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THE EQUIVALENCE OF FUNCTIONAL CENTRAL LIMIT THEOREMS FOR COUNTING PROCESSES AND ASSOCIATED PARTIAL SUMS

BY DONALD L. IGLEHART[1] AND WARD WHITT[2]
Stanford University and Yale University

1. Introduction. Let $\{u_n, n \geqq 1\}$ be a sequence of nonnegative random variables, not necessarily independent or identically distributed, with an associated counting process or point process $\{N(t), t \geqq 0\}$, defined by

$$N(t) = \max\{k : u_1 + \cdots + u_k \leqq t\}, \qquad u_1 \leqq t \tag{1.1}$$
$$= 0, \qquad u_1 > t.$$

We shall show that functional central limit theorems (invariance principles) for $N(t)$ are equivalent to corresponding statements for the sequence of partial sums of the u_n's. This equivalence exists because the counting process and the partial sum process are essentially inverses of each other.

Let $\{X_n, n \geqq 1\}$ be the usual sequence of random functions in $D \equiv D[0, r]$, $0 < r < \infty$, induced by the sequence of partial sums; that is, let

$$X_n(t) = (\sigma^2 n)^{-\frac{1}{2}} \sum_{i=1}^{[nt]} (u_i - \mu), \qquad 0 \leqq t \leqq r, \tag{1.2}$$

where μ and σ^2 are positive constants. Let $\{N_n, n \geqq 1\}$ be the corresponding sequence of random functions induced in D by $N(t)$:

$$N_n(t) = (\sigma^2 \mu^{-3} n)^{-\frac{1}{2}} [N(nt) - nt/\mu], \qquad 0 \leqq t \leqq r. \tag{1.3}$$

Finally, let W be the Wiener measure on D. It is easy to show that weak convergence of $\{X_n\}$ or $\{N_n\}$ to any limit in $D[0, r]$ for some r, $0 < r < \infty$, implies weak convergence in $D[0, r]$ for any r, $0 < r < \infty$. This in turn implies weak convergence in $D[0, \infty)$, cf. [6], [7], and [10]. (We use $\Rightarrow$ to denote weak convergence of probability measures. When stochastic processes or ordinary random variables appear in such an expression, we mean the measures induced by these functions.) Billingsley has shown that if $X_n \Rightarrow W$, then $N_n \Rightarrow W$, cf. Theorem 17.3 of [1]. We shall prove the converse: if $N_n \Rightarrow W$, then $X_n \Rightarrow W$. Naturally this converse is not very useful for showing that $X_n \Rightarrow W$ when the conditions implying that $N_n \Rightarrow W$ also directly imply that $X_n \Rightarrow W$, which is the case when $\{u_n\}$ is i.i.d., cf. [1] Theorem 16.1. For nontrivial applications of this converse to queueing problems, see Section 8 of [3] and Lemma 1(iii) of [8]. The application to queues in [8] is of some general interest

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because it involves a functional central limit theorem for the partial sum process associated with superposition of several renewal counting processes.

We shall actually prove the necessary and sufficient conditions for weak convergence described above in a more general setting. We shall consider a double sequence of nonnegative random variables and allow the limit measure to be any measure on D which concentrates on $C \equiv C[0, r]$ with probability one. This greater generality is motivated in part by applications in queueing theory. In particular, the extension of Theorem 17.3 of [1] to double sequences and Theorem 3.1 of Prohorov [5] are used heavily in [4]. For applications where the limit is not W, see Theorem 8.4 of [3] and Section 3.3 of [4]. Our result also applies to sequences of random variables which are not nonnegative if instead of the sequence of partial sums $\{S_n, n \geqq 1\}$, where $S_n = u_1 + \cdots + u_n$, we consider the associated sequence of maxima $\{M_n, n \geqq 1\}$, where $M_n = \max\{S_k, 1 \leqq k \leqq n\}$. Since we have no i.i.d. assumptions, $\{M_n\}$ can be regarded as a sequence of partial sums in its own right. The associated counting process is then a first passage time process. Our equivalence theorem is applied this way in [12] to obtain functional central limit theorems in k-dimensional renewal theory. Finally, we remark that our result here concerns the case in which (1.2) and (1.3) have a positive translation term. The case of no translation term is treated in [11].

