The Annals of Mathematical Statistics
1971, Vol. 42, No. 3, 961–975

THE EXACT ERROR IN SPECTRUM ESTIMATES

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Since the asymptotic expression for the variance of estimators of the spectrum of a stationary time series was derived, it has often been used as an approximation to the variance of estimators using finite samples. Little attempt seems to have been made to investigate the nature of the convergence to the asymptotic form. In this paper an exact expression for the variance is derived on the additional assumption that the time series is a normal process, and is used to study estimators of various different spectra. A philosophy for choosing spectrum estimators is proposed which attempts to place the two forms of error, bias and variance, in their true perspective.

1. Background. Let $\{X_t;\ t = \cdots -2, -1, 0, 1, 2, \cdots\}$ be a real-valued, weakly stationary, normally distributed, discrete stochastic process (time series) with zero mean, *covariance function* $R(v) = E[X_t X_{t+v}] = R(-v)$, and *spectral density*

$$f(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} R(v) \cos v\omega \ (-\pi \leqq \omega \leqq \pi). \tag{1.1}$$

The relation

$$R(v) = \textstyle\int_{-\pi}^{\pi} f(\omega) \cos v\omega \, d\omega \tag{1.2}$$

holds. Given a sample $\{X_t;\ t = 1, 2, \cdots T\}$ from the process, the almost universally adopted form of spectrum estimator is

$$f_T{}^*(\omega) = \frac{1}{\pi} \sum_{v=0}^{T-1} k_T{}^*(v) R_T(v) \cos v\omega \qquad (-\pi \leqq \omega \leqq \pi) \tag{1.3}$$

where $R_T(v)$ is the *sample covariance function*

$$R_T(v) = \frac{1}{T} \sum_{t=1}^{T-v} X_t X_{t+v}, \qquad (0 \leqq v \leqq T), \tag{1.4}$$

and $k_T{}^*(v)$ is the *lag window*, usually formed from a *lag window generator* $k^*(\cdot)$ by

$$k_T{}^*(v) = k^*\left(\frac{v}{M_T}\right) \qquad (v \neq 0) \tag{1.5}$$
$$k_T{}^*(0) = 0.5,$$

where $k^*(\cdot)$ is a nonnegative, bounded, even function satisfying $k^*(\theta) = 0$ if $\theta > 1$, and where M_T is the *truncation point* of the estimator. It is shown in Neave

Received October 29, 1968; revised July 13, 1970.

(1970) that asymptotically as $T \to \infty$ and $M_T/T \to \lambda$, a nonnegative constant, the variance of $f_T{}^*(\omega)$ is given by

$$\text{(1.6)} \quad \lim_{T\to\infty} \operatorname{Var}[f_T{}^*(\omega)] = 2\frac{M_T}{T}(1+\delta_{0,\omega}+\delta_{\pi,\omega})f^2(\omega)\int_0^1 k^{*2}(\theta)(1-\lambda\theta)\,d\theta$$

where $\delta_{a,b} = 1$ if $a = b$ and 0 otherwise. A better-known asymptotic result has $\lambda = 0$ in (1.6).

It will be convenient subsequently to use the *unbiased sample covariance function*

$$\text{(1.7)} \qquad \tilde{R}_T(v) = \frac{1}{T-v}\sum_{t=1}^{T-v} X_t X_{t+v}, \qquad (0 \leqq v \leqq T)$$

in our spectrum estimators. Accordingly we define

$$\text{(1.8)} \qquad \tilde{f}_T(\omega) = \frac{1}{\pi}\sum_{v=0}^{T-1} k_T(v)\tilde{R}_T(v)\cos v\omega.$$

It is apparent that $f_T{}^*(\omega) \equiv \tilde{f}_T(\omega)$ if and only if

$$k_T(v) = (1-v/T)k_T{}^*(v).$$

Recalling that $k_T{}^*(\cdot)$ is generated from the function $k^*(\cdot)$ by (1.5), clearly $k_T(\cdot)$ may be similarly generated from $k(\theta) = (1-\lambda\theta)k^*(\theta)$ where $\lambda = M_T/T$. Assuming consistency of the estimator, which from (1.6) requires $M_T/T \to 0$, $k^*(\cdot)$ and $k(\cdot)$ are asymptotically identical.

