}_i^{\mu\nu}$ is defined by $\tilde{S}_i^1 = \{\tilde{s}_i^{\mu\nu}\}$. We therefore have the right-hand side of (3.34) is given by

(r.h.s.) (3.34) =

$$\prod_{k=1}^{m}\omega_k^2(t_i)(1+O(\Delta_i))\int_{t_i}^{\min(s,t)}\frac{(s-u)_+^{m-1}(t-u)_+^{m-1}}{(m-1)!(m-1)!}du$$

$$(3.40)\quad -\prod_{k=1}^{m}\omega_k^2(t_i)(1+O(\Delta_i))\times\sum_{\mu,\nu=0}^{m-1}\int_{t_i}^{s}\frac{(s-u)_+^{m-1}(t_{i+1}-u)_+^{m-\mu-1}}{(m-1)!(m-\mu-1)!}du$$

$$\cdot\tilde{s}_i^{\mu\nu}\int_{t_i}^{t}\frac{(t-v)_+^{m-1}(t_{i+1}-v)_+^{m-\nu-1}}{(m-1)\,(m-\nu-1)!}dv.$$

From (3.32), (3.40) and Lemma 2, we obtain

$$\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} B_i(s,t)\,ds\,dt = \frac{(m!)^2}{(2m)!(2m+1)!}\alpha(t_i)(t_{i+1}-t_i)^{2m+1}$$

$$(3.41)\quad +O(\Delta_i)\left\{\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} ds\,dt\left[\int_{t_i}^{\min(s,t)} \frac{(s-u)_+^{m-1}(t-u)_+^{m-1}}{(m-1)!(m-1)!}du\right\}\right.$$

$$+O(\Delta_i)\left\{\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} ds\,dt \sum_{\mu,\nu=0}^{m-1}\left[\int_{t_i}^{s}\frac{(s-u)_+^{m-1}}{(m-1)!}\frac{(t_{i+1}-u)_+^{m-\mu-1}}{(m-\mu-1)!}du\,\tilde{s}_i^{\mu\nu}\right.\right.$$

$$\left.\left.\times\int_{t_i}^{t}\frac{(t-v)_+^{m-1}}{(m-1)!}\frac{(t_{i+1}-v)^{m-\nu-1}}{(m-\nu-1)!}dv\right]\right\}.$$

Since the first term in curly brackets in (3.41) is greater than the second term in curly brackets (which is nonnegative), we have, upon evaluating the first term in curly brackets

$$\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} B_i(s,t)\,ds\,dt$$

$$(3.42)\quad = \frac{(m!)^2}{(2m)!(2m+1)!}\alpha(t_i)(t_{i+1}-t_i)^{2m+1} + O(\Delta_i)(t_{i+1}-t_i)^{2m+1}$$

$$= \frac{(m!)^2}{(2m)!(2m+1)!}\alpha(t_i)(t_{i+1}-t_i)^{2m+1}(1+O(\Delta_i))$$

and the Lemma is proved.

THEOREM 2. *Let* $EX(s)X(t) = Q(s,t)$, $s, t \in [0,1]$, *where* $X(t)$ *satisfies*

$$L_m X(t) = dW(t)/dt$$

$$X^{(\nu)}(0) = \xi_{\nu+1}, \qquad \nu = 0, 1, 2, \cdots, m-1$$

where $W(t)$ *is a Wiener process,* $\{\xi_\nu\}_{\nu=1}^m$ *are* m *linearly independent, normal, zero mean random variables independent of* $W(t)$, *and* L_m *is an* mth *order differential operator with null space spanned by an* ECT *system of continuity class* C^{2m}.

Let $f(s) = \int_0^1 Q(s,t)\rho(t)\,dt$ *and*

$$\lim_{s\downarrow t}\frac{\partial^{2m-1}}{\partial s^{2m-1}}Q(s,t) - \lim_{s\uparrow t}\frac{\partial^{2m-1}}{\partial s^{2m-1}}Q(s,t) = (-1)^m\alpha(t).$$

Suppose ρ *is strictly positive and has a bounded first derivative on* $[0,1]$. *Let*

$$(3.43)\quad \int_0^{t_{in}^*}[\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}du = \frac{i}{n}\int_0^1[\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}du \qquad i = 1, 2, \cdots, n$$

$$t_{0n}^* = 0.$$

Then T_n^* is an asymptotically optimal sequence, and

$$\|f-P_{m,T_n^*}f\|_Q^2=\frac{1}{n^{2m}}\frac{(m!)^2}{(2m)!(2m+1)!}\left[\int_0^1[\rho^2(\theta)\alpha(\theta)]^{(2m+1)^{-1}}\,d\theta\right]^{2m+1}+O\left(\frac{1}{n^{2m}}\right). \tag{3.44}$$

PROOF. Let $\Delta=\max_i|t_{i+1}-t_i|$. We know that for any asymptotically optimal sequence, with $\rho>0$, $\lim_{n\to\infty}\Delta=0$, since otherwise $\|f-P_{m,T_n}f\|_Q^2$ will not tend to zero. The following argument is similar to that in [8].

