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THE EXACT DISTRIBUTION OF WILKS' CRITERION[1]

BY A. M. MATHAI AND P. N. RATHIE
McGill University and University of Waterloo

1. Summary. In this paper the exact distribution of Wilks' likelihood ratio criterion for testing linear hypothesis about regression coefficients is discussed. The exact distribution, in the most general case, is given in simple algebraic functions which can be computed without much difficulty. Explicit expressions for the density function as well as for the cumulative distribution function are given under the null hypothesis.

2. Introduction. Consider a multivariate linear setup under normality. That is, let $x_1, \cdots, x_N$ be a set of N observations, x_α being drawn from $N(\boldsymbol{\beta} Z_\alpha, \Sigma)$. The vectors Z_α, with t components, are known and the $p \times p$ matrix Σ and the $p \times t$ matrix $\boldsymbol{\beta}$ are unknown. Let $N \geqq p+t$ and the rank of $\mathbf{Z} = (Z_1, \cdots, Z_N)$ be t. Let $\boldsymbol{\beta}$ be partitioned into,

$$\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$$

where $\boldsymbol{\beta}_1$ has t_1 columns and $\boldsymbol{\beta}_2$ has t_2 columns. Consider the problem of testing the hypothesis,

$$H: \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^*$$

where $\boldsymbol{\beta}_1^*$ is a given matrix. Let $U = \lambda^{2/N}$ where λ is the likelihood ratio criterion for testing H. The moments of U, when the hypothesis is true, are evaluated for several cases ([1] page 192–194). They are,

$$(1) \qquad E(U^h) = \prod_{i=1}^{p} [\{\Gamma[(n+1-i)/2+h]\Gamma[(n+t_1+1-i)/2]\}/\{\Gamma[(n+1-i)/2] \cdot \Gamma[(n+t_1+1-i)/2+h]\}],$$

where $n = N-t$. If U is denoted as $U_{p,t_1,n}$, then it is pointed out in ([1] page 193) that the distribution of $U_{p,t_1,N-t_1-t_2}$ is the same as that of $U_{t_1,p,N-p-t_2}$, when the hypothesis is true. Hence, when obtaining the distribution of $U_{p,q,n}$, without loss of generality, we need consider only the cases where $q \geqq p$.

The exact distribution of $U_{p,q,n}$ is obtained by several authors for particular values of p and q. When $p = 1$, it is easy to see that $n(1-U_{1,q,n})/qU_{1,q,n}$ has an F-distribution with q and n degrees of freedom. $(n+1-p)(1-U_{p,1,n})/pU_{p,1,n}$ has an F-distribution with p and $n+1-p$ degrees of freedom. Wilks (1932) who introduced the statistic $U_{p,q,n}$ obtained the exact distributions for the cases $p = 1, 2, 3$, $q = 3$; $p = 4$, $q = 4$ by direct methods. Schatzoff (1964), (1966) first considered the representation of $(-\log U)$ as a sum of independently distributed random

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variables and derived the exact expressions for its distribution in all cases for p and q by taking successive convolutions. He showed how to compute numerical values of the coefficients in the derived expressions (for both the density and distribution functions) by recursive computational techniques. Nair (1938) considered for the first time the application of moment sequence in deriving distributions. Consul (1966) used the technique of Mellin Transforms to derive exact expressions for the cases $p = 1, 2, 3, 4$ and $q = 3, 4, 5, 6, 7, 8$. Pillai and Gupta (1969) used the method of successive convolutions as suggested by Schatzoff, but on a different representation of U than that considered $U_{p,q,n}$ as a product of p independent beta random variables $X_1, \cdots, X_p$ with X_i distributed as $B((n-i+1)/2, q/2)$. Pillai and Gupta used the representations:

(a) $U_{2r,q,n}$ distributed like $Y_1^2, \cdots, Y_r^2$ where Y_i are independent $B(n+1-2i, q)$, $i = 1, \cdots, r$ and

(b) $U_{2s+1,q,n}$ distributed like $Z_1^2, \cdots, Z_s^2, Z_{s+1}$, where the Z_i are independently distributed as $B(n+1-2i, q)$ and Z_{s+1} is independently distributed as $B((n+1-p)/2, q/2)$.

