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ON A CLASS OF BIVARIATE DISTRIBUTIONS FOLLOWING A CERTAIN STOCHASTIC STRUCTURE[1]

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0. Introduction. Consider two independent random variables X, Y with $E(X) = E(Y) = 0$ and two other random variables X^*, Y^* following a Stochastic Structure:

$$X^* = AX + BY; \tag{1}$$

$$Y^* = CX + DY;$$

where A, B, C, D are nonzero constants.

Laha ([2], [4]) studied the problem of characterizing the distributions of X and Y through regression properties of X^* and Y^*. In particular, he showed that if $AD = BC$, $E(Y^* \mid X^*) = \beta X^*$ almost surely whatever may be the distributions of X and Y, where $\beta = DB^{-1}$. If $AD \neq BC$, both X and Y have symmetric stable distributions with the same characteristic exponent $\alpha(1 < \alpha \leqq 2)$, if and only if

(i) $E(Y^* \mid X^*) = \beta X^*$ for all $0 < |A| \leqq \delta$ for some $\delta > 0$ and

(ii) $$\beta = (CA^{-1}\alpha_1 |A|^\alpha + DB^{-1}\alpha_2 |B|^\alpha)$$

$$(\alpha_1 |A|^\alpha + \alpha_2 |B|^\alpha)^{-1}; \tag{2}$$

where α_1 and α_2 are the scale parameters of the distributions of X and Y respectively.

The object of this article is to make some extensions of these results. In Section 1, we consider a stochastic structure similar to (1) when $\mathbf{X}$ is a $p \times 1$ random vector, $\mathbf{Y}$ is a $q \times 1$ random vector, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are matrices of order $q \times p$, $q \times q$, $r \times p$, $r \times q$ respectively. Assuming the matrix $\mathbf{B}$ to be nonsingular we show that if $\mathbf{C} = \mathbf{DB}^{-1}\mathbf{A}$, $E(\mathbf{Y}^* \mid \mathbf{X}^*) = \boldsymbol{\beta}\mathbf{X}^*$ almost surely where $\boldsymbol{\beta} = \mathbf{DB}^{-1}$. In Section 2, we confine ourselves to the case when $p = q = 2$. In this case, if $\mathbf{C} \neq \mathbf{DB}^{-1}\mathbf{A}$ and some additional conditions are satisfied, it is possible to characterize a class of bivariate distributions through the regression properties of $\mathbf{X}^*$ and $\mathbf{Y}^*$. These distributions are not necessarily stable as defined by Lévy ([3] Section 63). In Section 3, we consider a special case of the class of bivariate distributions introduced in the previous section. The latter class is stable and includes bivariate normal distribution.

1. Some preliminary results. Suppose, $\mathbf{X}$ and $\mathbf{Y}$ are random vectors of order $p \times 1$ and $q \times 1$ respectively such that $E(\mathbf{X})$ and $E(\mathbf{Y})$ exist. $\varphi(\mathbf{U}, \mathbf{V})$ is the joint

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characteristic function of **X** and **Y** where **U** and **V** are matrices or order $p \times 1$ and $q \times 1$ respectively of real quantities, so that

(3) $$\varphi(\mathbf{U}, \mathbf{V}) = E(\exp(i\mathbf{U}'\mathbf{X} + i\mathbf{V}'\mathbf{Y})).$$

Define:

(4) $$e_{1j} = \left[\frac{\partial\varphi(\mathbf{U}, \mathbf{V})}{\partial v_j}\right]_{V=0}; \qquad j = 1, 2, \ldots, q;$$

$$e_{2j} = \frac{\partial(\mathbf{U}, \mathbf{0})}{\partial u_j}; \qquad j = 1, 2, \ldots, p;$$

where u_j, v_j are jth components of **U** and **V** respectively. Suppose,

(5) $$\mathbf{E}_1' = (e_{11}, e_{12}, \ldots e_{1q}); \qquad \mathbf{E}_2' = (e_{21}, e_{22}, \ldots e_{2p}).$$

We first prove the following result:

LEMMA 1. *Suppose* $E(\mathbf{X}) = \mathbf{0}$; $E(\mathbf{Y}) = \mathbf{0}$.

Then,

$$E(\mathbf{Y} \mid \mathbf{X}) = \boldsymbol{\beta}\mathbf{X} \textit{ almost everywhere} \text{ if and only if}$$
$$\mathbf{E}_1 = \boldsymbol{\beta}\mathbf{E}_2; \textit{ where}$$
$$\boldsymbol{\beta} = (\beta_{ij}) \textit{ is a } q \times p \textit{ matrix of real quantities.}$$

PROOF. For $p = q = 1$, the lemma is proved in [2]. Since the proof for the general case can be derived in a similar way, we omit the details.

First we assume that $E(\mathbf{Y} \mid \mathbf{X}) = \boldsymbol{\beta}\mathbf{X}$ almost surely. This can be rewritten as

$$E(\mathbf{Y} \mid \mathbf{X}) = \begin{bmatrix} \boldsymbol{\beta}_1\mathbf{X} \\ \boldsymbol{\beta}_2\mathbf{X} \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{\beta}_q\mathbf{X} \end{bmatrix}$$

where $\boldsymbol{\beta}_j = (\beta_{j1}, \beta_{j2}, \ldots \beta_{jp}); j = 1, 2 \ldots q$.

Hence, $E(y_j \mid \mathbf{X}) = \boldsymbol{\beta}_j\mathbf{X}; j = 1, 2, \ldots q$; where y_j is the jth component of **Y**.

(6) $$e_{1j} = i \int \exp(i\mathbf{U}'\mathbf{X})[\boldsymbol{\beta}_j\mathbf{X}]\, dF_1(\mathbf{X}), \qquad j = 1, 2, \ldots q;$$

where $\mathbf{F}_1(\mathbf{X})$ is the marginal distribution function of **X** and the single integral in (6) stands for a p-fold integral.

(7) $$e_{2k} = i \int x_k \exp(i\mathbf{U}'\mathbf{X}), dF_1(\mathbf{X}); \qquad k = 1, 2, \ldots p;$$

where x_k is the kth component of **X** and the single integral in (7) stands for a p-fold integral. Now,

$$\boldsymbol{\beta}_j\mathbf{E}_2 = \sum_{k=1}^{p} \beta_{jk}e_{2k} = i \sum_{k=1}^{p} \beta_{jk} \int x_k \exp(i\mathbf{U}'\mathbf{X})dF_1(\mathbf{X})$$
$$= i \int \boldsymbol{\beta}_j\mathbf{X} \exp(i\mathbf{U}'\mathbf{X})\, dF_1(\mathbf{X}) = e_{1j} \qquad (j = 1, 2, \ldots, q).$$

Thus, $\mathbf{E}_1 = \boldsymbol{\beta}\mathbf{E}_2$, which proves the "only if" part of the proposition.

