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ON THE ASYMPTOTIC OPTIMALITY OF SPECTRAL ANALYSIS FOR TESTING HYPOTHESES ABOUT TIME SERIES

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0. Summary. Classification of a sample from a zero mean, stationary, Gaussian time series into populations distinguished by characteristics of the spectrum can be done with a decision theoretic procedure or spectral analysis. Decision theory requires that each population be characterized by a probability distribution on the space of spectral density functions. In this paper, we relate the two methods by showing that under many conditions, as the sample length increases, the expected cost of the Bayes test formed from spectral estimates by approximating their sampling distribution by a product of chi-squared distributions approaches the expected cost of the Bayes test formed from the original data. The amount of smoothing that can be used in the spectral estimates depends on the prior knowledge of the order of differentiability of the spectrum. This result is related to but weaker than the notion that spectral estimates are asymptotically sufficient statistics for the second order properties of a stationary Gaussian time series.

1. Introduction. First, we define measurable sets of spectral density functions. Let $f = \{f(\omega), \ |\omega| \leqq \pi\}$ denote a spectral density function, and $R(\tau)$ $(= \int_{-\pi}^{\pi} e^{i\omega\tau} f(\omega)\, d\omega)$, the corresponding covariance function. Let H, η, K_1, and α_1 be finite positive constants that satisfy $H > \eta$, $\alpha_1 \leqq 1$, and let p be a positive integer. The space F_p is the set of f such that $\sup_\omega |f(\omega)| \leqq H, \inf_\omega |f(\omega)| \geqq \eta$, and

$$\sum_{|\tau| \geqq N} |\tau|^p |R(\tau)| \leqq K_1 N^{-\alpha_1}. \tag{1.1}$$

To F_p, we assign the norm $\|f\| = \sup_\omega |f(\omega)|$, and so F_p is a subspace of the Banach space $C[-\pi, \pi]$. The σ-field of F_p is the smallest σ-algebra of subsets of F_p that contains all the open sets of F_p[7]. Since the vector $\{R(\tau), 0 \leqq \tau \leqq T < \infty\}$ is a continuous function of f, it is measurable [7].

We consider multiple alternative classification problems for real, zero mean, stationary, Gaussian time series $\{X(t), t = \cdots, -1, 0, 1, \cdots\}$. The data is a finite sample of the time series, $X' = (X(1), X(2), \cdots, X(T))$. The probability densities that distinguish the $L+1$ alternative hypotheses are mixtures of zero mean Gaussian densities. The $L+1$ mixing distributions $\mu_l (l = 0, 1, \cdots, L)$ are defined on the space of spectral density functions F_p. The cost of choosing hypothesis k when l is true $C(k \mid l)$ and the prior probabilities of the various hypotheses q_l are given. If the covariance matrix $(R(j-k))$ is denoted by Σ and the probability density for a real, zero mean, Gaussian random vector is denoted by $n_r(X \mid 0, \Sigma)$, the posterior probabilities of the hypotheses are given by

$$P(k \mid X) = q_k \int n_r(X \mid 0, \Sigma)\, d\mu_k / \sum_{l=0}^{L} q_l \int n_r(X \mid 0, \Sigma)\, d\mu_l \tag{1.2}$$

from which the Bayes test follows [1].

For the problems described above, the spectral analysis approach would form estimates of the spectrum at several frequencies, $\{\hat{f}(\omega_m), m = 0, 1, 2, \cdots, M-1\}$, and then use these to form test statistics [3], [10]. If these spectral estimates are suitably formed, then the sampling distribution of $(n_0\hat{f}(\omega_0), n_1\hat{f}(\omega_1), \cdots, n_{M-1}\cdot\hat{f}(\omega_{M-1}))$ is given approximately by the density $\prod_{m=0}^{M-1} w_c(n_m\hat{f}(\omega_m) \,|\, f(\omega_m), 1, n_m)$, where w_c is the complex Wishart density, f is the true spectrum, and n_m depends on the way the spectral estimates are computed. Since

$$w_c(\hat{y} \,|\, y, 1, n) = \hat{y}^{n-1}y^{-n}e^{-\hat{y}/y}/\Gamma(n), \tag{1.3}$$

the distribution of $2n_m\hat{f}(\omega_m)/f(\omega_m)$ is approximately χ^2 with $2n_m$ degrees of freedom. This conclusion was stated by Goodman [3] who studied the complex Gaussian and complex Wishart distributions. Wahba [10] showed that a wide class of test statistics approach in mean square test statistics that are formed from spectral estimates that actually have a complex Wishart distribution (apart from the constant factor n_m).

