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BOOK REVIEW

Correspondence concerning reviews should be addressed to the Book Review Editor, Professor James F. Hannan, Department of Statistics, Michigan State University, East Lansing, Michigan 48823.

YU. A. ROZANOV. *Stationary Random Processes*. Holden-Day, San Francisco, 1967. 211 pp. $10.95.

Review by P. MASANI
Indiana University

About three-fourths of this book deals with multivariate weakly stationary[1] stochastic processes (WSSP) with discrete or continuous time and their harmonic analysis, linear prediction, filtering and interpolation, and about a fourth with multivariate strictly stationary stochastic processes (SSSP) and their ergodic and asymptotic properties. In content it covers the usual ground as well as many topics available so far only in the periodical literature, especially in the author's own research papers. In organization and attitude the book bears the wholesome impress of the author's teacher, A. N. Kolmogorov, and of his thirty-year-old masterpiece on "Stationary sequences in Hilbert space."[2] In style and treatment it is well written and exhaustive. The book is perhaps the most up-to-date one on the subject, but since its origins go back to the author's 1959–1960 lectures at Moscow University, it naturally needs up-dating in certain respects.

We would strongly recommend the book to all concerned. The following detailed description and commentary are designed to guide prospective readers.

In a way the present translation was wasted effort, for an equally good English translation of the same (1963) Russian edition had been made two years earlier by M. Ravindranathan, and issued in bound mimeographed form by the Indian Statistical Institute, Calcutta, in 1965. This could have easily been improved and published in print. Some of the inaccuracies and lacunae commented upon by Dr. Ravindranathan remain without comment in the present translation. Thus on page 66, to complete the argument in the penultimate paragraph, one needs the lemma on page 173 of the Helson–Lowdenslager paper ([2] Part I). On page 76 in the opening equation Γ^{-1} should be replaced by $\Gamma^{-1}(e^{-i\lambda})$. On page 80, to justify the first inequality in the proof of Lemma 7.1, one needs that given in Lemma 1.4 of the Wiener–Masani paper ([1] Part II). On p. 81 the reasoning which just precedes Lemma 7.3 is not clear: $G(\cdot)$ is in L^1_{0-}, L^1_{0+} however, and this suffices to show its constancy. On page 89 in Theorem 8.3 for "linearly stationary" read "linearly singular." On page 97, line 6, insert "$\hat{\xi}_k(t)$" after the word "projections."

Let us turn to the opening chapter. In the spirit of [K] this deals exclusively with parts of the theory depending only on the group properties of the time-domain

[1] "Stationary in the wide sense" in the terminology of Doob's book "Stochastic Processes," Wiley, New York (1953). This book will be referred to as [D] in this review.

[2] *Bull. Moscow State Univ.* **2** (1941) 1–40. This paper will be referred to as [K] in this review.

$\mathbb{N}$ or $\mathbb{R}$[3], and which consequently generalize to stationary random fields on arbitrary locally compact abelian groups. After defining WSSP in Section 1, the author points out that the probabilistic setting is dispensable and the concept can be broached conveniently as a *stationary sequence or curve in Hilbert space*, which of course is the attitude adopted in [K]. A nice feature of the treatment is the very early discussion of *random measures and stochastic integration*, Section 2. The former is a set-function Φ on the Borel field of $\mathbb{R}$ whose values $\Phi(\Delta)$ are random variables in $\mathscr{H} = L_2(\Omega, \mathscr{B}, P)$, which for the greater part are assumed to be orthogonally scattered. The latter is integration with respect to such a measure. The author also defines the *Bochner integral* of a WSSP, and proves a useful Fubini Theorem for integration with respect to the product measure $\Phi \times$ Lebesgue (Theorem 2.4). In Section 4 he proves the q-variate extension of the Kolmogorov–Karhunen result that a q-ple WSSP is governed by a *unitary shift group* U_τ, $\tau \in \mathbb{N}$ or $\mathbb{R}$. By Stone's Theorem there is an associated spectral measure $E(\cdot)$ over the dual group, $\hat{\mathbb{N}} = C$ or $\hat{\mathbb{R}} = \mathbb{R}$. Following [K] the author uses this $E(\cdot)$ to define the $\mathscr{H}$-valued measure Φ and the $q \times q$ nonnegative hermitian matrix-valued measure F, which provide the *spectral representations* of ξ and of its matricial covariance function $B(\cdot)$, Sections 4, 5. In Lemma 5.1 he establishes the q-variate extension of Kolmogorov's Lemma 2 ([K] Section 1) on the covariance function of any (non-stationary) process, with some unhappy hand-waving in treating the parameter set $\mathbb{R}$.