2. The results. Let $\{u_i^j;\, i, j \geqq 1\}$ be a double sequence of nonnegative random variables with no independence or common distribution assumptions. For each $j \geqq 1$, form the counting process $\{N^j(t), t \geqq 0\}$, defined by

$$N^j(t) = \max\{k : u_1^j + \cdots + u_k^j \leqq t\}, \qquad u_1^j \leqq t \tag{2.1}$$
$$= 0, \qquad u_1^j > t.$$

Now construct the (single) sequences of random functions $\{X_n\}$ and $\{N_n\}$ in $D[0, \infty)$:

$$X_n(t) = (1/c_n)\textstyle\sum_{i=1}^{[a_n t]}(u_i^n - b_n), \qquad t \geqq 0, \tag{2.2}$$

and

$$N_n(t) = (b_n/c_n)[N^n(a_n b_n t) - a_n t], \qquad t \geqq 0, \tag{2.3}$$

where $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences of positive constants. For the proofs in this paper we shall let $b_n \equiv 1$. To see that we can do this without loss of generality, consider the transformation: $u_i^{n\prime} = u_i^n/b_n$, $c_n' = c_n/b_n$, $b_n' = 1$, and $a_n' = a_n$. Note that with double sequences of random variables weak convergence of $\{X_n\}$ or $\{N_n\}$ in $D[0, r]$ for some r, $0 < r < \infty$, does not necessarily imply weak convergence in $D[0, s]$ for any $s > r$, but it is to be expected. Again weak convergence in $D[0, \infty)$ holds if there is weak convergence in $D[0, r]$ for all r, $0 < r < \infty$.

THEOREM 1. *Let r and s be arbitrary positive finite constants; let $c_n/b_n \to +\infty$, and $a_n b_n/c_n \to +\infty$; and let $Y \in D[0, r+s]$ with $P\{Y \in C[0, r+s]\} = 1$. If either $X_n \Rightarrow Y$ or $N_n \Rightarrow -Y$ in $D[0, r+s]$, then*

$$(X_n, N_n) \Rightarrow (Y, -Y) \quad in \quad D[0, r] \times D[0, r].$$

COROLLARY 1. *Let* $a_n b_n / c_n \to +\infty$, $c_n / b_n \to +\infty$, *and* $Y \in D[0, \infty)$ *with* $P\{Y \in C[0, \infty)\} = 1$. *Then* $X_n \Rightarrow Y$ *in* $D[0, \infty)$ *if and only if* $N_n \Rightarrow -Y$ *in* $D[0, \infty)$.

COROLLARY 2. *Let* X_n *and* N_n *be generated by a single sequence of random variables* $\{u_n\}$; *let r be an arbitrary positive constant*; $c_n/b_n \to \infty$, $a_n b_n / c_n \to \infty$, *and* $Y \in D[0, r]$ *with* $P\{Y \in C[0, r)\} = 1$. *If either* $X_n \Rightarrow Y$ *or* $N_n \Rightarrow -Y$ *in* $D[0, r]$, *then for any s*, $0 < s < \infty$,

$$(X_n, N_n) \Rightarrow (Y, -Y) \quad in \quad D[0, s] \times D[0, s].$$

If we have a single sequence $\{u_n\}$ and if $Y = W$, $a_n = n$, $b_n = \mu$, and $c_n = \sigma n^{\frac{1}{2}}$, then the result mentioned in the introduction is obtained from Theorem 1. To see this, note that

$$(\mu/\sigma n^{\frac{1}{2}})[N(\mu n t) - nt] \equiv (\sigma^2 \mu^{-3} m)^{-\frac{1}{2}}[N(mt) - mt/\mu]$$

if $m = n\mu$.