Next we assume that $\mathbf{E}_1 = \boldsymbol{\beta}\mathbf{E}_2$. Simple calculation leads to:

$$e_{1j} = i \int \exp(i\mathbf{U}'\mathbf{X})\, E[y_j \mid \mathbf{X}]dF_1(\mathbf{X}); \qquad j = 1, 2, \dots, q. \tag{8}$$

Since $e_{1j} = \boldsymbol{\beta}_j\mathbf{E}_2$, it follows from (7) that

$$e_{1j} = i \int \exp(i\mathbf{U}'\mathbf{X})\, \boldsymbol{\beta}_j\mathbf{X}dF_1(\mathbf{X}); \qquad j = 1, 2, \dots, q. \tag{9}$$

From (8) and (9) we obtain

$$\int \exp(i\mathbf{U}'\mathbf{X})[E(y_j \mid \mathbf{X}) - \boldsymbol{\beta}_j\mathbf{X}]dF_1(\mathbf{X}) = 0; \qquad j = 1, 2, \dots, q.$$

By the uniqueness of Fourier's transform it follows that $E(y_j \mid \mathbf{X}) = \boldsymbol{\beta}_j\mathbf{X}$, $(j = 1, 2, \dots, q)$ almost surely.

Thus, $E(\mathbf{Y} \mid \mathbf{X}) = \boldsymbol{\beta}\mathbf{X}$, almost surely, which proves the "if" part of the proposition.

Consider a stochastic structure with random vectors $\mathbf{X}(p \times 1)$, $\boldsymbol{\xi}(q \times 1)$, $\boldsymbol{\eta}(r \times 1)$ of the following form:

$$\mathbf{X}^* = \mathbf{A}\mathbf{X} + \boldsymbol{\xi}; \qquad Y^* = \mathbf{C}\mathbf{X} + \boldsymbol{\eta}; \tag{10}$$

where $\mathbf{A} = (a_{ij})$, $\mathbf{C} = (c_{ij})$ are matrices or order $q \times p$ and $r \times p$ respectively. The results of the lemma can now be used to derive a sufficient condition for $E(\mathbf{Y}^* \mid \mathbf{X}^*)$ to be of the form $\boldsymbol{\beta}\mathbf{X}^*$ where $\boldsymbol{\beta}$ is a $r \times q$ matrix of real quantities.

THEOREM 1. *Suppose a stochastic structure with random vectors* $\mathbf{X}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ *is given by equation* (10) *where* $E(\mathbf{X}) = \mathbf{0}$; $E(\boldsymbol{\xi}) = \mathbf{0}$; $E(\boldsymbol{\eta}) = \mathbf{0}$. *Further, let* $E(\boldsymbol{\eta} \mid \boldsymbol{\xi}) = \boldsymbol{\beta}\boldsymbol{\xi}$ *and the distribution of* $\mathbf{X}$ *is independent of the joint distribution of* $\boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$. *Then* $E(\mathbf{Y}^* \mid \mathbf{X}^*) = \boldsymbol{\beta}\mathbf{X}^*$ *provided* $\boldsymbol{\beta}\mathbf{A} = \mathbf{C}$.

PROOF. Let $\Phi(\mathbf{U}, \mathbf{V})$, $\varphi(\mathbf{U}, \mathbf{V})$ and $\varphi_1(\cdot)$ be the characteristic functions of $(\mathbf{X}^{*\prime}, \mathbf{Y}^{*\prime})$, $(\boldsymbol{\xi}', \boldsymbol{\eta}')$ and $\mathbf{X}'$ respectively where $\mathbf{U}$ is a $q \times 1$ and $\mathbf{V}$ is a $r \times 1$ matrix. Then,

$$\Phi(\mathbf{U}, \mathbf{V}) = \varphi(\mathbf{U}, \mathbf{V})\varphi_1(\mathbf{A}'\mathbf{U} + \mathbf{C}'\mathbf{V}). \tag{11}$$

Suppose

$$e_{1j}^* = \left[\frac{\partial\Phi(\mathbf{U}, \mathbf{V})}{\partial v_j}\right]_{V=0}; \qquad e_{2j}^* = \frac{\partial\Phi(\mathbf{U}, \mathbf{0})}{\partial u_j};$$

$$e_{1j} = \left[\frac{\partial\varphi(\mathbf{U}, \mathbf{V})}{\partial v_j}\right]_{V=0}; \qquad e_{2j} = \frac{\partial\varphi(\mathbf{U}, \mathbf{0})}{\partial u_j}.$$

From (11) we obtain

$$e_{1j}^* = e_{1j}\varphi_1(\mathbf{A}'\mathbf{U}) + \varphi(\mathbf{U}, \mathbf{0})\left[\frac{\partial\varphi_1(\mathbf{A}'\mathbf{U} + \mathbf{C}'\mathbf{V})}{\partial v_j}\right]_{V=0}, \qquad j = 1, 2, \dots, r. \tag{12}$$

Simple calculation leads to:

$$\begin{aligned} e_{3j} &= \left[\frac{\partial\varphi_1(\mathbf{A}'\mathbf{U} + \mathbf{C}'\mathbf{V})}{\partial v_j}\right]_{V=0} \\ &= i \int \exp(i\mathbf{U}'\mathbf{A}\mathbf{X})\, [\textstyle\sum_{k=1}^{p} c_{jk}x_k]dF_1(\mathbf{X}); \qquad j = 1, 2, \dots, r, \end{aligned} \tag{13}$$

where x_k is the kth component of $\mathbf{X}$, $\mathbf{F}_1(\mathbf{X})$ is the marginal distribution function of $\mathbf{X}$, and the single integral in (13) stands for a p-fold integral. Let

(14) $\mathbf{E}_1' = (e_{11}, \ldots, e_{1r});\qquad \mathbf{E}_1{}^{*\prime} = (e^*_{11}, \ldots, e^*_{1r});\qquad \mathbf{E}_3' = (e_{31}, \ldots, e_{3r}).$

Then, from (12), (13) and (14) we obtain

$$\mathbf{E}_1{}^* = \varphi_1(\mathbf{A}'\mathbf{U})\mathbf{E}_1 + \varphi(\mathbf{U}, \mathbf{0})\mathbf{E}_3. \tag{15}$$

Again from (11)

$$e^*_{2j} = e_{2j}\varphi_1(\mathbf{A}'\mathbf{U}) + \varphi(\mathbf{U}, \mathbf{0})\left[\frac{\partial\varphi_1(\mathbf{A}'\mathbf{U})}{\partial u_j}\right]; \qquad j = 1, 2, \ldots, q. \tag{16}$$

Simple calculation leads to

$$e_{4j} = \frac{\partial\varphi_1(\mathbf{A}'\mathbf{U})}{\partial u_j} = i\int \exp(i\mathbf{U}'\mathbf{A}\mathbf{X})\,[\textstyle\sum_{k=1}^{p} a_{jk}x_k]\,dF_1(\mathbf{X}); \quad j = 1, 2, \ldots, q; \tag{17}$$

and the single integral in (17) stands for a p-fold integral.