2. The exact expression for the variance. We shall be using the expression (1.8) for the spectrum estimator. Since (1.3) is much more common, we shall assume that (1.9) and (1.10) hold so that $\tilde{f}_T(\omega) \equiv f_T{}^*(\omega)$. Then

$$\text{(2.1)} \qquad E[\tilde{f}_T(\omega)] = \frac{1}{\pi}\sum_{v=0}^{M_T} k_T(v)R(v)\cos v\omega,$$

$$\text{(2.2)} \qquad E[\tilde{f}_T{}^2(\omega)] = \frac{1}{\pi^2}\sum_{v,u=0}^{M_T}\sum k_T(v)k_T(u)E[\tilde{R}_T(v)\tilde{R}_T(u)]\cos v\omega\cos u\omega.$$

Since $\operatorname{Var}[\tilde{f}_T(\omega)] = E[\tilde{f}_T{}^2(\omega)] - \{E[\tilde{f}_T(\omega)]\}^2$, we have

$$\text{(2.3)} \quad \operatorname{Var}[\tilde{f}_T(\omega)] = \frac{1}{\pi^2}\sum_{v,u=0}^{M_T}\sum k_T(v)k_T(u)\operatorname{Cov}[\tilde{R}_T(v), \tilde{R}_T(u)]\cos v\omega\cos u\omega$$

where

$$\text{(2.4)} \qquad \operatorname{Cov}[\tilde{R}_T(v), \tilde{R}_T(u)] = E[\tilde{R}_T(v)\tilde{R}_T(u)] - R(v)R(u).$$

Now

$$\text{(2.5)} \qquad E[\tilde{R}_T(v)\tilde{R}_T(u)] = \frac{1}{(T-v)(T-u)}\sum_{t=1}^{T-v}\sum_{s=1}^{T-u} E[X_t X_{t+v} X_s X_{s+u}].$$

It is proved in Isserlis (1918) that for normally distributed variates A, B, C, D, having zero means,

$$E[ABCD] = E[AB]E[CD]+E[AC]E[BD]+E[AD]E[BC]. \tag{2.6}$$

Writing X_t, X_{t+v}, X_s, X_{s+u} for A, B, C, D, (2.4), (2.5) and (2.6) lead to

$$\operatorname{Cov}[\tilde{R}_T(v), \tilde{R}_T(u)] = \frac{1}{(T-v)(T-u)} \sum_{t=1}^{T-v} \sum_{s=1}^{T-u} \{R(t-s)R(t-s+v-u) + R(t-s-u)R(t-s+v)\}. \tag{2.7}$$

Using the fact that $R(\cdot)$ is even, the change of variable $x = t-s$ produces

$$\begin{aligned}\operatorname{Cov}[\tilde{R}_T(v),\tilde{R}_T(u)] = {} & \frac{1}{T-v} \sum_{x=0}^{u-v} \{R(x)R(x+v-u)+R(u-x)R(x+v)\} \\ & + \frac{2}{(T-v)(T-u)} \sum_{x=u-v+1}^{T-v} (T-v-x) \\ & \cdot \{R(x)R(x+v-u)+R(x-u)R(x+v)\},\end{aligned} \tag{2.8}$$

taking $v \leqq u$. The combination of (2.3) and (2.8) produces the result

$$\begin{aligned}\operatorname{Var}[\tilde{f}_T(\omega)] = \frac{2}{\pi^2}\Bigg[& \sum_{v=0}^{M_T} \frac{k_T^2(v)\cos^2 v\omega}{(T-v)^2} \Bigg\{ \frac{T-v}{2}[R^2(0)+R^2(v)] \\ & + \sum_{x=1}^{T-v} (T-v-x)[R^2(x)+R(x-v)R(x+v)] \Bigg\} \\ & + 2 \sum\sum_{0 \leqq v<u \leqq M_T} \frac{k_T(v)k_T(u)\cos v\omega \cos u\omega}{(T-v)(T-u)} \\ & \cdot \Bigg\{ \frac{T-u}{2} \sum_{x=0}^{u-v} [R(x)R(x+v-u)+R(u-x)R(x+v)] \\ & + \sum_{x=u-v+1}^{T-v} (T-v-x)[R(x)R(x+v-u)+R(x-u)R(x+v)] \Bigg\} \Bigg].\end{aligned} \tag{2.9}$$

This expression enables us to compute the exact variance of the estimator of any given spectrum corresponding to a normal stochastic process; the covariance function $R(\cdot)$ is calculated from (1.2). The remainder of the paper discusses some actual computations of this type, and leads to some suggested modes of thought concerning the choices of spectrum estimators.

3. A computer study. A computer study was carried out to investigate the properties of estimators of the four spectra illustrated in Figure 1 (a–d) referred to in sequence by the letters A, B, C, D. A program was written to compute the variance, the expected value, the bias, and hence the mean square error, for each spectrum, given sample lengths T of 30, 60, 90 and 120, at the frequencies

$$\omega = \frac{\pi j}{180}, \qquad j = 0, 1, \cdots, 180 \tag{3.1}$$

and using the lag window generators

$$(3.2)\qquad (1)\ \text{Parzen}\quad k_P{}^*(\theta) = \begin{cases} 1-6\theta^2+6\theta^3, & (0 \leqq \theta < \tfrac{1}{2}) \\ 2(1-\theta)^3, & (\tfrac{1}{2} \leqq \theta \leqq 1). \end{cases}$$

$$(3.3)\qquad (2)\ \text{Tukey}\quad k_T{}^*(\theta) = \tfrac{1}{2}(1+\cos\pi\theta) \qquad (0 \leqq \theta \leqq 1)$$

with ratios of truncation point to sample size:

$$(3.4)\qquad \frac{M_T}{T} = \cdot 1,\ \cdot 2\ \text{ and }\ \tfrac{1}{3}.$$

The results are summarized in Table 1 (a–d) which give the averages of the squared bias, the variance and the mean square error in each case. Throughout, the results have been scaled as percentages relative to

$$\frac{1}{180}\left\{\tfrac{1}{2}f^2(0)+\sum_{j=1}^{179} f^2\left(\frac{\pi j}{180}\right)+\tfrac{1}{2}f^2(\pi)\right\}$$

as 100 per cent, this being an approximation from the frequencies considered of $1/\pi \int_0^\pi f^2(\omega)\, d\omega$. This base was chosen because, by (1.6), the average variance of the spectrum estimator is asymptotically proportional to it.