With this firm footing the author is able to handle the harmonic analysis of WSSP in a thorough way. In Section 5 he proves the q-variate extension of Khinchin's Theorem that a $q \times q$ matrix-valued function B on $\mathbb{N}$ or $\mathbb{R}$ is the covariance function of a q-ple WSSP, iff B is (matricially) positive-definite (Theorem 5.1), as well as the dual extension due to Cramér on the spectral distribution matrix of a q-ple WSSP (Theorem 5.2). Then follow the Ergodic Theorem for unitary groups and the "Lévy Inversion Formulae" for the recovery of Φ and F from ξ and B (Section 6), as also an interesting representation for band-limited ξ (Theorem 6.4). Next the author defines the space $L_2(F)$, introduces an inner product in it and shows that it becomes a Hilbert space (Lemma 7.1). He then reveals explicitly by stochastic integration the isomorphism between $L_2(F)$ and the subspace $\mathscr{H}_\xi$ of $\mathscr{H}$ spanned by ξ (Section 7). (In their full generality these results are due to the author himself and independently to M. Rosenberg.) In Section 8 Kolmogorov's *subordination theory* ([K] Section 4) is extended to the q-variate case. In Section 9 the author speaks of ξ as having *rank m*, iff its spectral distribution F is absolutely continuous and rank $F' = m$, a.e. on C or $\mathbb{R}$. This "spectral rank" concept is narrower than Zasukhin's concept of rank ρ, which is defined for all ξ. It would have been clearer had the author used some qualification such as "spectral" for his own concept. Of course, $m = \rho$ in many important cases. In

[3] $\mathbb{N}$ and $\mathbb{R}$ denote the sets of integers and real numbers, respectively. $\mathbb{C}$ will denote the set of complex numbers.

Theorem 9.1 the author proves that a ξ of rank m is always a (two-sided) moving-average and that $F' = \Psi\Psi^*$, where Ψ is a $q \times m$ matrix-valued function. The chapter concludes with a discussion of ξ for which F' is a rational matrix-valued function (Section 10).

What we have described so far holds for stationary sequences and curves in any Hilbert space $\mathscr{H}$. But in Chapter I the author also brings in some ideas which make sense only when $\mathscr{H} = L_2(\Omega, \mathfrak{A}, P)$. For instance, he calls ξ *ergodic*, iff its time-average is equal to the expectation of $\xi(0)$, and he characterizes this property in terms of the spectral distribution of ξ (Section 6).

In Chapter II the author turns to aspects of the theory of q-ple WSSP ξ over $\mathbb{N}$ which involve the *ordering of* $\mathbb{N}$, i.e. to prediction theory proper, confining himself only to *linear prediction*. Sections 1–6 cover the usual topics: Wold–Zasukhin decomposition, spectral criteria for pure nondeterminism at rank m, and at the full rank $m = q$, in terms of the factorizability of F' and the integrability of log det . F'. Interspersed are remarks on functions of the Hardy class. It would have been easier on the reader had these remarks been consolidated and relegated to an appendix. In Section 7 are discussed ξ for which F is absolutely continuous and $c_1 I \leqq F' \leqq c_2 I$, where $0 < c_1 \leqq c_2 < \infty$, which as the author has shown elsewhere is the n.&s.c. that the $\xi_k(t)$, $t \in \mathbb{N}$, $1 \leqq k \leqq q$, form an (*unconditional*) *basis* for the subspace $\mathscr{H}_\xi$ of $\mathscr{H}$ spanned by the $\xi_k(t)$. Unfortunately, only the sufficiency of this condition is proved in the book (Theorem 7.1). He then presents the iterative algorithm for the optimal factor of F' and for the predictor in the spectral domain given in Part II of the Acta Mathematica paper of Wiener and the reviewer. In Theorem 8.1 he gives Matveev's spectral n.&s.c. for ξ to be purely non-deterministic and of rank m, involving the beschränktartige functions of Nevanlinna. Then follows a detailed study of the case $m = 1$. Next comes *linear filtering*, i.e. the linear prediction of ξ on the basis of another purely non-deterministic, full-rank q-ple WSSP η stationarily correlated with ξ (Section 9). This is the first step towards the "filtering with noise" of Wiener, not discussed in the book. In Section 10 the author turns to the *linear interpolation* of ξ. This very hard subject logically precedes prediction, since it is really independent of the ordering of $\mathbb{N}$. The author gives spectral criteria for non-interpolability, minimality, as well as some expressions for the linear interpolator discovered by himself. In Section 11 he shows that the WSSP ξ for which the $\xi_k(t)$, $t \in \mathbb{N}$, form a *conditional basis* of the subspace $\mathscr{H}_\xi$ may be characterized in terms of minimality.