Let

(18) $\mathbf{E}_2' = (e_{21}, \ldots, e_{2q});\qquad \mathbf{E}_2{}^{*\prime} = (e^*_{21}, \ldots, e^*_{2q});\qquad \mathbf{E}_4' = (e_{41}, \ldots, e_{4q}).$

Then, from (16), (17) and (18) it follows that

$$\mathbf{E}_2{}^* = \varphi_1(\mathbf{A}'\mathbf{U})\mathbf{E}_2 + \varphi(\mathbf{U}, \mathbf{0})\mathbf{E}_4. \tag{19}$$

Since $E(\boldsymbol{\eta} \mid \boldsymbol{\xi}) = \boldsymbol{\beta}\boldsymbol{\xi}$, by Lemma 1

$$\mathbf{E}_1 = \boldsymbol{\beta}\mathbf{E}_2. \tag{20}$$

Using the condition $\boldsymbol{\beta}\mathbf{A} = \mathbf{C}$, simple calculation leads to

$$\mathbf{E}_3 = \boldsymbol{\beta}\mathbf{E}_4. \tag{21}$$

From (15), (19), (20), (21), it follows that $\mathbf{E}_1{}^* = \boldsymbol{\beta}\mathbf{E}_2{}^*$.

The proof of the theorem now follows from Lemma 1.

COROLLARY. *Suppose we have a stochastic structure*:

$$\mathbf{X}^* = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{Y}; \qquad \mathbf{Y}^* = \mathbf{C}\mathbf{X} + \mathbf{D}\mathbf{Y}; \tag{22}$$

where $\mathbf{X}^*$, $\mathbf{Y}^*$, $\mathbf{X}$, $\mathbf{A}$, $\mathbf{C}$ *are defined as in Theorem* 1. $\mathbf{Y}$ *is a* $q \times 1$ *random vector independent of* $\mathbf{X}$, *with* $E(\mathbf{Y}) = \mathbf{0}$. $\mathbf{D}$ *is a* $r \times q$ *matrix and* $\mathbf{B}$ *is a* $q \times q$ *nonsingular matrix. Then*

$$E(\mathbf{Y}^* \mid \mathbf{X}^*) = \mathbf{D}\mathbf{B}^{-1}\mathbf{X}^* \quad \textit{almost surely provided} \quad \mathbf{C} = \mathbf{D}\mathbf{B}^{-1}\mathbf{A}.$$

The proof of the corollary follows immediately from Theorem 1, if we substitute $\boldsymbol{\xi} = \mathbf{B}\mathbf{Y}$ and $\boldsymbol{\eta} = \mathbf{D}\mathbf{Y}$.

Theorem 1 and the corollary following it give the multivariate extension of Theorem 1 given in [2]. Given the stochastic structure (22), a sufficient condition for $E(\mathbf{Y}^* \mid \mathbf{X}^*) = \boldsymbol{\beta}\mathbf{X}^*$ is $\mathbf{C} = \mathbf{D}\mathbf{B}^{-1}\mathbf{A}$. In Section 2, we investigate under what conditions $E(\mathbf{Y}^* \mid \mathbf{X}^*) = \boldsymbol{\beta}\mathbf{X}^*$ when $\mathbf{C} \neq \mathbf{D}\mathbf{B}^{-1}\mathbf{A}$ assuming $p = q = 2$.

2. Characterization of a class of bivariate distributions.

THEOREM 2. *Consider two* 2×1 *independent random vectors* **X** *and* **Y** ($E(\mathbf{X}) = E(\mathbf{Y}) = \mathbf{0}$) *with characteristic functions* $f(\mathbf{U})$ *and* $g(\mathbf{U})$ *respectively where* $\mathbf{U}' = (u_1, u_2)$. *Suppose, for some real constants* C_1, C_2, D_1, D_2, $f(\mathbf{U})$ *and* $g(\mathbf{U})$ *can be represented for* all *real* u_1, u_2 *as*:

$$\log f(\mathbf{U}) = C_1 h_1(\mathbf{c}'\mathbf{U}) + C_2 h_2(\mathbf{d}'\mathbf{U}); \tag{23}$$

$$\log g(\mathbf{U}) = D_1 h_1(\mathbf{f}'\mathbf{U}) + D_2 h_2(\mathbf{g}'\mathbf{U});$$

where $\mathbf{c}' = (c_1, c_2)$; $\mathbf{d}' = (d_1, d_2)$; $\mathbf{f}' = (f_1, f_2)$; $\mathbf{g}' = (g_1, g_2)$ *are vectors of real quantities*; *and* $c_1/c_2 \neq d_1/d_2$; $h_1(\cdot)$, $h_2(\cdot)$ *are real valued functions such that* $dh_i(x)/dx (i = 1, 2)$ *are continuous for all* x.

Suppose, $\mathbf{X}^*$, $\mathbf{Y}^*$ *are* 2×1 *random vectors defined according to* (22) *where* **A**, **B**, **C**, **D** *are real* 2×2 *matrices with* **A** *and* **B** *being nonsingular. Further, let*

$$\mathbf{Ac} = \mathbf{Bf} = \mathbf{S}; \qquad \mathbf{Ad} = \mathbf{Bg} = \mathbf{T}; \tag{24}$$

where $\mathbf{S}' = (s_1, s_2)$; $\mathbf{T}' = (t_1, t_2)$ *are real vectors.*

Then,

$$E(\mathbf{Y}^* \mid \mathbf{X}^*) = \boldsymbol{\beta}\mathbf{X}^* \text{ almost surely.}$$

Further,

$$\boldsymbol{\beta} = \mathbf{PQ}^{-1}, \tag{25}$$

where $\mathbf{P} = (C_1\mathbf{Cc} + D_1\mathbf{Df} \quad C_2\mathbf{Cd} + D_2\mathbf{Dg})$; $\mathbf{Q} = \mathbf{A}((C_1 + D_1)\mathbf{c} \quad (C_2 + D_2)\mathbf{d})$.