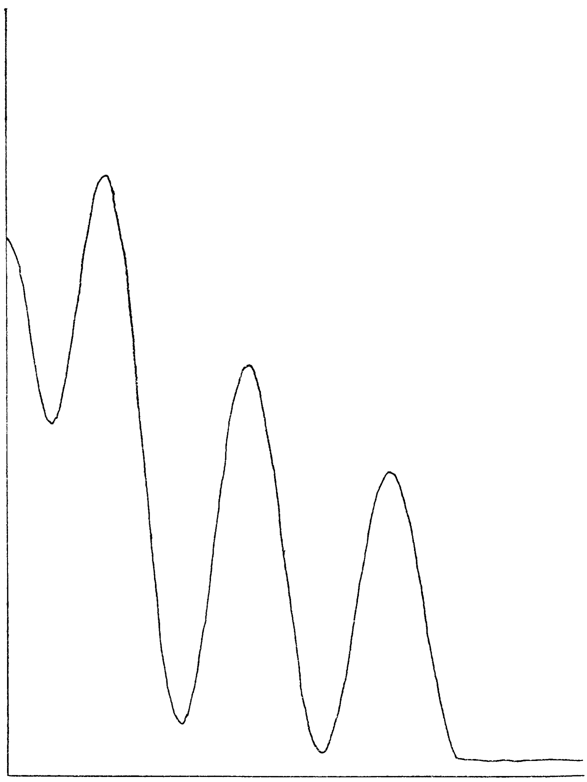

FIG. 1a. Spectrum A.

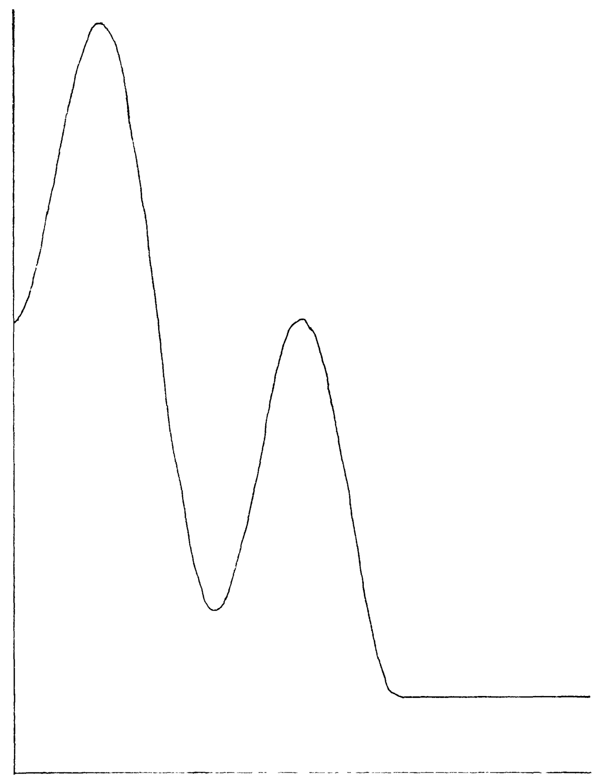

FIG. 1b. Spectrum B.

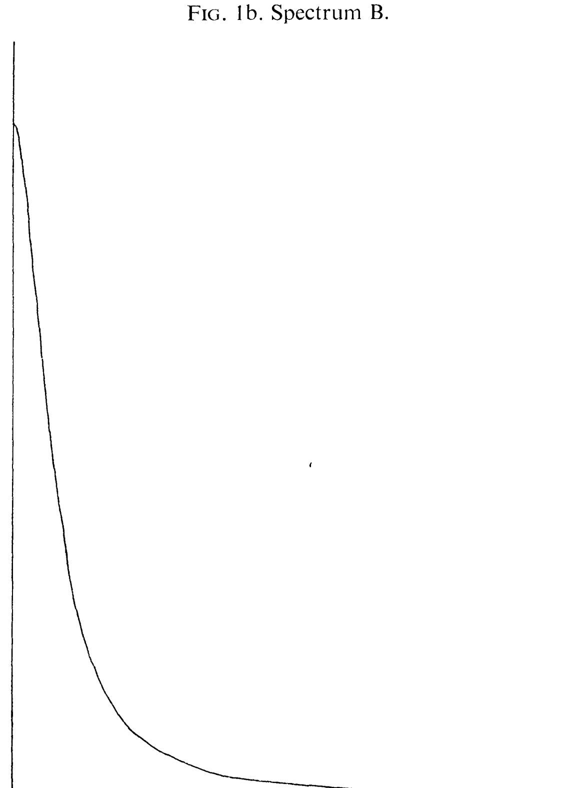

FIG. 1c. Spectrum C.