They obtained the exact distribution for $p = 3, 4, 5, 6$ and also computed the percentage points which supplemented some tables by Pillai (1960). Approximations and asymptotic expansions are given by Bartlett (1938), Box (1949) and Rao (1948). The technique used by Consul (1966) is to invert the Gamma products of (1) with the help of inverse Mellin Transforms. By this technique the density of U is uniquely determined because the moment sequence in (1) uniquely determines the distribution due to the fact that $0 < u < 1$. He then used some special properties of hypergeometric series to arrive at the exact distributions in the above mentioned cases. In this paper we will use the technique of Mellin Transforms. But before taking the inverse Mellin Transform of the Gamma products in (1) we will eliminate the Gammas by cancelling the common factors and then splitting the factors with the help of partial fraction methods. For convenience we will consider the cases p-even, q-even; p-odd, q-even; p-even, q-odd; and p-odd, q-odd separately, thereby exhausting all the cases. Since the distribution of λ is easily obtained from the distribution of U we will consider only the density and the cumulative distribution function of U. From (1) we can obtain the expected value of U^{s-1} as,

$$E(U^{s-1}) = C\prod_{i=1}^{p}\{\Gamma[(n+1-i)/2+s-1]/\Gamma[(n+1+q-i)/2+s-1]\} \tag{2}$$

where,

$$C = \prod_{i=1}^{p}\{\Gamma[(n+1+q-i)/2]/\Gamma[(n+1-i)/2]\}. \tag{3}$$

Now the density of U is given by the usual Mellin inversion formula,

$$f(u) = (1/2\pi i)\int_L u^{-s}E(U^{s-1})\,ds \tag{4}$$

where L is a suitably selected contour and $i = (-1)^{\frac{1}{2}}$.

3. Case I (*q*-**even,** *p*-**even).** Let $q = 2m \geqq p$. Now $E(U^{s-1})$, excluding the constant C is,

$$(5)\qquad \{\Gamma(n/2+s-1)\Gamma(n/2+s-1-\tfrac{1}{2}) \cdots \Gamma[n/2+s-1-(p-1)/2]\}$$
$$\div \{\Gamma(n/2+m+s-1)\Gamma(n/2+m+s-1-\tfrac{1}{2}) \cdots \Gamma[n/2+m+s-1-(p-1)/2]\}.$$

Here m is a positive integer and therefore, for example, the first two ratios in (5) above are, $[(s+n/2+m-2)(s+n/2+m-3) \cdots (s+n/2-1)]^{-1}$ and

$$[(s+n/2+m-\tfrac{5}{2}) \cdots (s+n/2-\tfrac{3}{2})]^{-1},$$

respectively. Now cancelling the common factors in (5) and then collecting the factors obtained from the alternate Gamma ratios in (5), we get the following structure of factors.

$$(6)\qquad \begin{array}{l} (s+n/2+m-2)(s+n/2+m-3) \cdots (s+n/2-1) \\ (s+n/2+m-3)(s+n/2+m-4) \cdots (s+n/2-2) \\ \cdots \\ (s+n/2+m-p/2-1) \cdots (s+n/2-p/2) \end{array}$$

and

$$(7)\qquad \begin{array}{l} (s+n/2+m-\tfrac{5}{2})(s+n/2+m-\tfrac{7}{2}) \cdots (s+n/2-\tfrac{3}{2}) \\ \cdots \\ (s+n/2+m-p/2-\tfrac{3}{2})(s+n/2+m-p/2-\tfrac{5}{2}) \cdots (s+n/2-p/2-\tfrac{1}{2}). \end{array}$$