PROOF. Let $\Phi(\mathbf{U}, \mathbf{V})$ be the joint characteristic function of $\mathbf{X}^*$ and $\mathbf{Y}^*$. Clearly,

$$\Phi(\mathbf{U}, \mathbf{V}) = f(\mathbf{A}'\mathbf{U} + \mathbf{C}'\mathbf{V})g(\mathbf{B}'\mathbf{U} + \mathbf{D}'\mathbf{V}); \tag{26}$$

$$e_{1j} = \left[\frac{\partial f(\mathbf{A}'\mathbf{U} + \mathbf{C}'\mathbf{V})}{\partial v_j}\right]_{V=0} g(\mathbf{B}'\mathbf{U}) + \left[\frac{\partial g(\mathbf{B}'\mathbf{U} + \mathbf{D}'\mathbf{V})}{\partial v_j}\right]_{V=0} f(\mathbf{A}'\mathbf{U}). \tag{27}$$

From (23) and (24) we obtain

$$\log f(\mathbf{A}'\mathbf{U} + \mathbf{C}'\mathbf{V}) = C_1 h_1(\mathbf{S}'(\mathbf{U} + \mathbf{WV})) + C_2 h_2(\mathbf{T}'(\mathbf{U} + \mathbf{WV})); \tag{28}$$

$$\log g(\mathbf{B}'\mathbf{U} + \mathbf{D}'\mathbf{V}) = \mathbf{D}_1 h_1(\mathbf{S}'(\mathbf{U} + \mathbf{ZV})) + \mathbf{D}_2 h_2(\mathbf{T}'(\mathbf{U} + \mathbf{ZV}));$$

where $\mathbf{W} = (\mathbf{A}')^{-1}\mathbf{C}'$; $\mathbf{Z} = (\mathbf{B}')^{-1}\mathbf{D}'$.

Hence,

$$\frac{1}{f(\mathbf{A}'\mathbf{U} + \mathbf{C}'\mathbf{V})} \frac{\partial f(\mathbf{A}'\mathbf{U} + \mathbf{C}'\mathbf{V})}{\partial v_j} = C_1 \frac{\partial h_1(\mathbf{S}'(\mathbf{U} + \mathbf{WV}))}{\partial(\mathbf{S}'(\mathbf{U} + \mathbf{WV})}$$

$$(W_{1j}s_1 + \mathbf{W}_{2j}s_2) + C_2 \frac{h_2(\mathbf{T}'(\mathbf{U} + \mathbf{WV}))}{\partial(\mathbf{T}'(\mathbf{U} + \mathbf{WV}))} (W_{1j}t_1 + W_{2j}t_2); \qquad j = 1, 2. \tag{29}$$

Due to continuity of the partial derivatives,

$$\frac{1}{f(\mathbf{A}'\mathbf{U})}\left[\frac{\partial f(\mathbf{A}'\mathbf{U}+\mathbf{C}'\mathbf{V})}{\partial v_j}\right]_{V=0}$$

$$= C_1 \frac{\partial h_1(\mathbf{S}'\mathbf{U})}{\partial(\mathbf{S}'\mathbf{U})}(W_{1j}s_1 + W_{2j}s_2) + C_2 \frac{\partial h_2(\mathbf{T}'\mathbf{U})}{\partial(\mathbf{T}'\mathbf{U})}(W_{1j}t_1 + W_{2j}t_2); \tag{30}$$

$j = 1, 2.$

Similarly,

$$\frac{1}{g(\mathbf{B}'\mathbf{U})}\left[\frac{\partial g(\mathbf{B}'\mathbf{U}+\mathbf{D}'\mathbf{V})}{\partial v_j}\right]_{V=0}$$

$$= D_1 \frac{\partial h_1(\mathbf{S}'\mathbf{U})}{\partial(\mathbf{S}'\mathbf{U})}(Z_{1j}s_1 + Z_{2j}s_2) + D_2 \frac{\partial h_2(\mathbf{T}'\mathbf{U})}{\partial(\mathbf{T}'\mathbf{U})}(Z_{1j}t_1 + Z_{2j}t_2); \tag{31}$$

$j = 1, 2.$

From (27), (30), (31) we obtain

$$\begin{aligned} e_{1j} = f(\mathbf{A}'\mathbf{U})g(\mathbf{B}'\mathbf{U}) \Big\{ &[C_1(W_{1j}s_1 + W_{2j}s_2) + D_1(Z_{1j}s_1 + Z_{2j}s_2)] \\ &\frac{\partial h_1(\mathbf{S}'\mathbf{U})}{\partial(\mathbf{S}'\mathbf{U})} + [C_2(W_{1j}t_1 + W_{2j}t_2) + D_2(Z_{1j}t_1 + Z_{2j}t_2)] \\ &\frac{\partial h_2(\mathbf{T}'\mathbf{U})}{\partial(\mathbf{T}'\mathbf{U})} \Big\}; \qquad j = 1, 2. \end{aligned} \tag{32}$$

Let

$$\mathbf{E}_1' = (e_{11}, e_{12}).$$

Then, from (32), we obtain

$$\mathbf{E}_1 = f(\mathbf{A}'\mathbf{U})g(\mathbf{B}'\mathbf{U})(C_1\mathbf{W}'\mathbf{S} + D_1\mathbf{Z}'\mathbf{S} \quad C_2\mathbf{W}'\mathbf{T} + D_2\mathbf{Z}'\mathbf{T}) \begin{bmatrix} \dfrac{\partial h_1(\mathbf{S}'\mathbf{U})}{\partial(\mathbf{S}'\mathbf{U})} \\ \dfrac{\partial h_2(\mathbf{T}'\mathbf{U})}{\partial(\mathbf{T}'\mathbf{U})} \end{bmatrix} \tag{33}$$

$$= f(\mathbf{A}'\mathbf{U})g(\mathbf{B}'\mathbf{U})\mathbf{P} \begin{bmatrix} \dfrac{\partial h_1(\mathbf{S}'\mathbf{U})}{\partial(\mathbf{S}'\mathbf{U})} \\ \dfrac{\partial h_2(\mathbf{T}'\mathbf{U})}{\partial(\mathbf{T}'\mathbf{U})'} \end{bmatrix},$$

since

$$\begin{aligned} &(C_1\mathbf{W}'\mathbf{S} + D_1\mathbf{Z}'\mathbf{S} \quad C_2\mathbf{W}'\mathbf{T} + D_2\mathbf{Z}'\mathbf{T}) \\ &= (C_1\mathbf{C}\mathbf{A}^{-1}\mathbf{S} + D_1\mathbf{D}\mathbf{B}^{-1}\mathbf{S} \quad C_2\mathbf{C}\mathbf{A}^{-1}\mathbf{T} + D_2\mathbf{D}\mathbf{B}^{-1}\mathbf{T}) \\ &= (C_1\mathbf{C}\mathbf{c} + D_1\mathbf{D}\mathbf{f} \quad C_2\mathbf{C}\mathbf{d} + D_2\mathbf{D}\mathbf{g}) \\ &= \mathbf{P}. \end{aligned} \tag{34}$$