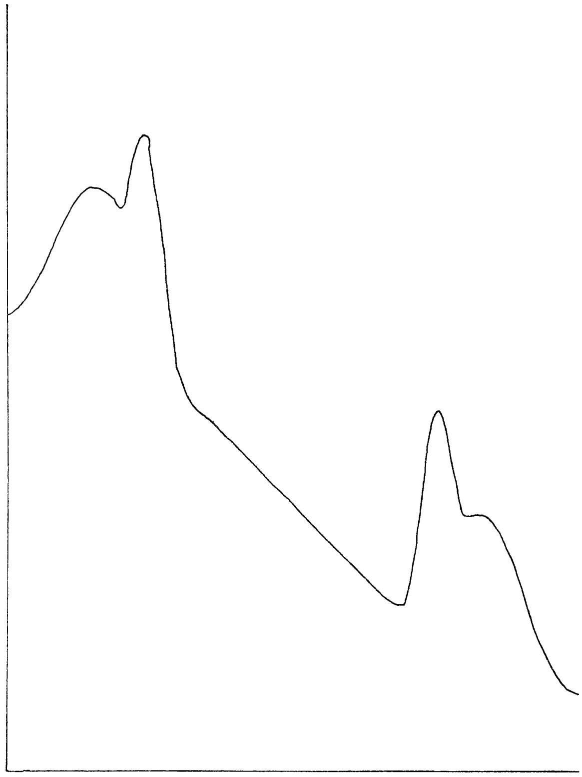

FIG. 1d. Spectrum D.

TABLE 1(a)

Sample size	$\frac{M_T}{T} =$	Parzen .1	Parzen .2	Parzen $\frac{1}{3}$	Tukey .1	Tukey .2	Tukey $\frac{1}{3}$
30	Square Bias	22.91	17.25	15.14	19.41	16.09	12.99
	Variance	8.66	14.04	20.19	10.89	17.76	25.98
	M.S.E.	31.57	31.30	35.33	30.31	33.85	38.97
60	Square Bias	17.09	13.30	6.46	15.96	9.60	3.21
	Variance	7.29	12.24	18.78	9.26	16.12	24.89
	M.S.E.	24.38	25.54	25.24	25.22	25.72	28.10
90	Square Bias	15.48	7.53	2.40	13.80	3.70	1.02
	Variance	6.62	11.74	18.32	8.60	15.68	24.42
	M.S.E.	22.10	19.27	20.72	22.40	19.39	25.43
120	Square Bias	13.12	3.90	1.06	9.23	1.60	0.43
	Variance	6.29	11.49	18.04	8.34	15.41	24.13
	M.S.E.	19.40	15.39	19.11	17.57	17.01	24.56

TABLE 1(b)

Sample size	$\frac{M_T}{T} =$	Parzen .1	Parzen .2	Parzen $\frac{1}{3}$	Tukey .1	Tukey .2	Tukey $\frac{1}{3}$
30	Square Bias	15.58	10.15	7.13	11.84	9.10	4.48
	Variance	8.89	14.40	20.38	11.19	18.05	25.94
	M.S.E.	24.47	24.55	27.52	23.03	27.15	30.42
60	Square Bias	10.00	5.13	1.66	8.97	2.56	0.70
	Variance	7.38	12.09	18.28	9.30	15.73	24.21
	M.S.E.	17.39	17.22	19.93	18.27	18.29	24.92
90	Square Bias	7.65	1.96	0.52	4.93	0.77	0.21
	Variance	6.59	11.38	17.75	8.44	15.15	23.78
	M.S.E.	14.24	13.34	18.27	13.37	15.92	24.00
120	Square Bias	4.93	0.86	0.22	2.31	0.31	0.09
	Variance	6.17	11.08	17.53	8.07	14.91	23.59
	M.S.E.	11.09	11.94	17.74	10.39	15.22	23.68

TABLE 1(c)

Sample size	$\frac{M_T}{T} =$	Parzen .1	Parzen .2	Parzen $\frac{1}{3}$	Tukey .1	Tukey .2	Tukey $\frac{1}{3}$
30	Square Bias	53.57	31.34	17.03	44.66	22.47	10.64
	Variance	9.76	18.35	28.26	13.24	24.48	36.71
	M.S.E.	63.34	49.69	45.29	57.90	46.95	47.35
60	Square Bias	30.60	11.90	4.55	21.59	6.62	2.22
	Variance	9.72	17.63	26.15	13.06	22.97	33.31
	M.S.E.	40.32	29.53	30.70	34.65	29.59	35.53
90	Square Bias	18.25	5.24	1.66	11.10	2.45	0.73
	Variance	9.46	16.62	24.27	12.54	21.38	30.86
	M.S.E.	27.71	21.85	25.92	23.64	23.83	31.59
120	Square Bias	11.33	2.60	0.75	6.07	1.09	0.32
	Variance	9.15	15.74	22.92	11.99	20.17	29.27
	M.S.E.	20.48	18.34	23.67	18.06	21.25	29.59