In Chapter III the author carries out for continuous parameter q-ple WSSP ξ roughly what he accomplished in Chapter II for the discrete parameter case. His treatment resembles that in [D]. He associates with ξ a discrete parameter $\tilde{\xi}$ by using the Cayley transform of the ($\mathscr{H}$-valued) spectral measure Φ of ξ. After showing that ξ, $\tilde{\xi}$ are alike in being deterministic, purely non-deterministic, etc., he deduces theorems concerning ξ from those established in Chapter II for $\tilde{\xi}$. In Section 4 he makes a detailed study of the predictor of a univariate ξ having a rational spectral density F'. In Section 5 he discusses linear filtering, primarily for such ξ. In Section 6 he takes up the difficult subject of *continuous-time interpolation.* As he puts it (p. 131):

> ... the method of solution of the problem of the linear interpolation of a multi-dimensional stationary process ... for the discrete-parameter case, leads in the continuous-parameter case to complicated integral equations, and at present no effective solution has been found for this problem in its full generality.

He accordingly restricts himself to the case of a rational spectral density. Finally comes forecasting on the basis of a bounded subinterval of the past, a topic of interest to application-oriented workers in the subject (Section 7). Some original results of the author in this direction are given.

In Chapter IV the author shifts from WSSP to SSSP, covering in the first half the familiar territory available for instance in [D]. But there are technical differences. In Sections 1–3 he uses the coordinate (or function-space) version $\tilde{\xi}$ of the q-ple SSSP ξ over $(\Omega, \mathfrak{A}, P)$ to infer the existence of a group S_τ, $\tau \in \mathbb{N}$ or $\mathbb{R}$, of P-measure preserving transformations on $\mathfrak{A}_\xi$, (the σ-algebra generated by ξ) onto $\mathfrak{A}_\xi$, the S_τ being well-defined up to sets of zero P-measure and such that for $N \geqq 1$, Borel subsets $\Gamma_1, \cdots, \Gamma_n$ of $\mathbb{C}$ and $t_1, \cdots, t_N$ in $\mathbb{N}$ or $\mathbb{R}$,

$$S_\tau[\bigcap_{i=1}^{N} \{\xi_{k_i}(t_i)\}^{-1}(\Gamma_i)] = \bigcap_{i=1}^{N} \{\xi_{k_i}(t_i-\tau)\}^{-1}(\Gamma_i).$$

He calls S_τ, $\tau \in \mathbb{N}$ or $\mathbb{R}$, the *shift group* of ξ. He then shows that the transformations U_τ induced by the S_τ on the space of $\mathfrak{A}_\xi$-measurable rv's on Ω themselves form a group when rv's differing on sets of zero P-measure are identified. In Section 4 he considers the Hilbert space $\mathscr{H}_\xi$ of these rv's h for which $E(|h|^2) < \infty$, the inner product (h_1, h_2) being $E(h_1 \cdot \bar{h}_2)$. The U_τ, when restricted to $\mathscr{H}_\xi$, constitute of course a unitary group. For discrete time, $\mathscr{H}_\xi$ is separable. For continuous time the author defines *stochastically continuous* ξ for which $\mathscr{H}_\xi$ turns out to be separable. He also defines *measurable* ξ for which the U_τ-group becomes measurable, and such that for measurable and stochastically continuous ξ the U_τ-group becomes strongly continuous. In Section 5 the author proves the *Individual Ergodic Theorem* that if ξ is a q-ple SSSP on $(\Omega, \mathfrak{A}, P)$ with induced shift-group U_τ, $\eta_0 \in L_1(\Omega, \mathfrak{A}_\xi, P)$ and $\eta_\tau = U_\tau(\eta_0)$, then the time-averages of P-almost all trajectories of the SP η is $E(\eta_0 \mid \mathfrak{J})$, where $\mathfrak{J}$ is the σ-algebra of invariant sets. He then deals with *ergodic* ξ in much the same spirit as in [D]. He shows that ξ is ergodic, iff all the eigenvalues of the U_τ-group are simple (Theorem 6.1').