From (26)

$$e_{2j} = \frac{\partial \Phi(\mathbf{U}, \mathbf{0})}{\partial u_j} = g(\mathbf{B}'\mathbf{U}) \frac{\partial f(\mathbf{A}'\mathbf{U})}{\partial u_j} + f(\mathbf{A}'\mathbf{U}) \frac{\partial g(\mathbf{B}'\mathbf{U})}{\partial u_j}. \tag{35}$$

From (23) and (24)

$$\frac{1}{f(\mathbf{A}'\mathbf{U})} \frac{\partial f(\mathbf{A}'\mathbf{U})}{\partial u_j} = C_1 \frac{\partial h_1(\mathbf{S}'\mathbf{U})}{\partial(\mathbf{S}'\mathbf{U})} s_j + C_2 \frac{\partial h_2(\mathbf{T}'\mathbf{U})}{\partial(\mathbf{T}'\mathbf{U})} t_j; \tag{36}$$

$$\frac{1}{g(\mathbf{B}'\mathbf{U})} \frac{\partial g(\mathbf{B}'\mathbf{U})}{\partial u_j} = D_1 \frac{\partial h_1(\mathbf{S}'\mathbf{U})}{\partial(\mathbf{S}'\mathbf{U})} s_j + D_2 \frac{\partial h_2(\mathbf{T}'\mathbf{U})}{\partial(\mathbf{T}'\mathbf{U})} t_j; \qquad j = 1, 2.$$

From (35) and (36)

$$e_{2j} = f(\mathbf{A}'\mathbf{U}) g(\mathbf{B}'\mathbf{U}) \left\{ [C_1 s_j + D_1 s_j] \frac{\partial h_1(\mathbf{S}'\mathbf{U})}{\partial(\mathbf{S}'\mathbf{U})} + [C_2 t_j + D_2 t_j] \frac{\partial h_2(\mathbf{T}'\mathbf{U})}{\partial(\mathbf{T}'\mathbf{U})} \right\}; \qquad j = 1, 2.$$

Denoting by $\mathbf{E}_2' = (e_{21}, e_{22})$, we observe that

$$\mathbf{E}_2 = f(\mathbf{A}'\mathbf{U}) g(\mathbf{B}'\mathbf{U}) \mathbf{Q} \begin{bmatrix} \dfrac{\partial h_1(\mathbf{S}'\mathbf{U})}{\partial(\mathbf{S}'\mathbf{U})} \\ \dfrac{\partial h_2(\mathbf{T}'\mathbf{U})}{\partial(\mathbf{T}'\mathbf{U})} \end{bmatrix} \tag{37}$$

since $((C_1 + D_1)\mathbf{S} \quad (C_2 + D_2)\mathbf{T}) = \mathbf{A}((C_1 + D_1)\mathbf{c} \quad (C_2 + D_2)\mathbf{d}) = \mathbf{Q}$.

It may be observed that the condition $c_1/c_2 \neq d_1/d_2$ implies that the matrix $\mathbf{Q}$ is nonsingular. Since the representation (23) is valid for all real u_1 and u_2 neither $f(\mathbf{A}'\mathbf{U})$ nor $g(\mathbf{B}'\mathbf{U})$ has any real zero. Thus from (33) and (37), we obtain $\mathbf{E}_1 = \beta \mathbf{E}_2$. The proof of the theorem now follows from Lemma 1.

Theorem 2 thus introduces a class of bivariate distributions for which $E(\mathbf{Y}^* \mid \mathbf{X}^*) = \beta \mathbf{X}^*$ where $\mathbf{X}^*$ and $\mathbf{Y}^*$ have the stochastic structure given by (22). It may be pointed out that the theorem requires some conditions on the matrices $\mathbf{A}$ and $\mathbf{B}$ while $\mathbf{C}$ and $\mathbf{D}$ can be arbitrary. The characteristic functions $f(\mathbf{U})$ and $g(\mathbf{U})$ depend upon u_1 and u_2 only through the functions $h_1(\cdot)$ and $h_2(\cdot)$. In Section 3, we consider some specific forms of these functions.

We next try to prove the converse of the theorem. Suppose,

$$\mathbf{X}^* = \mathbf{AX} + \mathbf{BY}; \quad \mathbf{Y}^* = \mathbf{CX} + \mathbf{DY}; \quad E(\mathbf{X}) = E(\mathbf{Y}) = \mathbf{0}; \quad E(\mathbf{Y}^* \mid \mathbf{X}^*) = \beta \mathbf{X}^*; \tag{38}$$

where $\mathbf{X}$, $\mathbf{Y}$, ($\mathbf{X}$, $\mathbf{Y}$ independent) $\mathbf{X}^*$, $\mathbf{Y}^*$ are 2×1 random vectors; $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, are 2×2 matrices of real quantities. Let the matrices $\mathbf{A}$, $\mathbf{B}$, $(\mathbf{C} - \beta\mathbf{A})$, be nonsingular. This automatically excludes the possibility that $\mathbf{C} = \mathbf{DB}^{-1}\mathbf{A}$. Let us also introduce the matrix

$$\mathbf{G} = (g_{ij}) = -\mathbf{A}(\mathbf{C} - \beta\mathbf{A})^{-1}(\mathbf{D} - \beta\mathbf{B})\mathbf{B}^{-1}, \tag{39}$$
$$= -\mathbf{A}(\mathbf{C} - \beta\mathbf{A})^{-1}(\mathbf{DB}^{-1} - \beta).$$

THEOREM 3. *Let* $f(\mathbf{U})$ *and* $g(\mathbf{U})$ *be the characteristic functions of* **X** *and* **Y** *respectively. Suppose,*

(i) *For given* **A**, **B**, **C**, **D** *the relationships given in* (38) *and the assumptions following it are true.*

(ii) $f(\mathbf{A}'\mathbf{U})$ *and* $g(\mathbf{B}'\mathbf{U})$ *do not have any real zero.*

(40) (iii)
$$g_{21} \neq 0; \qquad g_{11} \neq g_{22}.$$

(iv) *Either* g_{12} *and* g_{21} *are of same sign or*

$$|g_{22} - g_{11}| > 2(-g_{12}g_{21})^{\frac{1}{2}}. \tag{41}$$

Then, $f(\mathbf{U})$ *and* $g(\mathbf{U})$ *must be of the form given by* (23) *and the conditions given in* (24) *must also be satisfied.*