TABLE 1(d)

Sample size	$\frac{M_T}{T}$ =	Parzen .1	Parzen .2	Parzen $\frac{1}{3}$	Tukey .1	Tukey .2	Tukey $\frac{1}{3}$
30	Square Bias	6.98	3.74	2.26	4.90	2.90	1.51
	Variance	8.93	14.07	20.14	11.12	17.67	25.95
	M.S.E.	15.91	17.81	22.40	16.02	20.56	27.46
60	Square Bias	3.65	1.66	0.82	2.80	1.05	0.53
	Variance	7.16	11.96	18.32	9.04	15.72	24.36
	M.S.E.	10.81	13.63	19.14	11.84	16.77	24.88
90	Square Bias	2.40	0.90	0.42	1.56	0.57	0.24
	Variance	6.42	11.36	17.82	8.35	15.19	23.90
	M.S.E.	8.82	12.26	18.24	9.91	15.77	24.14
120	Square Bias	1.60	0.57	0.24	0.99	0.34	0.12
	Variance	6.08	11.10	17.59	8.04	14.96	23.69
	M.S.E.	7.68	11.67	17.82	9.03	15.30	23.81

4. General considerations. The average error of an estimator $\hat{\theta}$ of a value θ has two components, its *bias* $E[\hat{\theta}]-\theta$ and its *variance* $E(\hat{\theta}-E[\hat{\theta}])^2$. The mean square error is one convenient combination of these two, given by

$$E(\hat{\theta}-\theta)^2 = E^2(\hat{\theta}-\theta)+E(\hat{\theta}-E[\hat{\theta}])^2 \tag{4.1}$$
$$= \text{bias}^2+\text{variance}.$$

Recommendations are to be found in several publications (e.g. Granger and Hatanaka (1964) page 61; Parzen (1964) page 942) for values of the ratio M_T/T. Such an approach is in effect taking account only of the (asymptotic) variance of the estimator (cf. (1.6)) and not of the bias. The bias depends essentially on M_T and only very slightly on T; for by (2.1) and (1.10), the expected value of the standard type of estimator (1.3) is

$$E[f_T{}^*(\omega)] = \frac{1}{\pi}\sum_{v=0}^{M_T} k_T{}^*\left(\frac{v}{M_T}\right)\left(1-\frac{vM_T}{T}\right)R(v)\cos v\omega. \tag{4.2}$$

In contrast to the above, we submit that in fact one should concentrate initially on how large M_T needs to be in order that the bias be sufficiently small. This approach is more allied to the discussions of bandwidth by e.g. Priestley (1962), (1965) and Jenkins and Watts (1968) page 284. There is no point in advocating that the variance (and thus M_T/T) should be small if the estimator's mean is not close to the true value; one then has a precise estimator of the wrong quantity. But even bandwidth cannot tell the complete story. We believe that a user of spectral methods should familiarise himself with (4.2), computing and plotting $E[f_T{}^*(\omega)]$

for various spectra and values of M_T—the factor $(1-vM_T/T)$ can be approximated to one without significant loss, thus making the expression independent of T. Simple methods for doing this are suggested in Appendix 1. In particular, one can then get a feeling for what values of M_T are needed in order that peaks of various widths be present at least in the expected value curve—if they are not adequately represented there, one cannot expect the estimator to have them. Some typical graphs are shown in Figure 2 of expected value curves for estimators of spectrum A with the Tukey generator (3.3). In this case clearly $M_T = 10$ is of no use, while the main characteristics are beginning to appear at $M_T = 20$, and the fit is reasonably close with $M_T = 40$. For spectra having more and/or narrower peaks, larger values of M_T would be necessary. Since $k_P^*(\theta)$ damps down quicker than $k_T^*(\theta)$, use of the Parzen generator also requires higher values of M_T.

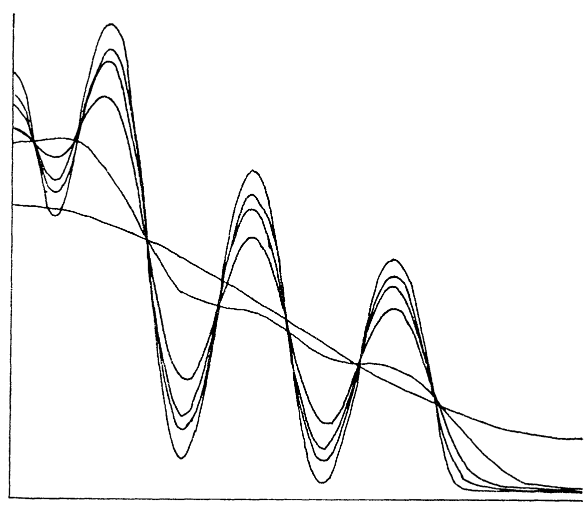

FIG. 2. Expected value curves.