If we take the data to be spectral estimates formed from X, then using the approximate sampling distribution we could take the posterior probabilities of the hypotheses to be

$$P(k \,|\, \hat{f}) = \frac{q_k \int \prod_{m=0}^{M-1} w_c(n_m\hat{f}(\omega_m) \,|\, f(\omega_m), 1, n_m)\, d\mu_k}{\sum_{l=0}^{L} q_l \int \prod_{m=0}^{M-1} w_c(n_m\hat{f}(\omega_m) \,|\, f(\omega_m), 1, n_m)\, d\mu_l} \tag{1.4}$$

and from these form a Bayes test. Our objective is to show that for large T this test is nearly the same as the one based on $P(k \mid X)$ given in (1.2).

Consider estimates of the algebraic type [8]. Let

$$\hat{f}(\omega) = (2\pi)^{-1} \sum_{|\tau| < M_T} e^{-i\omega\tau} h(\tau/M_T) R_T(\tau), \tag{1.5}$$

where $h(u)$ is bounded, even, defined for all real u, zero for $|u| \geqq 1$, and satisfies

$$|1 - h(u)|/|u|^p < K_h, \tag{1.6}$$

where M_T is the greatest integer less than or equal to T^δ, $1 \geqq \delta > p^{-1}(1 - \frac{1}{4}\alpha_1/(p+1+\alpha_1))$, and where

$$\begin{aligned} R_T(\tau) &= T^{-1} \textstyle\sum_{t=1}^{T-|\tau|} X(t)X(t+|\tau|) && \text{for } |\tau| < T \\ &= 0 && \text{for } |\tau| \geqq T. \end{aligned} \tag{1.7}$$

We take $n_m = \frac{1}{2}T/M_T$ and $\omega_m = \pi(m+\frac{1}{2})/M_T$. Let X be Gaussian with spectrum f_0 that belongs to F_p. Applying mild restrictions to μ_k, we show that as $T \to \infty$, $P(k \,|\, \hat{f}) - P(k \mid X) \to 0$ a.s. for $f_0 \in F_p$. This implies that the expected cost of the test based on $P(k \,|\, \hat{f})$ approaches the expected cost of the test based on $P(k \mid X)$. The smaller δ is, the larger the amount of smoothing used in the spectral estimates $\hat{f}$. The lower bound on δ decreases as p increases which in turn increases as the order of differentiability of the members of the space of spectra increases.

2. Asymptotic equivalence of the posterior probabilities for the periodogram case. In this section we prove the theorem that forms the basis of our results.

THEOREM 1. *As defined above, let F_1 be the measure space of spectral density functions, and let μ_l, $l = 0, 1, 2, \cdots, L$ be probability distributions on F_1. Let $\{X(t)\}$ be a real, zero mean, stationary, Gaussian time series with spectral density function f_0, covariance function $R_0(\tau)$, and covariance matric Σ_0, where $f_0 \in F_1$. Let $X' = (X(1), X(2), \cdots, X(T))$ be a sample from it. We assume that*

$$\lim_{T\to\infty} \exp\{\gamma T^{\beta}\} \textstyle\sum_{l=0}^{L} q_l \mu_l(E_T') = \infty, \tag{2.1}$$

where

$$E_T' = \{f \mid \|f - f_0\| \leqq K_n^{\frac{1}{2}} T^{-\frac{1}{2}}, f \in F_1\},$$

$K_n > 0, 0 < \gamma < \frac{1}{4} K_n H^{-2}$, *and* $\frac{1}{2} < \beta < 1$. *Let $\xi(\omega)$ be the periodogram,*

$$\xi(\omega) = (2\pi T)^{-1} |\textstyle\sum_{t=1}^{T} X(t) e^{-i\omega t}|^2, \tag{2.2}$$

and let $S(f, T) = \sum_{j=0}^{T-1} \log f(2\pi j/T) + \xi(2\pi j/T)/f(2\pi j/T)$ (*for* $3\pi > \omega > \pi$, $f(\omega) = f(\omega - 2\pi)$). *Then,*

$$P(k \mid X) - q_k \textstyle\int \exp\{-\frac{1}{2} S(f, T)\}\, d\mu_k / \sum_{l=0}^{L} q_l \int \exp\{-\frac{1}{2} S(f, T)\}\, d\mu_l \to 0 \text{ a.s.}$$

The event E_T' is the occurrence of f_0 or a spectrum near f_0. Condition (2.1) prevents the prior probability of E_T' from approaching zero too rapidly.