PROOF. Suppose, $\Phi(\mathbf{U}, \mathbf{V})$ is the characteristic function of the joint distribution of $\mathbf{X}^*$ and $\mathbf{Y}^*$. Then,

$$\Phi(\mathbf{U}, \mathbf{V}) = f(\mathbf{A}'\mathbf{U}+\mathbf{C}'\mathbf{V})g(\mathbf{B}'\mathbf{U}+\mathbf{D}'\mathbf{V}). \tag{42}$$

Let

$$e_{1j} = \left[\frac{\partial\Phi(\mathbf{U}, \mathbf{V})}{\partial v_j}\right]_{V=0}; \qquad j = 1, 2.$$

$$e_{2j} = \frac{\partial\Phi(\mathbf{U}, \mathbf{0})}{\partial u_j}; \qquad j = 1, 2.$$

$$\mathbf{E}_1{}' = (e_{11}, e_{12}); \ \mathbf{E}_2{}' = (e_{21}, e_{22}).$$

Then by Lemma 1,

$$\mathbf{E}_1 = \beta\mathbf{E}_2. \tag{43}$$

Simple calculation leads to:

$$\left[\frac{f(\mathbf{A}'\mathbf{U}+\mathbf{C}'\mathbf{V})}{\partial v_j}\right]_{V=0} = i(c_{j1}m_1+c_{j2}m_2); \ \frac{\partial f(\mathbf{A}'\mathbf{U})}{\partial u_j} = i(a_{j1}m_1+a_{j2}m_2);$$

$$\left[\frac{\partial g(\mathbf{B}'\mathbf{U}+\mathbf{D}'\mathbf{V})}{\partial v_j}\right]_{V=0} = i(d_{j1}n_1+d_{j2}n_2); \ \frac{\partial g(\mathbf{B}'\mathbf{U})}{\partial u_j} = i(b_{j1}n_1+b_{j2}n_2); \tag{44}$$

where,

$$m_j = \int\int x_j \exp(i\mathbf{U}'\mathbf{A}\mathbf{X})dF_1(\mathbf{X});$$

$$n_j = \int\int y_j \exp(i\mathbf{U}'\mathbf{B}\mathbf{Y})dF_2(\mathbf{Y}); \qquad j = 1, 2. \tag{45}$$

$$\mathbf{M}' = (m_1, m_2); \ \mathbf{N}' = (n_1, n_2);$$

$F_1(\mathbf{X})$, $F_2(\mathbf{Y})$ being the marginal distribution functions of **X** and **Y** respectively. After some trivial calculations, we obtain from (42)–(45)

$$g(\mathbf{B}'\mathbf{U})(\mathbf{C}-\beta\mathbf{A})\mathbf{M}+f(\mathbf{A}'\mathbf{U})(\mathbf{D}-\beta\mathbf{B})\mathbf{N} = \mathbf{0}. \tag{46}$$

Since, $f(\mathbf{A}'\mathbf{U})g(\mathbf{B}'\mathbf{U}) \neq 0$ for all real u_1, u_2

$$(47) \qquad \frac{\mathbf{M}}{f(\mathbf{A}'\mathbf{U})} = -(\mathbf{C}-\beta\mathbf{A})^{-1}(\mathbf{D}-\beta\mathbf{B}) \frac{\mathbf{N}}{g(\mathbf{B}'\mathbf{U})}.$$

Let $f(\mathbf{A}'\mathbf{U}) = f^*(\mathbf{U})$; $g(\mathbf{B}'\mathbf{U}) = g^*(\mathbf{U})$.

From (44) we obtain

$$(48) \qquad \begin{bmatrix} \dfrac{\partial f^*(\mathbf{U})}{\partial u_1} \\ \dfrac{\partial f^*(\mathbf{U})}{\partial u_2} \end{bmatrix} = i\mathbf{AM}; \qquad \begin{bmatrix} \dfrac{\partial g^*(\mathbf{U})}{\partial u_1} \\ \dfrac{\partial g^*(\mathbf{U})}{\partial u_2} \end{bmatrix} = i\mathbf{BN}.$$

From (39), (47) and (48), we obtain

$$\frac{1}{f^*(\mathbf{U})} \begin{bmatrix} \dfrac{\partial f^*(\mathbf{U})}{\partial u_1} \\ \dfrac{\partial f^*(\mathbf{U})}{\partial u_2} \end{bmatrix} = \frac{1}{g^*(\mathbf{U})} \mathbf{G} \begin{bmatrix} \dfrac{\partial g^*(\mathbf{U})}{\partial u_1} \\ \dfrac{\partial g^*(\mathbf{U})}{\partial u_2} \end{bmatrix}$$

which can be rewritten as:

$$(49) \qquad \begin{aligned} \frac{\partial \log f^*(\mathbf{U})}{\partial u_1} &= g_{11} \frac{\partial \log g^*(\mathbf{U})}{\partial u_1} + g_{12} \frac{\partial \log g^*(\mathbf{U})}{\partial u_2} \\ \frac{\partial \log f^*(\mathbf{U})}{\partial u_2} &= g_{21} \frac{\partial \log g^*(\mathbf{U})}{\partial u_1} + g_{22} \frac{\partial \log g^*(\mathbf{U})}{\partial u_2}. \end{aligned}$$

We solve the system of partial differential equations given by (49) by using the techniques discussed in Petrovsky ([5] page 53). Let

$$(50) \qquad \begin{aligned} Z_1(\mathbf{U}) &= \log f^*(\mathbf{U}) - g_{11} \log g^*(\mathbf{U}), \\ Z_2(\mathbf{U}) &= \log f^*(\mathbf{U}) - g_{22} \log g^*(\mathbf{U}). \end{aligned}$$

Then, the system (49) can be rewritten as:

$$(51) \qquad \begin{aligned} \frac{\partial Z_1(\mathbf{U})}{\partial u_1} &= \frac{g_{12}}{g_{22}-g_{11}} \left[\frac{\partial Z_1(\mathbf{U})}{\partial u_2} - \frac{\partial Z_2(\mathbf{U})}{\partial u_2} \right]; \\ \frac{\partial Z_2(\mathbf{U})}{\partial u_1} &= \frac{g_{12}}{g_{22}-g_{11}} \frac{\partial Z_1(\mathbf{U})}{\partial u_2} + \left[\frac{g_{11}-g_{22}}{g_{21}} - \frac{g_{12}}{g_{22}-g_{11}} \right] \frac{\partial Z_2(\mathbf{U})}{\partial u_2}. \end{aligned}$$

Condition (40) ensures that the co-efficients of $\partial Z_1/\partial u_2$, $\partial Z_2/\partial u_2$ in the right-hand side of (51) are finite.