One can thus obtain an idea of the size of M_T which will satisfy one's individual requirements with respect to bias. Then if the sample has already been taken, the variance of the resulting estimator can be roughly approximated by (1.6), or more closely approximated by taking into account the findings of Section 5 with regard to the exact behaviour of the variance. If this variance seems large enough to destroy the degree of precision required, then it is unlikely that *any* estimator will have satisfactory properties: to lower M_T would result in the expected value curve being too blurred, and to increase M_T would result in even higher variance but with little improvement to the bias. It is much more sensible to carry out this investigation before the sample is taken, and then to choose T so that the variance is sufficiently small for the desired region of values of M_T.

5. Comments on the computer study. The "desired region of values of M_T" referred to in the previous section obviously depends on the intricacy of the spectrum and how much one is looking for. The figures of squared bias in Table 1 certainly do not present a complete picture. For example the smallest proportionate bias is for spectrum D, although the narrower peaks are only just beginning to show in the expected value curve at $M_T = 40$ for the Tukey generator. The reason for the small bias is that the spectrum mainly consists of

$$f(\omega) = A(1 + \cos \omega) \tag{5.1}$$

which is thus well approximated by just the first two terms of its Fourier series.

It is regrettably impractical to publish the many graphs actually plotted in the computer study; but a complete set of graphs of the expected values and standard deviations for each of the 24 estimators of spectrum A are given in Neave (1968). As an example though, Figure 3 shows the graphs for the Tukey estimator with $M_T = 12$ and $T = 60$.

The fact that the expected value curve is a smoothed version of the true spectrum is well-known (i.e. the peaks are not so high as they should be, and the troughs not so deep; further, the influence of narrow peaks and troughs spreads into neighboring frequencies). What is less well known, but is apparent from this study, is that, away from $\omega = 0$ and π, the same is true of the standard deviation, after the application of a scale factor, which from (1.6) is asymptotically

$$\left\{2 \frac{M_T}{T} \int_0^1 k^2(\theta) \left(1 - \frac{M_T}{T} \theta\right) d\theta\right\}^{\frac{1}{2}}. \tag{5.2}$$

Thus, away from the end-points, we verify the observation by Granger and Hughes (1968) that the variance is more truly proportional to $E^2[f_T{}^*(\omega)]$ than to $f^2(\omega)$ itself. The rise at the end-points indicated by (1.6) is visible in several cases, though not usually by quite as much as the factor of $2^{\frac{1}{2}}$ suggested by the asymptotic result. However, instead of being restricted to the end-points, this rise is also seen to contribute extensively to the standard deviation at neighboring frequencies, an effect of the overall smoothing just mentioned. In fact the "contamination" or "leakage" due to the standard deviation curve's smoothing often appears to affect at least as wide a frequency band as that of the expected value function (cf. Figure 2). Its effect is shown impressively in the case of spectrum C which is of a first-order autoregressive process. Table 1 shows that the variance in this case is proportionally much greater than in the other cases. This is almost entirely due to the peak in the spectrum at $\omega = 0$ being reinforced by the factor of about two, and the variance function being slow to throw off the effects of this factor as ω increases. To illustrate this effect, Figure 4 shows the standard deviation curve for the Parzen estimator with $M_T = 12$, $T = 120$ and the asymptotic standard deviation with $M_T/T = .1$; it is apparent that for the low frequency range, the true standard deviation is much greater than the approximation would indicate.

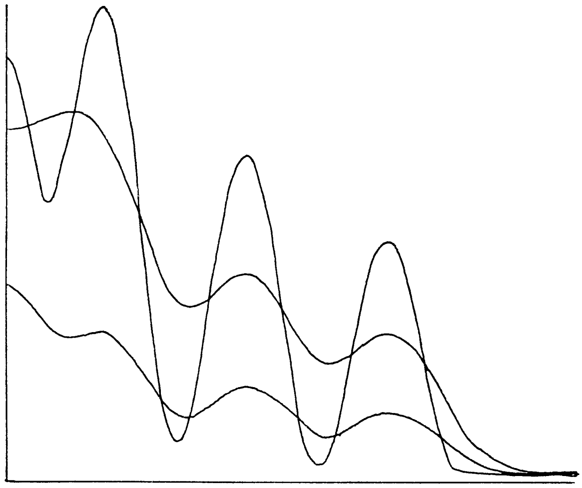

FIG. 3. Properties of a particular estimator.