Denoting (6) by X and (7) by Y, we have,

$$(8)\qquad E(U^{s-1}) = C/XY.$$

Several factors are repeated in the sets (6) and (7). In fact the sets (6) and (7) can be written as,

$$(9)\qquad (s+n/2+m-2)(s+n/2+m-3)^2 \cdots (s+n/2+m-1-p/2)^{p/2} \cdots$$
$$(s+n/2-1)^{p/2} \cdots (s+n/2-p/2+1)^2(s+n/2-p/2) \qquad \text{and}$$

$$(10)\qquad (s+n/2+m-\tfrac{5}{2})(s+n/2+m-\tfrac{7}{2})^2 \cdots (s+n/2+m-\tfrac{3}{2}-p/2)^{p/2}$$
$$\cdots (s+n/2-\tfrac{3}{2})^{p/2} \cdots (s+n/2-p/2+\tfrac{1}{2})^2(s+n/2-p/2-\tfrac{1}{2})$$

respectively. The exponents of the factors in (9) and (10) are in the sequence,

$$(11)\qquad 1, 2, \cdots, p/2-1, p/2, \cdots, p/2, p/2-1, \cdots, 3, 2, 1$$

where the exponent $p/2$ is repeated $m-p/2+1$ times. Now we will split $E(U^{s-1})$ into separate terms by using the technique of partial fractions. Let $\alpha = s+n/2$ and let $(\alpha-\delta)^j$ denote a factor in the denominator of (8). There will be j terms corresponding to $(\alpha-\delta)^j$ when $E(U^{s-1})$ is put into a sum by the method of partial fractions. Let σ_k be the coefficients of the kth term coming from $(\alpha-\delta)^j$. Then from the theory of partial fractions,

$$\sigma_k = [1/(j-k)!]\left[\frac{\partial^{j-k}}{\partial s^{j-k}}(\alpha-\delta)^j E(U^{s-1})\right], \quad \text{at} \quad \alpha = \delta, \tag{12}$$

$k = 1, 2, \cdots, j$.

In order to evaluate σ_k explicitly we will use the following technique. Let,

$$Z = (\alpha-\delta)^j E(U^{s-1}) \quad \text{and} \quad Z_0 = Z \quad \text{at} \quad \alpha = \delta. \tag{13}$$

Then,

$$\frac{\partial Z}{\partial s} = AZ \qquad \text{where} \quad A = \frac{\partial}{\partial s}\log Z. \tag{14}$$

That is, in case I,

$$A = (-1)[(\alpha+m-\tfrac{4}{2})^{-1}+(\alpha+m-\tfrac{5}{2})^{-1}+2(\alpha+m-\tfrac{6}{2})^{-1}+\cdots +(\alpha-p/2-\tfrac{1}{2})^{-1}] \tag{15}$$

and the term containing $j/(\alpha-\delta)$ is absent in (15). There are altogether $2m+p-3$ terms which are summed up in A in (15). Also,

$$\frac{\partial^n Z}{\partial s^n} = AZ^{(n-1)}+\binom{n-1}{1}A^{(1)}Z^{(n-2)}+\cdots+\binom{n-1}{n-1}A^{(n-1)}Z, \tag{16}$$

where $A^{(r)}$ denotes the rth derivative of A with respect to s.
Thus a recurrence relation is obtained as,

$$I_n = AI_{n-1}+\binom{n-1}{1}A^{(1)}I_{n-2}+\cdots+\binom{n-1}{n-1}I_0A^{(n-1)} \tag{17}$$

where, $I_n = \partial^n Z/\partial s^n$ and in case I,

$$A^{(r)} = (-1)^{r+1}r![(\alpha+m-\tfrac{4}{2})^{-r-1}+(\alpha+m-\tfrac{5}{2})^{-r-1} +2(\alpha+m-\tfrac{6}{2})^{-r-1}+\cdots+(\alpha-p/2-\tfrac{1}{2})^{-r-1}], \tag{18}$$

omitting the term containing $(\alpha-\delta)^{-r-1}$, and let $A_0{}^{(r)}$ denote $A^{(r)}$ at $\alpha = \delta$. (17) can be written as a single sum involving only Z and $A^{(r)}$, for $r \geqq 0$, but the above representation seems to be easier to handle. With the help of (12) to (17) we can evaluate the coefficients explicitly.