Let

$$k_{11} = \frac{g_{12}}{g_{22}-g_{11}}; \qquad k_{12} = \frac{-g_{12}}{g_{22}-g_{11}};$$

$$k_{21} = \frac{g_{12}}{g_{22}-g_{11}}; \qquad k_{22} = \frac{g_{11}-g_{22}}{g_{21}} - \frac{g_{12}}{g_{22}-g_{11}}.$$

Thus, (51) can be rewritten as:

(52)
$$\frac{\partial Z_1}{\partial u_1} = k_{11}\frac{\partial Z_1}{\partial u_2} + k_{12}\frac{\partial Z_2}{\partial u_2}; \qquad \mathbf{K} = (k_{ij});$$
$$\frac{\partial Z_2}{\partial u_1} = k_{21}\frac{\partial Z_1}{\partial u_2} + k_{22}\frac{\partial Z_2}{\partial u_2}.$$

Define k_1, k_2 so that

$$\textstyle\sum_{i=1}^{2}\sum_{j=1}^{2} k_{ij}k_iZ_j = \lambda \sum_{i=1}^{2} k_iZ_i.$$

Equating the co-efficients of Z_j from both sides, we obtain

(53)
$$\textstyle\sum_{i=1}^{2} k_{ij}k_i = \lambda k_j; \qquad j = 1, 2.$$

For non-trivial solutions of k_1 and k_2 in (53) we must have

$$|\mathbf{K}' - \lambda \mathbf{I}| = 0.$$

Thus, λ must be a characteristic root of $\mathbf{K}'$. It follows from condition (41) that the characteristic roots λ_1, λ_2, of $\mathbf{K}'$ are real and unequal. Suppose k_1^* and k_2^* are the values of k_1 and k_2 when $\lambda = \lambda_1$ and k_1^{**}, k_2^{**} are values of k_1, k_2 when $\lambda = \lambda_2$. Suppose

$$Z_1^* = k_1^*Z_1 + k_2^*Z_2; \; Z_2^* = k_1^{**}Z_1 + k_2^{**}Z_2.$$

Then

(54)
$$\begin{aligned}\frac{\partial Z_1^*}{\partial u_1} &= k_1^*\frac{\partial Z_1}{\partial u_1} + k_2^*\frac{\partial Z_2}{\partial u_1}\\ &= k_1^*\left(k_{11}\frac{\partial Z_1}{\partial u_2} + k_{12}\frac{\partial Z_2}{\partial u_2}\right)\\ &\quad + k_2^*\left(k_{21}\frac{\partial Z_1}{\partial u_2} + k_{22}\frac{\partial Z_2}{\partial u_2}\right)\\ &= \textstyle\sum_{i=1}^{2}\sum_{j=1}^{2} k_i^*k_{ij}\frac{\partial Z_j}{\partial u_2}\\ &= \lambda_1 \frac{\partial}{\partial u_2}(k_1^*Z_1 + k_2^*Z_2)\\ &= \lambda_1 \frac{\partial Z_1^*}{\partial u_2}.\end{aligned}$$

Similarly,

(55)
$$\frac{\partial Z_2^*}{\partial u_1} = \lambda_2 \frac{\partial Z_2^*}{\partial u_2}.$$

The general solutions of (54) and (55) are given by

$$Z_1^* = h_1(\lambda_1u_1 + u_2); \qquad Z_2^* = h_2(\lambda_2u_1 + u_2);$$

where $h_1(\cdot)$ and $h_2(\cdot)$ are differentiable functions of u_1 and u_2.

Now

$$Z_j^* = L_{j1} \log f^*(\mathbf{U}) + L_{j2}^* \log g^*(\mathbf{U}); \qquad j = 1, 2.$$

where

$$L_{11} = k_1^* + k_2^*; \qquad L_{12} = -k_1^* g_{11} - k_2^* g_{22};$$

$$L_{21} = k_1^{**} + k_2^{**}; \qquad L_{22} = -k_1^{**} g_{11} - k_2^{**} g_{22}.$$

Let $\mathbf{L} = (L_{ij})$. Since $\mathbf{K}' - \lambda_j \mathbf{I}$ is singular for $j = 1, 2$, it is possible to choose k_j^*, $k_j^{**}(j = 1, 2)$ in such a way that the matrix $\mathbf{L}$ is nonsingular. Hence,

(56) $$\begin{pmatrix} \log f^*(\mathbf{U}) \\ \log g^*(\mathbf{U}) \end{pmatrix} = \mathbf{L}^{-1} \begin{pmatrix} h_1(\lambda_1 u_1 + u_2) \\ h_2(\lambda_2 u_1 + u_2) \end{pmatrix}.$$

Taking

$$\mathbf{L}^{-1} = \begin{pmatrix} C_1 & C_2 \\ D_1 & D_2 \end{pmatrix}$$

we obtain

(57) $$\log f^*(\mathbf{U}) = C_1 h_1(\lambda_1 u_1 + u_2) + C_2 h_2(\lambda_2 u_1 + u_2);$$

(58) $$\log g^*(\mathbf{U}) = D_1 h_1(\lambda_1 u_1 + u_2) + D_2 h_2(\lambda_2 u_1 + u_2).$$

Replacing $\mathbf{AU}$ by $\mathbf{U}$ in (57) and $\mathbf{BU}$ by $\mathbf{U}$ in (58) and defining

$$\mathbf{c}' = (\lambda_1, 1)(\mathbf{A}^{-1})'; \qquad \mathbf{d}' = (\lambda_2, 1)(\mathbf{A}^{-1})'; \qquad \mathbf{f}' = (\lambda_1, 1)(\mathbf{B}^{-1})';$$
$$\mathbf{g}' = (\lambda_2, 1)(\mathbf{B}^{-1})',$$

it is easy to see that $f(\mathbf{U})$ and $g(\mathbf{U})$ have the form given in (23). Further, the conditions given in (24) must also be satisfied. Since $\lambda_1 \neq \lambda_2$, $c_1/c_2 \neq d_1/d_2$; $f_1/f_2 \neq g_1/g_2$, this completes the proof of the theorem.

3. A class of bivariate stable distributions. According to Lévy ([3] Section 63)] the distribution function $F(\mathbf{X})$ of a 2×1 random vector $\mathbf{X}$ is bivariate stable if for all choices of 2×1 vectors $\mathbf{B}_1$, $\mathbf{B}_2$ and positive quantities a_1, a_2 there exist a 2×1 vector $\mathbf{B}$ and a positive quantity "a" such that

(59) $$F(a_1 \mathbf{X} + \mathbf{B}_1) * F(a_2 \mathbf{X} + \mathbf{B}_2) = F(a\mathbf{X} + \mathbf{B});$$

where "*" stands for convolution. Unlike the univariate case, it is easy to see that the class of bivariate distributions given by (23) does not, in general, belong to the class of bivariate stable laws. The natural question which now arises is, "Under what conditions is the family given by (23) stable?" The following theorem provides a partial answer to this question.