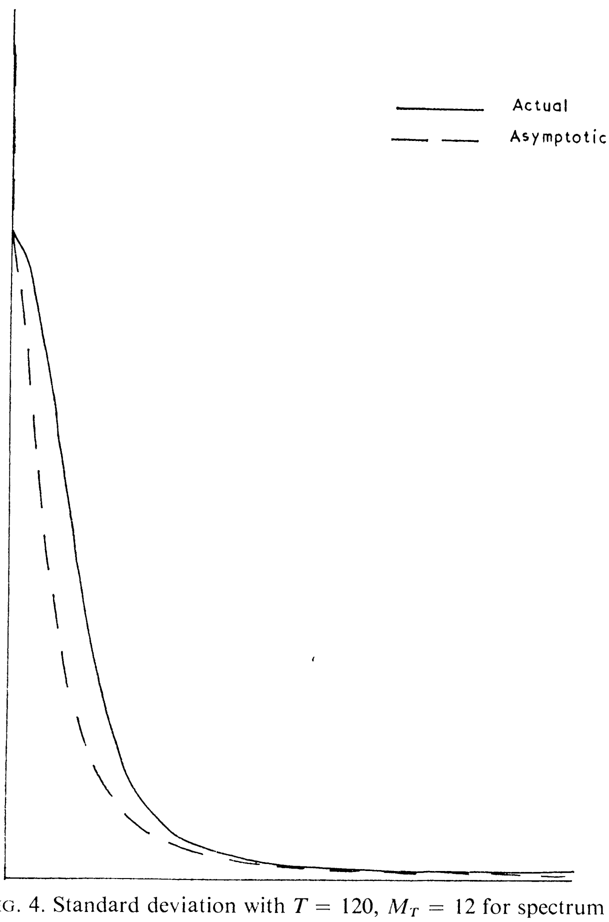

FIG. 4. Standard deviation with $T = 120$, $M_T = 12$ for spectrum C.

Due to the normalisation referred to in Section 3, the figures for the average variance in Table 1 should tend to limits of

$$(5.3) \qquad \frac{2M_T}{T}\int_0^1 k^2(\theta)\left(1-\frac{M_T}{T}\theta\right)d\theta \times 100\%$$

(but double at the end points), and these limits are shown in Table 2. For reference, the classical approximations

$$(5.4) \qquad \frac{2M_T}{T}\int_0^1 k^2(\theta)\,d\theta \times 100\%$$

are also given. It is seen that invariably the actual average variance exceeds the approximation, but for fixed M_T/T it decreases steadily as T increases. This is in contrast to the conclusions of Granger and Hughes (*op. cit.*), but in Appendix 2 we offer evidence that their results are incorrect. It is worth noticing that the classical approximations (5.4), which are larger than the more relevant (5.3), actually do

TABLE 2

	λ	$2\frac{M_T}{T}\int_0^1 k^2(\theta)(1-\lambda\theta)\,d\theta$	$2\frac{M_T}{T}\int_0^1 k^2(\theta)\,d\theta$
	.1	5.301%	5.383%
Parzen	.2	10.418%	10.786%
	$\frac{1}{3}$	16.953%	17.977%
	.1	7.328%	7.500%
Tukey	.2	14.310%	15.000%
	$\frac{1}{3}$	23.083%	25.000%

overestimate the figures for the higher values of T. It is also interesting to observe that, conversely to what might be expected, the larger the value of M_T/T, the quicker is the convergence to the limit. This is because several of the terms dispensed with during the proof of (1.6) are of the order M_T^{-1}; in other words, for fixed T, the *smaller* the value of M_T/T, the worse is the asymptotic formula as an approximation.

6. Conclusions. In conclusion, we reemphasize the need for awareness of just how much it is possible to learn from spectrum estimators with different truncation points; and to use this awareness to decide upon an adequate sample size. One should also be aware of the increased variability in quite substantial neighborhoods of both peaks and of the end-point frequencies. It appears further from this study and from Neave (1970) that the more relevant asymptotic approximation (1.6) for the variance is, on average, an underestimate of the true variance. In fact the true variance is usually only less than the approximation in the centre of spectral peaks: this is little comfort, seeing that the bias is comparatively large in these regions.

The results of this study tend to favor prewhitening (Blackman and Tukey (1958)), which would reduce the effect of the "double leakage" from peaks in both expected value and variability.

APPENDIX 1

Spectrum elements. The study of expected value functions suggested in Section 4 may be easily carried out using an idea from the computer study described. Basically this consists of specifying a few simple forms of spectrum, which we call *spectrum elements*, and then forming more complicated spectra by superposition of such spectrum elements. It is easy to see from (1.2) that the elementary covariances sum to give the covariance structure of the final spectrum, thus

$$\sum_j R_j(v) = \int_{-\pi}^{\pi} \sum_j f_j(\omega) \cos v\omega \, d\omega.$$

For the purpose of this study, four spectrum elements were used. Two of these are important as spectra in their own right, the other two are unlikely to occur individually in practice. However it is possible to closely approximate many spectra of interest by a sum of a fairly small number of these elements. The four elements (defined over $(0, \pi)$) and their covariance functions were