The central idea in Billingsley's proof of Theorem 17.3 of [1] is a random time change; [1] page 144. It is convenient for us to use a slight variation of Billingsley's argument. Let D_s consist of those functions Φ that are right-continuous and non-decreasing and satisfy $0 \leqq \Phi(t) \leqq s$ for all t, $0 \leqq t \leqq r$. Such functions represent transformations of the time interval $[0, r]$ into $[0, s]$. The space D_s is a closed subset of $D[0, r]$ and a complete separable metric space with Billingsley's version of the Skorohod metric on D, cf. Chapter 3 of [1]. Let d represent this metric and ρ the supremum metric on D and D_s.

Let $\{Z_n\}$ be any sequence of random functions in $D[0, s]$ and let $\{\Phi_n\}$ be any sequence of random functions in D_s, with Z_n and Φ_n defined on a common domain for each n. Assume that the prospective limits Z and Φ are also defined on a common domain. Billingsley's argument ([1] page 145 and Theorem 4.4) still applies to give

LEMMA 1. *Let r and s be positive, finite constants. If* $Z_n \Rightarrow Z$ *in* $D[0, s]$ *with* $P\{Z \in C[0, s]\} = 1$, *and* $\Phi_n \Rightarrow \Phi$ *in* D_s *where* Φ *is a constant function in* $C[0, r] \cap D_s$, *then*

$$(Z_n, Z_n \circ \Phi_n) \Rightarrow (Z, Z \circ \Phi) \quad in \quad D[0, s] \times D[0, r],$$

where $Z_n \circ \Phi_n \equiv Z_n(\Phi_n(t))$, $0 \leqq t \leqq r$, *and* $Z \circ \Phi \equiv Z(\Phi(t))$, $0 \leqq t \leqq r$.

We shall also use Lemma 1 to prove Theorem 1. In order to prove the converse of ([1] Theorem 17.3) and the corresponding part of Theorem 1, we shall prove

LEMMA 2. *Let r and s be positive, finite constants. Let* $Z_n \in D[0, s]$, $\Phi_n \in D_s$, *and* Φ *be a strictly increasing constant function in* $C \cap D_s$ *with* $\Phi(0) = 0$ *and* $\Phi(r) = r$. *If* $Z_n \circ \Phi_n \Rightarrow Z \circ \Phi$ *in* $D[0, r]$, $d(\Phi_n, \Phi) \Rightarrow 0$, *and* $\{Z_n\}$ *is C-tight in* $D[0, s]$, *then*

$$(Z_n, Z_n \circ \Phi_n) \Rightarrow (Z, Z \circ \Phi) \quad in \quad D[0, s] \times D[0, r].$$

C-tightness is discussed at the beginning of Section 3. For applications of Lemma 2 other than the proof of Theorem 1, see Section 7 of [3] and Section 5 of [9].

The connection between Theorem 1 and Lemmas 1 and 2 is made apparent by defining the sequence of random functions $\{Y_n\}$ in $D[0, r]$ (with $b_n = 1$):

$$Y_n(t) = (1/c_n)\sum_{i=1}^{N^n(a_n t)}(u_i^n - 1). \qquad 0 \leqq t \leqq r. \tag{2.4}$$

Notice that

$$Y_n(t) \leqq [a_n t - N^n(a_n t)]/c_n \leqq Y_n(t) + (1/c_n)u^n_{N^n(a_n t)+1} \tag{2.5}$$

from which we shall deduce that $\rho(Y_n, -N_n) \Rightarrow 0$. From the relationship above, it is clear that the material here is closely related to functional central limit theorems for random sums, cf. [1] Section 17, [2] Section 2, and [9] Section 2. Also, using the random time change Φ_n mapping $[0, r]$ onto $[0, r+s]$, where

$$\Phi_n(t) = \frac{N^n(a_n t)}{a_n} \wedge (r+s), \qquad 0 \leqq t \leqq r, \tag{2.6}$$

we shall show that $\rho(Y_n, X_n \circ \Phi_n) \Rightarrow 0$ and $\rho(\Phi_n, I) \Rightarrow 0$, where $I(t) = t, 0 \leqq t \leqq 1$. Hence, $\rho(-N_n, X_n \circ \Phi_n) \Rightarrow 0$ and Theorem 1 follows. We now fill in the details.