Using a Holder inequality on (3.27), gives, for any T_n that includes $t_0=0$, $t_n=1$,

$$\begin{aligned}\|f-P_{m,T_n}f\|_Q^2 &\geqq \frac{1}{n^{2m}}\frac{(m!)^2}{(2m)!(2m+1)!}\left[\sum_{i=0}^{n-1}\rho^{(2m+1)^{-2}}(t_i)\alpha^{(2m+1)^{-1}}(t_i)\right.\\ &\qquad\left.\cdot(1+O(\Delta_i))(t_{i+1}-t_i)\right]^{2m+1}\\ &=\frac{1}{n^{2m}}\frac{(m!)^2}{(2m)!(2m+1)!}\left[\int_0^1[\rho^2(t)\alpha(t)]^{(2m+1)^{-1}}\,dt\right]^{2m+1}+\frac{O(\Delta)}{n^{2m}}.\end{aligned} \tag{3.45}$$

Now, using (3.43) and the mean value theorem,

$$\begin{aligned}\rho^{(2m+1)^{-2}}(\theta_i^*)\alpha^{(2m+1)^{-1}}(\theta_i^*)(t_{i+1,n}^*-t_{in}^*) &=\int_{t_{in}^*}^{t_{i+1,n}^*}[\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}\,du\\ &=\frac{1}{n}\int_0^1[\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}\,du\end{aligned} \tag{3.46}$$

where θ_i^* is some number in $[t_{in}^*, t_{i+1,n}^*]$.

If, in Lemma 3 we use that

$$\rho^2(t_i)\alpha(t_i)(1+O(\Delta_i))=\rho^2(\theta_i^*)\alpha(\theta_i^*)(1+O(\Delta_i)) \tag{3.47}$$

we have

$$\begin{aligned}\|f-P_{m,T_n^*}f\|_Q^2 &=\frac{1}{n^{2m+1}}\frac{(m!)^2}{(2m)!(2m+1)!}\left[\int_0^1[\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}\,du\right]^{2m+1}\\ &\qquad\cdot\sum_{i=0}^{n-1}(1+O(\Delta_i))\\ &=\frac{1}{n^{2m}}\frac{(m!)^2}{(2m)!(2m+1)!}\left[\int_0^1[\rho^2(u)\alpha(u)]^{-(2m+1)^{-1}}\,du\right]+O\left(\frac{1}{n^{2m}}\right).\end{aligned} \tag{3.48}$$

Since $\|f-P_{m,T_n^*}f\|_Q^2$ achieves the lower bound (3.45) up to a vanishingly small term, T_n^* is an asymptotically optimal sequence. This completes the proof of Theorem 2.

It appears that the theorem can be proved under weaker conditions on ρ, similar to those considered in [10]. We do not carry this out.

4. Other related results. Suppose that a square root G of Q is known of the form

$$Q(t, t') = \int_0^1 G(t, u)G(t', u)\,du \tag{4.1}$$

where only $G(t, \cdot) \in L_2[0, 1]$, for every $t \in [0, 1]$. Then $X(t)$ has a representation $X(t) = \int_0^1 G(t, u)\,dW(u)$ and

$$\begin{aligned} f(t) &= \int_0^1 Q(t, t')\rho(t')\,dt' \\ f(\cdot) &\sim \int_0^1 \rho(t)X(t)\,dt \\ &= \int_0^1 \int_0^1 \rho(t')G(t', u)\,dt'\,dW(u) \\ &= \int_0^1 h(u)\,dW(u) \end{aligned} \tag{4.2}$$

where

$$h(u) = \int_0^1 G(t, u)\rho(t)\,dt. \tag{4.3}$$

If $X(t)$ has $m-1$ quadratic mean derivatives, then for any constants $\{c_{i\nu}\}$,

$$\begin{aligned} \|f - \textstyle\sum_{i=0}^n \sum_{\nu=0}^{m-1} c_{i\nu} Q_{t_i}^{(\nu)}\|_Q^2 &= E\{\int_0^1 \rho(t)X(t)\,dt - \textstyle\sum_{i=1}^n \sum_{\nu=0}^{m-1} c_{i\nu} X^{(\nu)}(t_i)\}^2 \\ &= \int_0^1 (h(u) - \textstyle\sum_{i=0}^n \sum_{\nu=0}^{m-1} c_{i\nu} G^{(\nu)}(t_i, u))^2\,du \end{aligned} \tag{4.4}$$

where
$$G^{(\nu)}(t_i, u) = \frac{\partial^\nu}{\partial t^\nu} G(t, u)\bigg|_{t=t_i}.$$

Hence the design problems we have considered are equivalent to the problem of best approximation of $h(u)$ by linear combinations of $\{G(t_i, u)\}_{i=1}^n$ or $\{G^{(\nu)}(t_i, u)\}_{\nu=0, i=0}^{m-1, n}$ in the L_2 norm.