(a) *Constant* (white noise process). $f(\omega) = c$,

$$R(0) = 2\pi c;$$
$$R(v) = 0 \; (v \neq 0).$$

(b) *First order autoregressive.* This has the covariance function:

$$R(v) = R(0)\alpha^{|v|}$$

for some α on $0 < \alpha < 1$; it stems from the generating process

$$X_t = \alpha X_{t-1} + \varepsilon_t$$

where ε_t is a white noise process. Then

$$f(\omega) = \frac{R(0)}{2\pi} \cdot \frac{1-\alpha^2}{1+\alpha^2-2\alpha \cos \omega}.$$

(c) *Half peak cosine wave.* This is defined by the spectrum

$$f(\omega) = \tfrac{1}{2}A\left(1+\cos\frac{\pi\omega}{a}\right), \qquad (\omega \leqq a)$$
$$= 0, \qquad (\omega \geqq a)$$
$$R(v) = A \sin va \cdot \frac{\pi^2}{v(\pi^2 - v^2a^2)}, \qquad \left(|v| \neq 0, \frac{\pi}{a}\right),$$
$$R(0) = Aa,$$
$$R\left(\frac{\pi}{a}\right) = \tfrac{1}{2}Aa \quad \text{if} \quad \frac{\pi}{a} \quad \text{is an integer.}$$

(d) *Whole peak cosine wave.* This is defined by the spectrum

$$f(\omega) = \tfrac{1}{2}A\left[1+\cos\frac{\pi}{a}(\omega-\omega_0)\right], \quad (\omega_0-a \leqq \omega \leqq \omega_0+a,)$$

$$= 0, \quad \text{(elsewhere)},$$

where $0 \leqq \omega_0-a \leqq \omega_0+a \leqq \pi$. Then

$$R(v) = \frac{2A\pi^2}{v(\pi^2-v^2a^2)}\cos v\omega_0 \sin va, \quad \left(|v| \neq 0, \frac{\pi}{a}\right),$$

$$R(0) = 2Aa,$$

$$R\left(\frac{\pi}{a}\right) = Aa\cos\left(\frac{\pi}{a}\omega_0\right) \quad \text{if } \frac{\pi}{a} \text{ is an integer.}$$

These results are all proved in Neave (1968).

A referee has pointed out that in order to detect strict harmonics, one needs to use high truncation points. To examine such situations, one should include among the study of expected value functions some spectra with very high and narrow peaks, if it is at all possible that processes being studied may have deterministic components. In practice, one should test for this possibility by periodogram analysis (Fisher (1929)).

APPENDIX 2

The white noise case. In the white noise case, i.e. where $R(v) = 0$ for all non-zero v, (2.9) reduces to

$$\mathrm{Var}\,[\tilde{f}(\omega)] = \frac{1}{\pi^2}R^2(0)\left\{\frac{1}{2T}+\sum_{v=1}^{M_T}\frac{1}{T-v}k_T^2(v)\cos^2 v\omega\right\}.$$

Making use of (1.9) to obtain the variance of the usual estimator (1.3), we have

$$\mathrm{Var}\,[f_T^*(\omega)] = \frac{1}{\pi^2}R^2(0)\left\{\frac{1}{2T}+\frac{1}{T}\sum_{v=1}^{M_T}k_T^{*2}(v)\left(1-\frac{v}{T}\right)\cos^2 v\omega\right\}$$

which by (1.5) gives

$$\mathrm{Var}\,[f_T^*(\omega)] = \frac{1}{\pi^2}R^2(0)\left\{\frac{1}{2T}+\frac{1}{T}\sum_{v=1}^{M_T}k^{*2}\left(\frac{v}{M_T}\right)\left(1-\frac{v}{T}\right)\cos^2 v\omega\right\}.$$

It is then easy to show that as $T \to \infty$ with $M_T/T \to \lambda$,

$$\mathrm{Var}\,[f_T^*(\omega)] \sim \frac{1}{2\pi^2}R^2(0)\frac{M_T}{T}\int_0^1 k^2(\theta)(1-\lambda\theta)\,d\theta$$

for $\omega \neq 0$ or π, and double this at the exceptional points. Since in the white noise case $f(\omega) = (2\pi)^{-1}R(0)$, the asymptotic formula (1.6) is obtained.

Various numerical illustrations of the above formula are given in Neave (1970), and again it is interesting to check how far the doubling effect at the end-points leaks into neighboring frequencies; for example the effect is still apparent at $\omega = \pi/6$ in the case of the Parzen generator with $M_T = 10$ and $T = 50$. In every case the true variance exceeds the relevant asymptotic approximation—which is consistent with the observations made in Section 5.

It is thus somewhat disturbing to find that Granger and Hughes (1968) report simulation results indicating that the asymptotic formula *over*-estimates the true variance by over 50 % in the central frequencies (in the case of the Parzen estimator with $M_T = 10$, $T = 30$). We investigate these results in detail in a forthcoming paper (Neave, 1971), and are forced to the conclusion, from theoretical and empirical evidence, that their results are in considerable error. Their paper goes on to consider various cross-spectral quantities, but in view of these initial mistakes, their results and conclusions must be regarded with some suspicion.

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