The proof of this theorem is closely related to proofs of the large sample equivalence of Bayes and maximum likelihood approaches [5], [6]. The numerator of the expression for $P(k \mid X)$ given in (1.2) can be written as

$$(2\pi)^{-T} \textstyle\int \exp\{-\frac{1}{2}[\log\det\Sigma + X'\Sigma^{-1}X - T\log 2\pi - S(f, T)]\} \cdot \exp\{-\frac{1}{2} S(f, T)\} d\mu_k. \tag{2.3}$$

In Lemmas 1 and 2, we prove that as $T \to \infty$ the first exponential in (2.3) remains smooth and bounded. In Lemma 3, we prove that the second exponential becomes peaked at $f = f_0$. Thus, the integral is dominated by the contribution from the neighborhood of f_0 and so we can evaluate the first exponential at f_0 and take it outside the integral sign. Theorem 1 follows from this.

LEMMA 1. *Let* $\frac{1}{2} < \beta' < \beta$, *and let*

$$E_T = \{f \mid \|f - f_0\| \leqq K_n^{\frac{1}{2}} T^{-\frac{1}{2}(1-\beta)}, f \in F_1\}.$$

Then,

$$T^{-\beta'}\{\log\det\Sigma - T\log 2\pi - \textstyle\sum_{j=0}^{T-1} \log f(2\pi j/T)\} \to 0 \tag{2.4}$$

uniformly for $f \in F_1$ and

$$\sup_{f\in E_T} |\log\det\Sigma - \textstyle\sum_{j=0}^{T-1} \log f(2\pi j/T) - \log\det\Sigma_0 + \sum_{j=0}^{T-1} \log f_0(2\pi j/T)| \to 0. \tag{2.5}$$

PROOF. By (1.1), we see that if $f \in F_p$, then $f^{(p)}$, the p *th* derivative of f, exists and $f^{(q)}(-\pi) = f^{(q)}(\pi)$ for $q \leqq p$. For $\alpha_2 < \alpha_1$, we have from (1.1)

$$\textstyle\sum_{n\geqq 1} n^{-1+\alpha_2} \sum_{|\tau|\geqq n} |\tau|^p |R(\tau)| = \sum_{|\tau|\geqq 1} (|\tau|^p \sum_{n=1}^{|\tau|} n^{-1+\alpha_2}) |R(\tau)| \leqq K_1 \sum_{n=1}^{\infty} n^{-1+\alpha_2-\alpha_1},$$

and so $\sum_{|\tau|\geqq 1}|\tau|^{p+\alpha_2}|R(\tau)|$ converges. This implies that for all $f\in F_p$, $f^{(p)}$ satisfies the Lipschitz condition

$$(2.6)\qquad |f^{(p)}(\omega_1)-f^{(p)}(\omega_2)|\leqq K_2|\omega_1-\omega_2|^{\alpha_2}$$

for some fixed constant K_2.

Lemma 1 follows almost immediately from a theorem due to Szegö [4] that concludes that

$$(2.7)\qquad \lim_{T\to\infty}\{\log\det\Sigma-T\log 2\pi-(2\pi)^{-1}T\textstyle\int_{-\pi}^{\pi}\log f(\omega)\,d\omega\} = (2\pi)^{-2}\sum_{n=1}^{\infty}n|r(n)|^2,$$

where

$$(2.8)\qquad r(n)=\textstyle\int_{-\pi}^{\pi}\log f(\omega)\,e^{i\omega n}\,d\omega.$$

The fact that the limit in (2.7) is uniform for $f\in F_1$ is not stated explicitly in [4]. However, it follows easily from the uniformity of the order relations used in the proof [2].

We will use repeatedly the fact that if $\{g(\omega),\ |\omega|\leqq\pi\}$ has a derivative of order p, $g^{(q)}(\pi)=g^{(q)}(-\pi)$ for $0\leqq q\leqq p$, and $g^{(p)}$ satisfies the Lipschitz condition (2.6), then [2]

$$(2.9)\qquad \left|\textstyle\int_{-\pi}^{\pi}e^{i\omega n}g(\omega)\,d\omega\right|\leqq K_2\pi^{1+\alpha_2}n^{-p-\alpha_2}.$$

Using (2.9), we see that

$$(2.10)\qquad \textstyle\sum_{j=0}^{T-1}\log f(2\pi j/T)=(2\pi)^{-1}T\sum_{k=-\infty}^{\infty}r(kT)\to(2\pi)^{-1}T\int_{-\pi}^{\pi}\log f(\omega)\,d\omega$$

uniformly for $f\in F_1$, from which (2.4) follows.