Let $g \in \mathscr{H}_Q$, then a quadrature formula for $\int_0^1 \rho(t)g(t)\,dt$ is given by $\int_0^1 \rho(t)P_{T_n}g(t)\,dt$, since this latter expression is a linear combination of the values of $g(t)$ at $t = t_i \in T_n$.

Then

$$\begin{aligned} &|\int_0^1 \rho(t)g(t)\,dt - \int_0^1 \rho(t)P_{T_n}g(t)\,dt|^2 \\ &\quad = |\langle f, g - P_{T_n}g\rangle_Q|^2 \\ &\quad = |\langle f - P_{T_n}f, g\rangle_Q|^2 \leqq \|g\|_Q \|f - P_{T_n}f\|_Q \\ &\quad \leqq \|g\|_Q \cdot \int_0^1 (h(u) - \textstyle\sum_{i=1}^n c_i G(t_i, u))^2\,du. \end{aligned} \tag{4.5}$$

In the case $G(t, u) = \dfrac{(t-u)_+^{m-1}}{(m-1)!}$, $\rho(t) = 1$, we have

$$h(u) = \frac{(1-u)^m}{m!}. \tag{4.6}$$

By making the change of variable $x = 1-u$ in (4.5) the problem of minimizing $\|f-P_{T_n}f\|_Q^2$ is equivalent to that of optimally approximating the monomial $x_m/m!$ by linear combinations of the functions

$$\left\{\frac{(x-\xi_i)_+^{m-1}}{(m-1)!}\right\}_{i=0}^{n},$$

($\xi_i = 1-t_i$), in the $L_2[0, 1]$ norm. Similarly, it is clear that the problem of minimizing $\|f-P_{m,T_n}f\|_Q^2$ is equivalent to optimally approximating $x^m/m!$ by linear combination of the functions

$$\left\{\frac{(x-\xi_i)_+^{m-1-\nu}}{(m-1-\nu)!}\right\}_{\nu=0,\, i=0}^{m-1\ \ n}$$

in the $L_2[0, 1]$ norm. Such linear combinations are known as spline functions. (See e.g. the volume in [11]). Functions of the form

$$s(x) = \frac{x^m}{m!} + \sum_{i=1}^{n} \sum_{\nu=0}^{m-1} c_{i\nu} \frac{(x-\xi_i)_+^{m-1-\nu}}{(m-1-\nu)!} \tag{4.7}$$

for some constants $\{c_{i\nu}\}$ are known in the approximation theory literature (see [5] [11]) as monosplines.

Monosplines of smallest L_2 norm have recently attracted attention in the context of establishing optimal quadrature formulae via minimizing the error bound of (4.5). Some of the results are relevant to the experimental design problem. These results are available when $(t_i-u)_+^{m-1-\nu}/(m-1-\nu)!$ is replaced by $G_{m,\nu}(t_i, u)$ of (2.6), $\nu = 0, 1, 2, \ldots, m-1$.[3] We state two relevant theorems, in our notation.

THEOREM. (Karlin [5] following Theorem 5). *Let Q be of the form (2.13), f given by (4.2) with $\rho(t) = 1$. Then, for every $T_n \in \mathscr{D}_n$, there exists a $\tilde{T}_{mn} \in \mathscr{D}_{mn}$ such that*

$$\|f-P_{\tilde{T}_{mn}}f\|_Q^2 \leqq \|f-P_{m,T_n}f\|_Q^2.$$

Professor Karlin informs us that it is sufficient for this Theorem that only $\rho(t) > 0$.

THEOREM. (Karlin [5] Theorem 5). *Let Q satisfy the hypotheses of the preceding theorem. Then*

$$\inf_{T_n \in \mathscr{D}_n} \|f-P_{T_n}f\|_Q^2 = \|f-P_{T_n^*}f\|_Q$$

where

(i) *T_n^* is unique.*

(ii) *T_n^* consists of n distinct points.*

(iii) $\langle f-P_{T_n^*}f, Q_{t_i^*}^{(1)}\rangle_Q = 0$, $t_i^* \in T_n^*$, $(m > 1)$.

[3] Linear combinations of the functions $\{G_{m-\nu}(t_i, u)\}_{\nu=0,\, i=1}^{m-1,\, n}$ are so called Tchebychev splines with respect to L^*, compare [4], Chapter 10, Section 3.

The statement (iii) is the remarkable result that, at the optimal design T_n^* for data without derivatives, the addition of first derivatives to the data set provides no new information.

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