3. The proofs. We first prove two technical lemmas. Since C-tightness is much easier to apply than D-tightness, cf. [1] page 55 and page 125, we would like a condition giving C-tightness based on weak convergence in D. In a sense, we want a converse to Theorem 15.5 of [1]. The property of C-tightness in C or D is expressed in terms of the modulus of continuity $w(\delta)$: $D \to R$, defined for any $x \in D[0, r]$ by

$$w_x(\delta) = \sup_{0 \leqq s,t \leqq r, |s-t| \leqq \delta} |x(t) - x(s)|. \tag{3.1}$$

LEMMA 3. *Let $Z_n \in D$, $Z \in D$, and $P\{Z \in C\} = 1$. If $Z_n \Rightarrow Z$, then $\{Z_n\}$ is C-tight: for all positive ε and η, there exists a $\delta(0 < \delta < 1)$ and an integer n_0 such that $P\{w_{Z_n}(\delta) > \varepsilon\} \leqq \eta$ for $n \geqq n_0$.*

PROOF. Theorem 5.1 of [1] implies that $w_{Z_n}(\delta) \Rightarrow w_Z(\delta)$ for each δ, but $w_Z(\delta) \Rightarrow 0$ as $\delta \downarrow 0$.

Let $J: D \to R$ be the maximum jump functional, defined for any $x \in D[0, r]$ by

$$J(x) = \sup_{0 \leqq t \leqq r}\{|x(t) - x(t-)|\}. \tag{3.2}$$

For $x \in C, J(x) = 0$.

LEMMA 4. *The function J is measurable and continuous almost everywhere with respect to any measure concentrating on C with probability one.*

PROOF. The modulus of continuity $w(\delta): D \to R$ is measurable because

$$\begin{aligned} w_x(\delta) &= \sup_{0 \leqq s,t \leqq r, |s-t| \leqq \delta} |x(t) - x(s)| \\ &= \sup_{0 \leqq s,t \leqq r, |s-t| \leqq \delta, s,t \in Q \cup \{r\}} |x(t) - x(s)|, \end{aligned}$$

where Q is the set of rational numbers. Obviously, $w_x(\delta) \geqq J(x)$ for all $\delta > 0$ and $w_x(\delta)$ decreases as δ decreases. Applying Lemma 1 of [1] page 110, we have

$\lim_{\delta \downarrow 0} w_x(\delta) = J(x)$. If we choose a sequence $\{\delta_n\}$ with $\delta_n \downarrow 0$, then J is the limit of a sequence of measurable functions, and is thus measurable.

To show continuity, suppose $d(x_n, x) \to 0$. For $x \in C$, $\rho(x_n, x) \to 0$, and

$$\begin{aligned} |J(x_n) - J(x)| &= J(x_n) \\ &\leqq \sup_{0 \leqq t \leqq r} |x_n(t) - x(t)| + \sup_{0 \leqq t \leqq r} |x_n(t-) - x(t-)| \\ &\leqq 2\rho(x_n, x) \to 0. \end{aligned}$$

PROOF OF LEMMA 2. If X is a weak limit of some weakly convergent subsequence $\{Z_{n'}\}$ of $\{Z_n\}$, then $Z_{n'} \circ \Phi_{n'} \Rightarrow X \circ \Phi$. Hence, $X \circ \Phi \sim Z \circ \Phi$. Now choose finitely many time points $t_1, \cdots, t_k$ in $[0, s]$. For each t_i, there is a unique u_i in $[0, r]$ such that $\Phi(u_i) = t_i$. Since all the projections are measurable, cf. [1] page 121, $[X(t_1), \cdots, X(t_k)] = [X \circ \Phi(u_1), \cdots, X \circ \Phi(u_k)] \sim [Z \circ \Phi(u_1), \cdots, Z \circ \Phi(u_k)] = [Z(t_1), \cdots, Z(t_k)]$. Hence, $X \sim Z$.