It is easily seen that the sequence of exponents in (9) and (10) are symmetric from both ends. For convenience we will denote the coefficients corresponding to the factors from either end in (9) by a_{ik}^j and c_{ik}^j and the coefficients corresponding to the factors in (10) by b_{ik}^j and d_{ik}^j. That is, a_{ik}^j denotes the coefficient of the term $(\alpha+m-1-i)^{-k}$ corresponding to the factor $(\alpha+m-1-i)^{-j}$. In $a_{ik}^j, \cdots, d_{ik}^j$, j does not denote a power. c_{ik}^j denotes the coefficient of the term $(\alpha-p/2-1+i)^{-k}$ corresponding to the factor $(\alpha-p/2-1+i)^{-j}$; $k = 1, \cdots, j$. Similarly b_{ik}^j and d_{ik}^j

stand for the coefficients corresponding to the factors in (10). By using this notation, we can write, $E(U^{s-1})$ in the following form.

$$(19)\qquad E(U^{s-1}) = C\{\textstyle\sum_{a\cup a'} a^j_{ik}/(\alpha+m-1-i)^k+\sum_c c^j_{ik}/[\alpha-(p+2)/2+i]^k + \sum_{b\cup b'} b^j_{ik}/(\alpha+m-\frac{3}{2}-i)^k+\sum_d d^j_{ik}/[\alpha-(p+3)/2+i]^k\},$$

where,

$$(20)\qquad a = \{(i,j,k)\,|\,j=i,\, 1 \leqq k \leqq i \leqq p/2-1\} = c = b = d,$$
$$a' = \{(i,j,k)\,|\,j=p/2,\, i=p/2, p/2+1, \cdots, m;\, 1\leqq k \leqq p/2\} = b'.$$

Now it is seen that σ_k corresponding to $\delta = i+1-m,\ i+\frac{3}{2}-m,\ p/2+1-i,\ p/2+\frac{3}{2}-i$, gives a^j_{ik}, b^j_{ik}, c^j_{ik} and d^j_{ik} respectively. Once we obtain Z_0 and $A_0{}^{(r)}$, $r \geqq 0$, for a^j_{ik}, b^j_{ik}, c^j_{ik} and d^j_{ik}, all the coefficients, that is, for $1 \leqq k \leqq j$, are obtained from (12) to (18). Hence, in the following table we give only the value of Z_0 corresponding to a's, b's, c's and the d's. A_0 and $A_0{}^{(r)}$ are easily obtained from (15) and (18). These Z_0's are calculated by direct substitution.

TABLE 1

	Z_0 p-even, q-even, ($q = 2m \geqq p$)
a^j_{ik} $(i,j,k)\in a$	$\{(-1)^{pm-j^2}2^{pm-j}1!3!\cdots(p-2j-1)!\}/\{[0!2!\cdots(2j-2)!][(2m+p-2j-1)!(2m+p-2j-3)!\cdots(2m-2j+1)!]\}$.
a^j_{ik} $j=p/2,\ i=p/2+r$ $0\leqq r\leqq m-p/2$	$\{(-1)^{pm-(p/2)(p/2+2r)}2^{pm-p/2}\}/\{[(p+2r-2)!(p+2r-4)!\cdots(2r)!][(2m-2r-1)!(2m-2r-3)!\cdots(2m-2r-p+1)!]\}$
c^j_{ik} $(i,j,k)\in c$	$\{(-1)^{j^2}2^{pm-j}0!2!\cdots(p-2j-2)!\}/\{[1!3!\cdots(2j-1)!][(2m+p-2j-2)!(2m+p-2j-4)!\cdots(2m-2j)!]\}$
b^j_{ik} $(i,j,k)\in b$	$\{(-1)^{pm-j(j+1)}2^{pm-j}0!2!\cdots(p-2j-2)!\}/\{[1!3!\cdots(2j-1)!][(2m+p-2j-2)!(2m+p-2j-4)!\cdots(2m-2j)!]\}$
b^j_{ik} $j=p/2,\ i=p/2+r$ $0\leqq r\leqq m-p/2$	$\{(-1)^{pm-(p/2)(p/2+2r+1)}2^{pm-p/2}\}/\{[(p+2r-1)!(p+2r-3)!\cdots(2r+1)!][(2m-2r-2)!(2m-2r-4)!\cdots(2m-p-2r)!]\}$
d^j_{ik} $(i,j,k)\in d$	$\{(-1)^{j(j-1)}2^{pm-j}1!3!\cdots(p-2j-1)!\}/\{[0!2!\cdots(2j-2)!][(2m+p-2j-1)!(2m+p-2j-3)!\cdots(2m-2j+1)!]\}$

In order to bring in symmetry in d^j_{ik} we can multiply $(-1)^{j(j-1)}$ by $(-1)^{2j}$ to obtain $(-1)^{j(j+1)}$. It is noticed from the above table that the coefficients can be easily written down due to the type of symmetry in the various factorials.