THEOREM 4. *Consider the random vector* $\mathbf{X}' = (x_1, x_2)$ *with* $E(\mathbf{X}) = \mathbf{0}$. *Suppose both* x_1 *and* x_2 *are stable and symmetric with the same characteristic exponent equal to* $\alpha > 1$. *Then, if the characteristic function of* $\mathbf{X}$ *is of the form given by* (23) *with* $c_1, d_1, c_2, d_2, C_1, C_2 \neq 0$ *and* $c_1/c_2 \neq d_1/d_2$ *it must be a bivariate stable distribution.*

PROOF. Let $f_1(u_1) = f(u_1, 0); f_2(u_2) = f(0, u_2)$ be characteristic functions of X_1 and X_2 respectively. Then, from (23)

$$\text{(60)} \qquad \log f_1(u_1) = C_1 h_1(c_1 u_1) + C_2 h_2(d_1 u_1);$$
$$\log f_2(u_2) = C_1 h_1(c_2 u_2) + C_2 h_2(d_2 u_2).$$

Eliminating $h_2(\cdot)$ from (60) we obtain

$$h_1(d_1 d_2^{-1} c_2 u_2) - h_1(c_1 u_2) = C_1^{-1} \log f_2(d_1 d_2^{-1} u_2) - C_1^{-1} \log f_1(u_2).$$

Writing $c_1 u_2 = u$; $d_1 d_2^{-1} c_2 u_2 = \theta u$; $d_1 d_2^{-1} u_2 = \theta^* u$; $u_2 = \theta^{**} u$, we obtain

$$\text{(61)} \qquad h_1(\theta u) - h_1(u) = C_1^{-1} \log f_2(\theta^* u) - C_1^{-1} \log f_1(\theta^{**} u).$$

Since, $c_1/c_2 \neq d_1/d_2$, it follows that $\theta \neq 1$.

Similarly,

$$\text{(62)} \qquad h_2(\gamma u) - h_2(u) = C_2^{-1} \log f_2(\gamma^* u) - C_2^{-1} \log f_1(\gamma^{**} u).$$

where

$$d_1 u_2 = u; \qquad c_1 c_2^{-1} d_2 u_2 = \gamma u; \qquad u_2 = \gamma^{**} u; \qquad c_1 c_2^{-1} u_2 = \gamma^* u; \qquad \gamma \neq 1.$$

Since, x_1 and x_2 are stable and symmetric with the same characteristic exponent $\alpha > 1$ and $E(x_1) = E(x_2) = 0$, it follows that

$$\text{(63)} \quad \log f_1(u) = -\gamma_1 |u|^\alpha; \qquad \log f_2(u) = -\gamma_2 |u|^\alpha; \qquad \gamma_1, \gamma_2 > 0; \qquad \alpha > 1.$$

From (61), (62) and (63) we observe that $h_1(\theta u) - h_1(u) = \gamma_3 |u|^\alpha$;

$$\text{(64)} \qquad h_2(\gamma u) - h_2(u) = \gamma_4 |u|^\alpha;$$

where γ_3 and γ_4 are functions of $\gamma_1, \gamma_2, C_1, C_2, \theta^*, \theta^{**}, \gamma^*, \gamma^{**}$ and α. We now solve the equations given in (64) using a technique given in Boole ([1] page 303).

First, we consider $h_1(\cdot)$. For a given "u", let us introduce a variable "t" and a function "U_t" such that for some integral value of "t", $U_t = u$; $U_{t+1} = \theta u$. Then, U_t must satisfy the difference equation:

$$U_{t+1} - \theta U_t = 0.$$

Hence, the general form of U_t is $U_t = A\theta^t$, where A is some constant. Let $h_1(u) = h_1(A\theta^t) = V_t$. Then, for integral values of t

$$V_{t+1} - V_t = \gamma_3 |u|^\alpha = \gamma_5 B^t$$

where $B = |\theta|^\alpha$; $\gamma_5 = \gamma_3 |A|^\alpha$. The general form V_t is given by

$$\text{(65)} \qquad V_t = h_1(A\theta^t) = h_1(u) = (B-1)^{-1} \gamma_5 B^t.$$

The relation $|u| = |A|\,|\theta|^t$ leads to:

$$\text{(66)} \qquad t = [\log|\theta|]^{-1} [\log|u| - \log|A|].$$

Introducing (66) in (65) we obtain

$$(67)\qquad h_1(u) = \frac{\gamma_5}{B-1} \cdot B^{(\log|u|-\log|A|)/(\log|\theta|)} = \gamma_6 |u|^{\alpha};$$

where

$$\gamma_6 = B^{-((\log|A|)/(\log|\theta|))} \frac{\gamma_5}{B-1}$$

Similarly, $h_2(u)$ must be of the form

$$(68)\qquad h_2(u) = \gamma_7 |u|^{\alpha}.$$

In troducing (67) and (68) in (23), we observe that $f(\mathbf{U})$ must be of the form:

$$(69)\qquad \log f(\mathbf{U}) = -\alpha_1 |u_1+\delta_1 u_2|^{\alpha} - \alpha_2 |u_1+\delta_2 u_2|^{\alpha};\ \max(\alpha_1, \alpha_2) > 0; \quad \min(\alpha_1, \alpha_2) \geqq 0; \quad 1 < \alpha \leqq 2.$$

The characteristic function $f(\mathbf{U})$ given by equation (69) clearly represents the characteristic function of a bivariate stable distribution as defined by equation (59). This completes the proof of the theorem.

It may be observed from (69) that for $\alpha = 2$, $f(\mathbf{U})$ reduces to the characteristic function of a bivariate normal distribution with means zero and dispersion matrix

$$2\begin{pmatrix} \alpha_1+\alpha_2 \alpha_1\delta_1+\alpha_2\delta_2 \\ \alpha_1\delta_1+\alpha_2\delta_2 \alpha_1\delta_1^2+\alpha_2\delta_2^2 \end{pmatrix}.$$

Finally, we observe that Theorem 3 and Theorem 4 lead to the following interesting corollary:

COROLLARY. *Suppose the conditions stated in Theorem* 3 *are true. Then, if the marginal distributions of either* $\mathbf{X}' = (x_1, x_2)$ *or* $\mathbf{Y}' = (y_1, y_2)$ *are stable, symmetric with the same characteristic exponent* $\alpha > 1$, *the characteristic functions of both* $\mathbf{X}$ and $\mathbf{Y}$ *have the form given by* (69).

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