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ON A THEOREM OF G. L. SIEVERS

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In [3] Sievers proved a rather general theorem on the probability of large deviations. In this note a simpler method of proof is used to show that this theorem holds under weaker assumptions than those made by Sievers. In particular, no assumptions on the rate of convergence of the underlying sequences are necessary.

THEOREM. Let $\{W_n\}_{n=1,2,\ldots}$ *be a sequence of random variables on a probability space* $(\Omega, \mathscr{A}, P)$ *which satisfies the following assumptions*:

(i) $m_n(t) = \int e^{tW_n}\,dP < \infty$, $t \in [0, T)$, $T > 0$.

(ii) $\lim_{n\to\infty} n^{-1}\Psi_n^{(k)}(t) = c_k(t) < \infty$, $t \in [0, T)$, $k = 0, 1, 2$, *where* $\Psi_n(t) = \ln m_n(t)$.

(iii) $c_2(t) > 0$, $t \in [0, T)$.

(iv) $n^{-1}\Psi_n^{(3)}(t)$ *is locally bounded on* $(0, T)$. *Then for any sequence* $\{a_n\}_{n=1,2,\ldots}$ *of real numbers with* $\lim_{n\to\infty} a_n = a \in \{c_1(t)\colon t \in (0, T)\}$ *it holds that*

$$\lim_{n\to\infty} [P(W_n > na_n)]^{1/n} = \exp[c_0(h) - ha],$$

where $h \in (0, T)$ *is the unique solution of* $a = c_1(h)$.

PROOF. From Hölder's inequality $\int |f|^\tau |g|^{1-\tau}\,dP \geqq \int |f|^\tau |g|^{1-\tau}\,dP_A \geqq [\int |f|\,dP_A]^\tau \cdot [\int |g|\,dP_A]^{1-\tau}$, $\tau > 1$, $P_A(B) = P(A \cap B)$ for $A, B \in \mathscr{A}$, it follows for $f = e^{tW_n}/m_n(t)$, $t \in (0, T)$, $g \equiv 1$, $A = \{W_n > na_n\}$ that

$$P(W_n > na_n)\cdot[P_t(W_n > na_n)]^{\tau/(1-\tau)} \geqq [m_n(t\tau)/m_n^{\tau}(t)]^{1/(1-\tau)} \tag{1}$$

holds for $\tau > 1$ and $t\tau \in (0, T)$.

Here P_t is defined to be the conjugate distribution of P, i.e. $P_t(B) = \int_B f\,dP$ for $B \in \mathscr{A}$ and $t \in (0, T)$. Fixing t for $t\tau \in (0, T)$ and expanding $[1/(1-\tau)] \ln m_n(t + (\tau-1)t)$ in a Taylor series about the point t and using (i), (ii) and (iv) one gets the existence of a real number $\sigma = \sigma(t)$, such that

$$[m_n(t\tau)/m_n^{\tau}(t)]^{1/(1-\tau)} = \exp n[(c_0(t) - tc_1(t)) + o(1) + (\tau-1)O(1)] \tag{2}$$

holds for $\tau > 1$ and $\tau - 1 < \sigma(t)$. From the trivial inequality $P_t(W_n \leqq na_n) \leqq P_t[e^{tW_n}/m_n(t) \leqq (e^{(t-h)na_n + hW_n})/m_n(t)] \leqq [m_n(h)/m_n(t)]\cdot e^{(t-h)na_n}$ for $t > h$ it follows from (i), (ii) and (iv) using Taylor series expansion of $\Psi_n(t)$ about the point h that

$$P_t(W_n \leqq na_n) \leqq \exp n[[(t-h)^2/2]\cdot(-c_2(h) + [(t-h)/3]\cdot C) + o(1)]$$

holds for $t > h$ in a neighborhood $U(h)$ of h. Here $C = C(h)$ is a constant (i.e. independent of n and $t \in U(h)$) satisfying $n^{-1}\,|\Psi_n^{(3)}(t)| \leqq C$ for $t \in U(h)$. Therefore

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(iii) implies for any $\varepsilon > 0$ the existence of a neighborhood $U'(h) \subset U(h)$ of h and of an integer $N = N(t, \varepsilon)$, such that

$$P_t(W_n \leqq na_n) \leqq \varepsilon \tag{3}$$

holds for $t \in U'(h)$, $t > h$ and $n \geqq N$.

Letting first n tend to infinity and then τ tend to 1, one obtains from (1), (2) and (3) the result

$$\liminf_{n\to\infty} [P(W_n > na_n)]^{1/n} \geqq \exp [c_0(t) - tc_1(t)]$$

for $t \in U'(h)$ and $t > h$. Assumption (iv) implies that $c_k(t)$, $k = 0, 1$, are continuous functions on $(0, T)$. Hence

$$\liminf_{n\to\infty} [P(W_n > na_n)]^{1/n} \geqq \exp [c_0(h) - ha] \tag{4}$$

holds. The relation

$$\limsup_{n\to\infty} [P(W_n > na_n)]^{1/n} \leqq \exp [c_0(h) - ha] \tag{5}$$

follows from the extended Tschebycheff inequality ([1] page 42), which yields $P(W_n > na_n) \leqq \exp [\Psi_n(t) - tna_n]$ for $t \in (0, T)$. Inequalities (4) and (5) imply the assertion of the theorem. The uniqueness of h follows from the strict monotonicity of $c_1(t)$ as in [3].

REMARK 1. Replacing (iv) by the assumption

(iv′) $n^{-1}\Psi_n^{(2)}(t)$ converges locally uniformly to $c_2(t)$ for $t \in (0, T)$

and carrying the Taylor expansion of the proof only to the second derivative the results in (2) and (3) hold, because (iv′) implies that $c_2(t)$ is continuous on $(0, T)$ and the continuity of $c_2(t)$ together with (iv′) implies for a fixed $t \in (0, T)$ and $\eta > 0$ the existence of a neighborhood $U_\eta(t)$ of t and of an integer $N = N(\eta)$, such that

$$|n^{-1}\Psi_n^{(2)}(t') - c_2(t)| < \eta$$

holds for $t' \in U_\eta(t)$ and $n \geqq N$. Hence the result of the theorem holds, if (iv) is replaced by the assumption (iv′) (which is weaker than (iv)), because (iv′) implies the continuity of $c_k(t)$ on $(0, T)$ for $k = 0, 1$. This answers an interesting question raised by a referee, to whom I wish to express my thanks for some helpful suggestions.

REMARK 2. From this theorem together with Theorem 2 of [2] follows Theorem 3 (i) of [2] about the rate of convergence of Markov processes.

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