From (1.1), we see that $\sum|\tau|\,|R(\tau)|$ is uniformly convergent in F_1 so that $||f_j-f||\to 0$ and $f_j\in F_1$ implies that $f\in F_1$. Thus, F_1 is closed, and by the Ascoli lemma it is compact [7]. The supremum in (2.5) is attained for $f=\tilde f_T$, where $\tilde f_T\in E_T$. Since (2.7) and (2.10) hold uniformly in F_1, the proof of (2.5) is complete when we show that

$$(2.11)\qquad \textstyle\sum_{n=1}^{\infty}n|\tilde r(n)|^2\to\sum_{n=1}^{\infty}n|r_0(n)|^2,$$

where $\tilde r(n)$ and $r_0(n)$ are defined as in (2.8) for $\tilde f=f_T$ and $f=f_0$, respectively. (2.11) holds because by (2.9), $\sum_{n=1}^{\infty}n|r(n)|^2$ is uniformly convergent.

LEMMA 2.

$$(2.12)\qquad T^{-\beta'}\sup_{f\in F_1}\{X'\Sigma^{-1}X-\textstyle\sum_{j=0}^{T-1}\xi(2\pi j/T)/f(2\pi j/T)\}\to 0\text{ a.s.}$$

and

$$(2.13)\qquad \sup_{f\in E_T}\left|X'\Sigma^{-1}X-\textstyle\sum_{j=0}^{T-1}\xi(2\pi j/T)/f(2\pi j/T)-X'\Sigma_0^{-1}X+\sum_{j=0}^{T-1}\xi(2\pi j/T)/f_0(2\pi j/T)\right|\to 0\text{ a.s.}$$

PROOF. Consider (2.12) first. From (2.2), it follows that

$$X'\Sigma^{-1}X - \textstyle\sum_{j=0}^{T-1} \xi(2\pi j/T)/f(2\pi j/T) = X'(\Sigma^{-1} - \bar{\Sigma}^{-1})X, \tag{2.14}$$

where the (t, t') element of $\bar{\Sigma}$ is

$$\bar{R}(t-t') = \textstyle\sum_{k=-\infty}^{\infty} R(t-t'+kT).$$

Letting $\sigma^{tt'}$ and $\bar{\sigma}^{tt'}$ be the elements of Σ^{-1} and $\bar{\Sigma}^{-1}$, respectively, we have

$$\begin{aligned} T^{-\beta'}X'(\Sigma^{-1} - \bar{\Sigma}^{-1})X &\leqq [T^{-\beta'}\sup_t |X(t)|^2] \textstyle\sum_t \sum_{t'} |\sigma^{tt'} - \bar{\sigma}^{tt'}| \\ &\leqq [T^{-\beta'}\sup_t |X(t)|^2][\sup_m \textstyle\sum_t |\sigma^{tm}|][\sup_n \sum_{t'} |\bar{\sigma}^{nt'}|] \\ &\quad \cdot [\textstyle\sum_m \sum_n |\bar{R}(m-n) - R(m-n)|], \end{aligned} \tag{2.15}$$

where the domain of m, n, t, t' is $[1, T]$.

Since

$$\begin{aligned} \text{Prob}[T^{-\beta'}\sup_t |X(t)|^2 > \varepsilon] &= \text{Prob} \textstyle\bigcup_t [|X(t)|^2 > \varepsilon T^{\beta}] \\ &\leqq \text{Prob}[|X(t)|^2 > \varepsilon T^{\beta'}] \\ &= 2T(2\pi R_0(0))^{-\frac{1}{2}} \textstyle\int_{(\varepsilon T^{\beta'})^{\frac{1}{2}}}^{\infty} \exp\{-\frac{1}{2}x^2/R_0(0)\}\, dx, \end{aligned} \tag{2.16}$$

we have by the Borel–Cantelli lemma $T^{-\beta'}\sup_t |X(t)|^2 \to 0$ a.s.