REMARK. In order to maintain symmetry and to facilitate easy computation, the complete simplification, of the quantities appearing in the various coefficients and factors, is not done throughout this article.

Now the density function of U, that is, $f(u)$ is obtained by taking the inverse Mellin Transform of (19). That is,

$$(21)\qquad f(u) = C\{\textstyle\sum_{a\cup a'} a^j_{ik} u^{n/2+m-1-i}\beta_k + \sum_c c^j_{ik} u^{n/2-p/2-1+i}\beta_k + \sum_{b\cup b'} b^j_{ik} u^{n/2+m-\frac{3}{2}-i}\beta_k + \sum_d d^j_{ik} u^{n/2-p/2-\frac{3}{2}+i}\beta_k\}$$

for $0 < u < 1$, where β_k stands for $(-\log u)^{k-1}/\Gamma(k)$. This result follows from Erdélyi ((1954) page 343 (16)). The authors have verified that for $p = 4$, $q = 4$ the above results agree with Consul's (1966) algebraic representations for $p = 4$, $q = 4$. In order to show the simplicity of our method we will tabulate a case which is not given by Consul (1966). The following table gives the coefficients for $p = 4$, $q = 10$.

TABLE 2

	Z_0 A_0Z_0 $\quad p = 4,\ q = 10$
a^1_{11}	$(-1)2^{19}1!/(0!11!9!)$
b^1_{11}	$2^{19}0!/(1!10!8!)$
a^2_{22} a^2_{21}	$2^{18}/(2!0!9!7!)$ $(-2)(1/2+1/1-2/1-2/2-2/3-2/4-2/5-2/6-2/7-1/8-1/9)Z_0$
b^2_{22} b^2_{21}	$2^{18}/(3!1!8!6!)$ $(-2)(1/3+1/2+2/1-2/1-2/2-2/3-2/4-2/5-2/6-1/7-1/8)Z_0$
a^2_{32} a^2_{31}	$2^{18}/(4!2!7!5!)$ $(-2)(1/4+1/3+2/2+2/1-2/1-2/2-2/3-2/4-2/5-1/6-1/7)Z_0$
b^2_{32} b^2_{31}	$2^{18}/(5!3!6!4!)$ $(-2)(1/5+1/4+2/3+2/2+2/1-2/1-2/2-2/3-2/4-1/5-1/6)Z_0$
a^2_{42} a^2_{41}	$2^{18}/(6!4!5!3!)$ $(-2)(1/6+1/5+2/4+2/3+2/2+2/1-2/1-2/2-2/3-1/4-1/5)Z_0$
b^2_{42} b^2_{41}	$2^{18}/(7!5!4!2!)$ $(-2)(1/7+1/6+2/5+2/4+2/3+2/2+2/1-2/1-2/2-1/3-1/4)Z_0$
a^2_{52} a^2_{51}	$2^{18}/(8!6!3!1!)$ $(-2)(1/8+1/7+2/6+2/5+2/4+2/3+2/2+2/1-2/1-1/2-1/3)Z_0$
b^2_{52} b^2_{51}	$2^{18}/(9!7!2!0!)$ $(-2)(1/9+1/8+2/7+2/6+2/5+2/4+2/3+2/2+2/1-1/1-1/2)Z_0$
c^1_{11}	$(-1)2^{19}0!/(1!10!8!)$
d^1_{11}	$2^{19}1!/(0!11!9!)$
C	$[\Gamma(n/2+10/2)\Gamma(n/2+9/2)\Gamma(n/2+8/2)\Gamma(n/2+7/2)]/[\Gamma(n/2)\Gamma(n/2-1/2)$ $\Gamma(n/2-2/2)\Gamma(n/2-3/2)]$

3.1. The cumulative distribution function. The cumulative distribution function $F(x)$ is obtained by integrating the density in (21) from 0 to x. That is,

$$F(x) = \int_0^x f(u)\,du. \tag{22}$$

In order to evaluate $F(x)$ we will derive the following general result.