The remaining articles of Chapter IV cover less familiar terrain. In Section 7 the author gives a n.&s.c. for the ergodicity of a SSSP ξ having a purely atomic spectral distribution. In Section 8 he proves a version of the 1932-theorem of von Neumann on the splitting of a measure-preserving transformation into ergodic components, his setting being an abstract but *perfect* probability space. In Section 9 a q-ple SSSP is called (*strictly*) *regular*, iff the "remote-past" σ-algebra $\mathfrak{A}_{-\infty} =$

$\cap_\tau \mathfrak{A}_\tau$, $\tau \in \mathbb{N}$ or $\mathbb{R}$, is trivial, $\mathfrak{A}_\tau$ being the "past and present" σ-algebra for the epoch τ. He proves that ξ is (strictly) regular, iff for all $A \in \mathfrak{A}_\xi$,

$$\sup_{B \in \mathfrak{A}_\tau} |P(A \cap B) - P(A)P(B)| \to 0 \qquad \text{as } \tau \to -\infty.$$

In Section 10 he considers *completely regular* ξ, i.e. ξ which satisfy Rosenblatt's condition

$$\sup_{A \in \mathfrak{A}_t, B \in \mathcal{B}_{t+\tau}} |P(A \cap B) - P(A)P(B)| \to 0, \qquad \text{as } \tau \to \infty,$$

$\mathcal{B}_t$ being the "present and future" σ-algebra for the epoch t. One of many interesting results proved is that a q-ple Gaussian SSSP is completely regular, iff $\rho(\tau) \to 0$ as $\tau \to \infty$, where $\rho(\tau)$ is so-to-speak the cosine of the angle between the "past and present" and "present and future" subspaces $H_\xi^-(t)$, $H_\xi^+(t+\tau)$. Finally in Section 11 the author presents his own generalization of the *Central Limit Theorem* for q-ple SSSP which are completely regular and have a bounded spectral density F', continuous at zero and with $F'(0)$ of full rank (Theorem 11.1).

There are certain respects in which the book could bear up-dating. The Wold–Zasukhin decomposition for discrete time is established *ab initio* in Chapter II, Section 3. But we now know that the result is a corollary of the Wold Decomposition Theorem[4] for isometric semigroups, and viewing it in this way gives us a clearer perspective on how and where prediction theory fits into functional analysis. For continuous time the author derives the decomposition in a round-about way by strong appeal to spectral ideas and results (Chapter III, Section 3). Thus the natural precedence of time-domain over spectral-domain analysis maintained in Chapter II is given up in Chapter III, Section 3 and even earlier in Lemma 2.1. The reader gets the impression that the only access to time-domain analysis for continuous time is by a round-about route. Actually a more direct, spectral-free approach to this analysis was suggested by Robertson and the reviewer in 1962.[5] It is simpler than Hanner's original analysis of 1950, and subsequent research has simplified it further.

Another respect in which the work could stand updating is in the explicit enunciation of the question of *concordance* between the Wold–Zasukhin Decomposition of ξ and the Lebesgue–Cramer decomposition of its spectral distribution F, and the inclusion of J. B. Robertson's result[6] that concordance prevails iff rank F' is steady, a.e. This theorem subsumes the fragmentary results on concordance of earlier origins. For instance, Theorem 8.3 of Chapter II is seen to be a rather special case. In the same vein we might mention H. Salehi's extension[7] of Theorem 10.2 (Chapter II) on "full-rank" minimality to cover minimality at any rank.

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[4] Brought into the open by Halmos in *Crelle J.* **208** (1961) 102–112.

[5] *Pacific J. Math.* **12** (1962) 1361–1378.

[6] *Canadian J. Math.* **20** (1968) 368–383, Theorem 5.2.

[7] *Ark. Mat.* **7** (1967) 305–311, Theorem 3.