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A COUNTEREXAMPLE ON TRANSLATION INVARIANT ESTIMATORS[1]

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It seems to be generally known that the proof of the continuity part of Theorem 1 in Hodges' and Lehmann's paper (1963) is incorrect. The fact that the theorem is incorrect is—perhaps—not so well known. We show this by constructing independent real random variables $X_1, \cdots, X_n$, each having the same non-atomic symmetric distribution, and an odd translation invariant estimator $h(X_1, \cdots, X_n)$ such that $P(h(X_1, \cdots, X_n) = 0) > 0$. h may be chosen symmetric provided $n \geqq 3$.

The construction. Define first three odd measurable functions ϕ, ψ and g from $]-\infty, +\infty[$ to $]-\infty, +\infty[$ such that

$$\phi(x) = 0.0x^{(1)}000x^{(2)}000x^{(3)}\cdots,$$

$$\psi(x) = 0.000x^{(1)}000x^{(2)}000x^{(3)}\cdots \qquad \text{and}$$

$$g(x) = 0.000x^{(4)}000x^{(8)}000x^{(12)}\cdots$$

when $x \in [0, 1[$ and $0, x^{(1)}x^{(2)}x^{(3)}\cdots$ is the dyadic expansion of x which has infinitely many zeroes. Note that $x_1 + g(x_2 - x_1) = 0$ when $x_1 \in \psi]-1, 0]$ and $x_2 \in \phi]0, 1]$. If $n = 2$, then we may proceed as follows. Define h as the map $(x_1, x_2) \to x_1 + g(x_2 - x_1)$. Choose independent random variables U_1 and U_2, each being uniform on $]-1, +1[$. Then put $X_i = \phi(U_i)$ or $\psi(U_i)$ with equal probability $\frac{1}{2}$,—the selections being independent and independent of the U's. It is easily seen that the distribution is non-atomic and symmetric and $P(h(X_1, X_2) = 0) \geqq P(X_1 \in \psi\,]-1, 0])P(\frac{1}{4} \leqq X_2 \in \phi[0, 1[) > 0$. In this inequality $h(X_1, X_2)$ could be replaced by $h(X_1, X_2) \wedge h(X_2, X_1)$—the latter being symmetric but not odd. (The only odd translation invariant and symmetric estimator is $(X_1 + X_2)/2$.)

If $n \geqq 3$, then counterexamples with odd translation invariant and symmetric estimators of the form $h(X_1, \cdots, X_n) = \text{median}\,(X_1, \cdots, X_n) + g((\wedge X_i + \vee X_i)/2 - \text{median}\,(X_1, \cdots, X_n))$ may be constructed. It suffices to do this for $n = 3$—the general case being very similar.

Choose independent random variables U_1, U_2 and U_3, each being uniform on $]-1, +1[$. Let $U_i^{(j)}$ denote the jth binary digit in the dyadic expansion of $|U_i|$. Then put

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$$X_i = \alpha(U_i) = [-0.U_i^{(1)}0000000U_i^{(2)}0000000U_i^{(3)} \cdots +2] \operatorname{sgn} \ U_i \text{ with probability } \tfrac{1}{3},$$

$$X_i = \beta(U_i) = [0.000U_i^{(1)}000U_i^{(2)}000U_i^{(3)} \cdots] \operatorname{sgn} U_i \text{ with probability } \tfrac{1}{3},$$

$$X_i = \gamma(U_i) = [0.0000U_i^{(1)}0000000U_i^{(2)}0000000U_i^{(3)} \cdots + 2] \operatorname{sgn} U_i \text{ with probability } \tfrac{1}{3},$$

the selections being independent and independent of the U's.

The event "$X_1 = \alpha(U_1)$ with $U_1 < 0$,

$X_2 = \beta(U_2)$ with $U_2 < 0$ and

$X_3 = \gamma(U_3)$ with $U_3 > 0$"

has positive probability and implies the event

"$X_1 \leqq X_2 \leqq X_3$,

$X_2 \in \psi]-1, 0]$ and $(X_1+X_3)/2 \in \phi[0, 1[$"

which in turn implies that $h(X_1, X_2, X_3) = 0$.

It would be interesting to know if there are singular probability measures P on the line such that $P \times P(\{(x_1, x_2): h(x_1, x_2) = 0\}) = 0$ for all translation invariant estimators h.

REFERENCE

[1] HODGES, J. L. and LEHMANN, E. L. (1963). Estimates of location based on rank tests. *Ann. Math. Statist.* **34** 598–611.