LEMMA 3.1.

$$\begin{aligned}\int_0^x u^\alpha(-\log u)^{k-1}\,du &= x^{\alpha+1}\textstyle\sum_{r=1}^k k(k-1)(k-2)\cdots(k-r+1)\\ &\quad\cdot(-\log x)^{k-r}/k(\alpha+1)^r\end{aligned} \tag{23}$$

$\alpha > 0$, *k—a positive integer*, $0 < u < 1$.

Now $F(x)$ can be written down by using (23). That is,

$$\begin{aligned}F(x) = C\{&\textstyle\sum_{a\cup a'}[a_{ik}^j x^{n/2+m-i}\mu_{kr}/(n/2+m-i)^r]\\ &+\textstyle\sum_{b\cup b'}[b_{ik}^j x^{n/2+m-1/2-i}\mu_{kr}/(n/2+m-1/2-i)^r]\\ &+\textstyle\sum_c[c_{ik}^j x^{n/2-p/2+i}\mu_{kr}/(n/2-p/2+i)^r]\\ &+\textstyle\sum_d[d_{ik}^j x^{n/2-p/2-1/2+i}\mu_{kr}/(n/2-p/2-\frac{1}{2}+i)^r]\}\end{aligned} \tag{24}$$

where μ_{kr} stands for $\sum_{r=1}^k k(k-1)\cdots(k-r+1)(-\log x)^{k-r}/\Gamma(k+1)$.

4. Case II (q-even, p-odd, $q = 2m \geqq p$). Here also, proceeding in a similar way as in Section 3 we obtain the reciprocal of $E(U^{s-1})$, excluding the constant C, as,

$$\begin{aligned}&(s+n/2+m-2)(s+n/2+m-3)\cdots(s+n/2-1)(s+n/2+m-\tfrac{5}{2})\cdots(s+n/2-\tfrac{3}{2})\\ &\times(s+n/2+m-3)(s+n/2+m-5)\cdots(s+n/2-2)(s+n/2+m-\tfrac{7}{2})\cdots(s+n/2-\tfrac{5}{2})\\ &\cdots\qquad\cdots\\ &\times(s+n/2-p/2+\tfrac{1}{2}+m-2)\cdots(s+n/2-p/2+\tfrac{1}{2}-1)\\ &\times(s+n/2-p/2+1+m-2)\cdots(s+n/2-p/2+1-1).\end{aligned} \tag{25}$$

The density is obtained as,

$$\begin{aligned}f(u) = C\{&\textstyle\sum_{a\cup a'} a_{ik}^j u^{n/2+m-1-i}\beta_k + \sum_c c_{ik}^j u^{n/2-p/2-\frac{3}{2}+i}\beta_k\\ &+\textstyle\sum_{b\cup b'} b_{ik}^j u^{n/2+m-\frac{3}{2}-i}\beta_k + \sum_d d_{ik}^j u^{n/2-p/2-1+i}\beta_k\},\end{aligned} \tag{26}$$

$0 < u < 1$, where β_k is given in (21) and

$$\begin{aligned}a &= \{(i,j,k)\,|\,j = i, 1 \leqq k \leqq j \leqq (p-1)/2\} = c,\\ a' &= \{(i,j,k)\,|\,j = (p+1)/2, 1 \leqq k \leqq j, i = (p+1)/2, (p+3)/2, \cdots, \cdots,\\ &\qquad\cdot(m-p/2+\tfrac{1}{2}) \quad \text{terms}\},\\ b &= \{(i,j,k)\,|\,j = i, 1 \leqq k \leqq j \leqq (p-3)/2\} = d,\\ b' &= \{(i,j,k)\,|\,j = (p-1)/2, 1 \leqq k \leqq j, i = (p-1)/2, (p+1)/2, \cdots, \cdots,\\ &\qquad\cdot m-(p-1)/2+1 \quad \text{terms}\}.\end{aligned} \tag{27}$$

Then when $\delta = 1+i-m$, $\frac{3}{2}+i-m$, $p/2+\frac{3}{2}-i$ and $p/2+1-i$, σ_k of (12) gives a_{ik}^j, b_{ik}^j, c_{ik}^j and d_{ik}^j respectively.