In order to show that $\sup_m \sum_t |\sigma^{tm}|$ is bounded, we consider Σ^{-1} as a linear operator from the space $l^{\infty}(T)$ into itself. We note that

$$||\Sigma^{-1}||_T = \sup_m \textstyle\sum_t |\sigma^{tm}|$$

and that $||\Sigma^{-1}||_T$ is bounded if $||\Sigma||_T$ is bounded away from zero [9]. The set of real numbers $A = \{a \mid a = ||\Sigma||_T, T \in [1, \infty], f \in F_1\}$ does not contain zero because Σ is positive definite. If A is closed, then it is bounded away from zero and

$$\sup_{f \in F_1} \sup_{1 \leqq T \leqq \infty} \sup_m \textstyle\sum_t |\sigma^{mt}| < \infty. \tag{2.17}$$

Consider a convergent sequence $\{a_n, a_n \in A\}$, the nth member of which is $||\Sigma||_T$ evaluated at $f = f_n$ and $T = T_n$. In the sequence $\{(f_n, T_n), n = 1, 2, \cdots\}$, there is a subsequence $\{(f_{n_j}, T_{n_j}), j = 1, 2, \cdots\}$ that converges to (f_∞, T_∞), where $f_\infty \in F_1$ and $T_\infty \in [1, \infty]$. By (2.9), we see that $a_{n_j} \to ||\Sigma||_T \mid_{f=f_\infty, T=T_\infty}$, — and so A is closed.

Consider the last two factors in (2.15). We have

$$\begin{aligned} \sup_n \textstyle\sum_{t'} |\bar{\sigma}^{nt'}| &= (2\pi)^{-2} \sup_n \textstyle\sum_{t'} |\sum_{k=-\infty}^{\infty} \rho(n-t'+kT)| \\ &\leqq (2\pi)^{-2} \textstyle\sum_{\tau=-\infty}^{\infty} |\rho(\tau)|, \end{aligned} \tag{2.18}$$

where

$$\rho(\tau) = \textstyle\int_{-\pi}^{\pi} e^{i\omega\tau} f^{-1}(\omega)\, d\omega. \tag{2.19}$$

From (2.9), it follows that $\sum_{\tau=-\infty}^{\infty} |\rho(\tau)|$ is bounded for $f \in F_1$. Since [10]

$$\textstyle\sum_m \sum_n |\bar{R}(m-n) - R(m-n)| \leqq 2K_2, \tag{2.20}$$

we have proved (2.12).

The proof of (2.13) is similar to the proof of (2.12). From the same considerations that led to (2.14) and (2.15), we have

$$\begin{aligned}
&|X'\Sigma^{-1}X-\textstyle\sum_{j=0}^{T-1}\xi(2\pi j/T)/f(2\pi j/T)-X'\Sigma_0^{-1}X \\
(2.21)\qquad &+\textstyle\sum_{j=0}^{T-1}\xi(2\pi j/T)/f_0(2\pi j/T)| \\
&\leqq \{T^{-\zeta}\sup_t|X(t)|^2\} \\
&\quad\cdot\{[T^{\zeta}\sup_m\textstyle\sum_t|\sigma^{tm}-\sigma_0{}^{tm}|][\sup_n\textstyle\sum_{t'}|\bar\sigma^{nt'}|][\textstyle\sum_m\textstyle\sum_n|\bar R(m-n)-R(m-n)|] \\
&\quad+[\sup_m\textstyle\sum_t|\sigma_0{}^{tm}|][T^{\zeta}\sup_n\textstyle\sum_{t'}|\bar\sigma^{nt'}-\bar\sigma_0{}^{nt'}|] \\
&\quad\cdot[\textstyle\sum_m\textstyle\sum_n|\bar R(m-n)-R(m-m)|] \\
&\quad+[\sup_m\textstyle\sum_t|\sigma_0{}^{tm}|][\sup_n\textstyle\sum_{t'}|\bar\sigma_0{}^{nt'}|][T^{\zeta}\textstyle\sum_m\textstyle\sum_n|\bar R(m-n)-R(m-n) \\
&\qquad-\bar R_0(m-n)+R_0(m-n)|]\},
\end{aligned}$$

where $\sigma_0{}^{tt'}$ and $\bar\sigma_0{}^{tt'}$ are the values at f_0 of $\sigma^{tt'}$ and $\bar\sigma^{tt'}$, respectively, and ζ is chosen so that $0<\zeta<\frac{1}{2}\alpha_1(1-\beta)/(2+\alpha_1)$. From (2.16), we see that $T^{-\zeta}\sup_t|X(t)|^2\to 0$ a.s. We must show that the supremum over E_T of the second factor on the right-hand side of (2.21) is bounded.