PROOF OF THEOREM 1. In one direction, the assertion is a consequence of Lemmas 5, 6, and 7 to come and Lemma 1. In the other direction, the assertion is a consequence of Lemmas 8, 9, 10, and 11 to come and Lemma 2. Throughout the following discussion assume $P\{Y \in C\} = 1$; $b_n = 1$, $c_n \to \infty$, $a_n/c_n \to \infty$ as $n \to \infty$; $\Phi_n(t) = [N^n(a_n t)/a_n] \wedge (r+s)$, $0 \leqq t \leqq r$; and r and s are positive finite constants.

LEMMA 5. *If $X_n \Rightarrow Y$ in $D[0, r+s]$, then $\Phi_n \Rightarrow I$ in D_s.*

PROOF. Since $X_n \Rightarrow Y$ and $a_n/c_n \to \infty$, Theorems 4.4 and 5.1 of [1] imply that

$$\sup_{0 \leqq t \leqq r+s} \left|(1/a_n) \textstyle\sum_{i=1}^{[a_n t]} (u_i^n - 1)\right| \Rightarrow 0,$$

where $[\cdot]$ is the integer part function. We now apply the basic relationship: $N^n(t) \geqq m$ if and only if $\sum_{i=1}^m u_i^n \leqq t$. We have $\inf_{0 \leqq t \leqq r} \{(1/a_n)[N^n(a_n t) - a_n t]\} \geqq -\varepsilon$ if and only if $N^n(a_n t) \geqq a_n(t - \varepsilon)$, $0 \leqq t \leqq r$, which holds if and only if

$$N^n(a_n t) \geqq [a_n(t-\varepsilon)], \qquad \varepsilon \leqq t \leqq r, \quad \text{or}$$

$$\textstyle\sum_{i=1}^{[a_n(t-\varepsilon)]} u_i^n \leqq a_n t, \qquad \varepsilon \leqq t \leqq r, \quad \text{or}$$

$$\textstyle\sum_{i=1}^{[a_n(t-\varepsilon)]} (u_i^n - 1) \leqq (a_n t - [a_n(t-\varepsilon)]), \qquad \varepsilon \leqq t \leqq r,$$

which holds if

$$\sup_{0 \leqq t \leqq r-\varepsilon} \{(1/a_n) \textstyle\sum_{i=1}^{[a_n t]} (u_i^n - 1)\} \leqq \varepsilon,$$

but we have just shown that the probability of this event approaches one as $n \to \infty$.

A similar argument shows that for all positive ε

$$\lim_{n \to \infty} P\{\sup_{0 \leqq t \leqq r} \{(1/a_n)[N^n(a_n t) - a_n t]\} < \varepsilon\} = 1.$$

LEMMA 6. *If $X_n \Rightarrow Y$ in $D[0, r+s]$, then $\rho(-N_n, Y_n) \Rightarrow 0$ in $D[0, r]$.*

PROOF. By the triangle inequality, $u_i^n/c_n \leqq |u_i^n - 1|/c_n + 1/c_n$; by Lemma 4, $J(X_n) \Rightarrow 0$; by assumption, $c_n \to \infty$; and by the proof of Lemma 5, $N^n(a_n r)/a_n \Rightarrow r$. Hence,

$$\rho(-N_n, Y_n) = \sup_{0 \leqq t \leqq r} \left|(1/c_n) u^n_{N^n(a_n t)+1}\right| \Rightarrow 0.$$

LEMMA 7. *If* $X_n \Rightarrow Y$ *in* $D[0, r+s]$, *then* $\rho(X_n \circ \Phi_n, -N_n) \Rightarrow 0$ *in* $D[0, r]$.

PROOF. By Lemma 6, it suffices to show that $\rho(X_n \circ \Phi_n, Y_n) \Rightarrow 0$ in $D[0, r]$. If $\Phi_n(r) < r+s$, then $Y_n = X_n \circ \Phi_n$. Since $\Phi_n \Rightarrow I$, $P\{\Phi_n(r) < r+s\} \to 1$.