The authors have verified the case $p = 3$, $q = 8$ and the results agree with the results obtained by Consul (1966). The cumulative distribution function in this case as well as in the following cases can be obtained in a similar fashion as in Sub-section 3.1. Hence the discussion is deleted.

5. Case III (*p*-even, *q*-odd, $q > p$). In this case we will combine the numerator Gammas and the denominator Gammas separately by using the duplication formula for the Gamma functions, namely,

$$\Gamma(2z) = \pi^{-\frac{1}{2}}2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}). \tag{28}$$

Now we get $E(U^{s-1})$ as,

$$\begin{aligned} E(U^{s-1}) &= C2^{pq/2}\{\Gamma(2s-2+n-1)\Gamma(2s-2+n-3)\cdots\Gamma(2s-2+n-p+1)\} \\ &\div\{\Gamma(2s-2+n-1\times q)\Gamma(2s-2+n-3+q)\cdots\Gamma(2s-2+n-p+1+q)\}. \end{aligned} \tag{29}$$

The reciprocal of $E(U^{s-1})$, excluding C, is,

$$\begin{aligned} &(s+n/2-2/2+q/2-1)(s+n/2-2/2+q/2-\tfrac{3}{2})\cdots(s+n/2-2/2+q/2-q/2-\tfrac{1}{2}) \\ &\cdot(s+n/2-2/2+q/2-2)(s+n/2-2/2+q/2-\tfrac{5}{2})\cdots(s+n/2-\tfrac{2}{2}-\tfrac{3}{2}) \\ &\cdot(s+n/2-\tfrac{2}{2}+q/2-3)(s+n/2-\tfrac{2}{2}+q/2-\tfrac{7}{2})\cdots(s+n/2-\tfrac{2}{2}-\tfrac{5}{2})\cdots\cdots \\ &\cdot(s+n/2-\tfrac{2}{2}+q/2-p/2)(s+n/2-\tfrac{2}{2}+q/2-p/2-\tfrac{1}{2})\cdots(s+n/2-\tfrac{2}{2}-p/2+\tfrac{1}{2}). \end{aligned} \tag{30}$$

The density is obtained as,

$$\begin{aligned} f(u) = C\{&\textstyle\sum_{a\cup a'} a_{ik}^j u^{n/2+q/2-1-i}\beta_k+\sum_c c_{ik}^j u^{n/2-p/2-\frac{3}{2}+i}\beta_k \\ &+\textstyle\sum_{b\cup b'} b_{ik}^j u^{n/2+q/2-\frac{3}{2}-i}\beta_k+\sum_d d_{ik}^j u^{n/2-p/2-1+i}\beta_k\}, \end{aligned} \tag{31}$$

$0 < u < 1$, where,

$$\begin{aligned} a &= \{(i,j,k)\,|\,j = i, 1 \leqq k \leqq j < p/2\} = b = c = d, \\ a' &= \{(i,j,k)\,|\,j = p/2, 1 \leqq k \leqq j, i = p/2, p/2+1, \cdots (q-p+\tfrac{3}{2}) \text{ terms}\}, \\ b' &= \{(i,j,k)\,|\,j = p/2, 1 \leqq k \leqq j, i = p/2, p/2+1, \cdots (q-p+1)/2 \text{ terms}\}. \end{aligned} \tag{32}$$

Now corresponding to $\delta = i$, $i+\frac{1}{2}$, $(p+q-3)/2-i$, $(p+q-3)/2+\frac{1}{2}-i$, a_{ik}^j, b_{ik}^j, c_{ik}^j, and d_{ik}^j are available from σ_k where α in this case is $s+(n-2+q)/2$. The authors have verified the result for $p = 4$, $q = 5$ and this agrees with Consul's (1966) result.

6. Case IV (*p*-odd, *q*-odd, $q \geqq p$). In this case since p is odd, $p-1$ is even, so we will separate the last Gamma ratio in $E(U^{s-1})$, namely, $\Gamma[s-1+n/2-(p-1)/2]\div\Gamma[s-1+n/2-(p-1)/2+q/2]$ and expand it in a series. That is,

$$\begin{aligned} &\Gamma[s-1+n/2-(p-1)/2]/\Gamma[s-1+n/2-(p-1)/2+q/2] \\ &= [1/\Gamma(q/2)]\Gamma[s-1+n/2-(p-1)/2]\Gamma(q/2)/\{\Gamma[s-1+n/2-(p-1)/2+q/2]\} \\ &= [1/\Gamma(q/2)]\textstyle\sum_{m=0}^{\infty}(-1)^m\binom{q/2-1}{m}/[s-1+n/2-(p-1)/2+m], \end{aligned} \tag{33}$$

by using the result (2) of page 8 in Erdélyi (1953). The conditions for the expansions are evidently satisfied by (33). Now,

$$E(U^{s-1}) = [C/\Gamma(q/2)] \textstyle\sum_{m=0}^{\infty} (-1)^m \binom{q/2-1}{m} [\Gamma(s-1+n/2)$$

$$\text{(34)} \qquad \cdot\Gamma(s-1+n/2-\tfrac{1}{2}) \cdots \Gamma(s-1+n/2-p/2+1)]/[\Gamma(s-1+n/2+q/2)$$

$$\cdot\Gamma(s-1+n/2-\tfrac{1}{2}+q/2) \cdots \Gamma(s-1+n/2-p/2+1+q/2)$$

$$\cdot(s-1+n/2-p/2+\tfrac{1}{2}+m)].$$

By comparing the result in the case p-even, q-odd, we can write down the density without much difficulty. By using the same notations as in the previous sections, the density is given as,

$$f(u) = C\{\textstyle\sum_{a\cup a'} a_{ik}^{j} u^{n/2+q/2-1-i} \beta_k + \sum_{b\cup b'} b_{ik}^{j} u^{n/2+q/2-\frac{3}{2}-i} \beta_k$$

$$\text{(35)} \qquad +\textstyle\sum_c c_{ik}^{j} u^{n/2-p/2-\frac{3}{2}+i} \beta_k + \sum_d d_{ik}^{j} u^{n/2-p/2-1+i} \beta_k$$

$$+f u^{n/2-p/2-\frac{1}{2}+m} + f_0 u^{n/2-p/2-\frac{1}{2}}\}, \qquad 0 < u < 1,$$

where,

$$a = \{(i,j,k) \mid j = i, 1 \leqq k \leqq j < (p-1)/2\} = c = d = b,$$

$$\text{(36)} \quad a' = \{(i,j,k) \mid j = (p-1)/2, 1 \leqq k \leqq j, i = (p-1)/2, (p-1)/2+1, \cdots$$

$$\cdot(q-p+4)/2 \quad \text{terms}\},$$

$$b' = \{(i,j,k) \mid j = (p-1)/2, 1 \leqq k \leqq j, i = (p-1)/2, (p-1)/2+1$$

$$\cdot \cdots (q-p+2)/2 \quad \text{terms}\}.$$

Corresponding to $\delta = i;\ \frac{1}{2}+i;\ (p+q+1)/2-i;\ (p+q-1)/2+\frac{1}{2}-i;\ (p+q-1)/2;$ $(p+q-1)/2-m;$ we get, $\sigma_k = a_{ik}^{j};\ b_{ik}^{j},\ m \neq (p+q-2)/2-i;\ c_{ik}^{j};\ d_{ik}^{j},\ m \neq i;\ f_0$ and f, $m \geqq (p+q)/2-1$, respectively, where in this case $\alpha = s+(n-2+q)/2$.

REMARK. The results given in this article also cover the exact distribution of the likelihood ratio criterion for testing independence in the multivariate normal case when a p-component vector is partitioned into $q = 2$ subvectors. The authors would like to thank the referee for some valuable comments which enabled them to modify the introduction and to shorten the article.

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