The proof of the first part of Theorem 1 is finished by applying Lemma 1 and Theorems 4.1 and 5.1 of [1]. We now turn to the second part of Theorem 1.

LEMMA 8. *If* $N_n \Rightarrow -Y$ *in* $D[0, r]$, *then* $\Phi_n \Rightarrow I$ *in* D_s.

PROOF. Recall that

$$\rho(\Phi_n, I) \leqq \sup_{0 \leqq t \leqq r} |N^n(a_n t)/a_n - t|,$$

but since $N_n \Rightarrow -Y$ and $a_n/c_n \to \infty$, Theorems 4.4 and 5.1 of [1] imply that $\sup_{0 \leqq t \leqq r} |N^n(a_n t)/a_n - t| \Rightarrow 0$. Hence, $\Phi_n \Rightarrow I$.

LEMMA 9. *If* $N_n \Rightarrow -Y$ *in* $D[0, r+s]$, *then* $\rho(-N_n, Y_n) \Rightarrow 0$ *in* $D[0, r]$.

PROOF. If $U_n = \sup_{0 \leqq t \leqq r} \{(1/c_n) u^n_{N^n(a_n t)+1}\}$, then we need to show that $U_n \Rightarrow 0$. Since $P\{-Y \in C\} = 1$, we have C-tightness for $\{N_n\}$ by virtue of Lemma 3. We shall show that this C-tightness would be violated if we did not have $U_n \Rightarrow 0$. For each $n \geqq 1$, there are time points $t_1 \in [0, r]$ and $t_2 \geqq t_1$ such that $|t_2 - t_1| = c_n U_n / a_n$ and $N^n(a_n t_2 -) - N^n(a_n t_1) = 0$. For $\delta < s$; the statements above imply that

$$w_{N_n}(\delta) \geqq U_n \wedge (a_n/c_n)\delta$$

where $a_n/c_n \to \infty$. Hence, $U_n \Rightarrow 0$, and the proof is complete.

LEMMA 10. *If* $N_n \Rightarrow -Y$ *in* $D[0, r+s]$, *then* $\rho(X_n \circ \Phi_n, -N_n) \Rightarrow 0$ *in* $D[0, r]$.

PROOF. Apply the argument of Lemma 7.

LEMMA 11. *If* $N_n \Rightarrow -Y$ *in* $D[0, r+s]$, *then* $\{X_n\}$ *is C-tight in* $D[0, r]$.

PROOF. Since $N_n \Rightarrow -Y$ where $P\{-Y \in C\} = 1$, the sequence $\{N_n\}$ is C-tight in $D[0, r+s]$ by virtue of Lemma 3. In other words, for all ε and $\eta > 0$, there exists a δ, $0 < \delta < 1$, and an n_0 such that $P(A_n \cup B_n) < \eta$ for $n \geqq n_0$, where
$A_n = \{N^n(a_n t_2) - N^n(a_n t_1) > [a_n(t_2 - t_1) + \varepsilon c_n]$, for some t_1 and t_2, $0 \leqq t_1, t_2 \leqq r+s$, $|t_2 - t_1| < \delta\}$ and
$B_n = \{N^n(a_n t_2) - N^n(a_n t_1) < [a_n(t_2 - t_1) - \varepsilon c_n] + 1$, for some t_1 and t_2, $0 \leqq t_1, t_2 \leqq r+s$, $|t_2 - t_1| < \delta\}$.

Now an argument similar to that used in Lemma 5 shows that if $N^n(a_n[r+s]) \geqq a_n r$, then $\{w_{X_n}(\delta) > 2\varepsilon\} \subseteq A_n \cup B_n$ for sufficiently large n, where the modulus of continuity is used in $D[0, r]$. In any case, for sufficiently large n

$$P\{w_{X_n}(\delta) \geqq 2\varepsilon\} \leqq P(A_n \cup B_n) + P\{N^n(a_n[r+s]) \leqq a_n r\}.$$

Since $N^n(a_n[r+s])/a_n \Rightarrow r+s$, the second term converges to zero as well as the first and we have the desired C-tightness for $\{X_n\}$ in $D[0, r]$.

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