We consider the factors in (2.21) that do not appear in (2.17), (2.18), or (2.20). We have

$$\begin{aligned}
\sup_m\textstyle\sum_t|\sigma^{tm}-\sigma_0{}^{tm}| &\leqq \sup_m\textstyle\sum_t\sum_j\sum_k|\sigma^{tj}|\,|R(j-k)-R_0(j-k)|\,|\sigma_0{}^{km}| \\
&\leqq[\sup_j\textstyle\sum_t|\sigma^{tj}|][\sup_m\textstyle\sum_k|\sigma_0{}^{km}|][\sup_k\textstyle\sum_j|R(j-k)-R_0(j-k)|]
\end{aligned}$$

and by (2.9)

$$\begin{aligned}
(2.22)\quad \sup_{f\in E_T}\sup_k\textstyle\sum_j|R(j-k)-R_0(j-k)| &\leqq \sup_{f\in E_T}\textstyle\sum_{|\tau|\leqq T^{\varepsilon}}|R(\tau)-R_0(\tau)| \\
&\quad+\sup_{f\in E_T}\textstyle\sum_{|\tau|>T^{\varepsilon}}|R(\tau)-R_0(\tau)| \\
&\leqq 2\pi K_n^{\frac{1}{2}}T^{\varepsilon-(1-\beta)/2}+2\pi^{1+\alpha_2}K_2(T^{\varepsilon}-\tfrac{1}{2})^{-\alpha_2},
\end{aligned}$$

where $\varepsilon=\frac{1}{2}(1-\beta)/(2+\alpha_1)$. Thus, since α_2 can be chosen arbitrarily close to α_1, $T^{\zeta}\sup_{f\in E_T}\sup_m\sum_t|\sigma^{tm}-\sigma_0{}^{tm}|$ is bounded. Since

$$(2.23)\qquad \textstyle\sum_{t'}|\bar\sigma^{nt'}-\bar\sigma_0{}^{nt'}|\leqq(2\pi)^{-2}\sum_{\tau=-\infty}^{\infty}|\rho(\tau)-\rho_0(\tau)|,$$

where $\rho(\tau)$ is defined in (2.19) and $\rho_0(\tau)$ is $\rho(\tau)$ evaluated at f_0 and since the reasoning applied to (2.22) also applies to (2.23), we see that $T^{\zeta}\sup_{f\in E_T}\sup_n\sum_{t'}|\bar\sigma^{nt'}-\bar\sigma_0{}^{nt'}|$ is bounded. Finally, we have

$$\begin{aligned}
&T^{\zeta}\sup_{f\in E_T}\textstyle\sum_t\sum_{t'}|\bar R(t-t')-R(t-t')-\bar R_0(t-t')+R_0(t-t')| \\
&\quad\leqq 2T^{\zeta}\sup_{f\in E_T}[\textstyle\sum_{|\tau|\leqq T^{\varepsilon}}|\tau||R(\tau)-R_0(\tau)|+\sum_{|\tau|>T^{\varepsilon}}|\tau|\,|R(\tau)-R_0(\tau)|] \\
&\quad\leqq 2T^{\zeta}[2\pi K_n^{\frac{1}{2}}T^{2\varepsilon-(1-\beta)/2}+2K_1T^{-\varepsilon\alpha_1}].
\end{aligned}$$

This completes the proof of Lemma 2.

LEMMA 3.

(2.24) $\sup_{f\in F_1}|T^{-\beta'}\sum_{j=0}^{T-1}[\xi(2\pi j/T)-f_0(2\pi j/T)]/f(2\pi j/T)|\to 0$ a.s.

PROOF. Using the Minkowski inequality, we obtain

$$\begin{aligned}
&E^{\frac{1}{2}}\{\sup_{f\in F_1}T^{-\frac{1}{2}}\textstyle\sum_{j=0}^{T-1}[\xi(2\pi j/T)-f_0(2\pi j/T)]/f(2\pi j/T)\}^2\\
&\quad=E^{\frac{1}{2}}\{\sup_{f\in F_1}(2\pi)^{-2}T^{\frac{1}{2}}\textstyle\sum_{\tau=0}^{T-1}[R_T(\tau)+R_T(\tau-T)-R_0(\tau)\\
&\qquad-R_0(\tau-T)]\textstyle\sum_{k=-\infty}^{\infty}\rho(\tau+kT)+T^{\frac{1}{2}}(2\pi)^{-2}\qquad(2.25)\\
&\qquad\cdot\textstyle\sum_{\tau=0}^{T-1}\sum_{k=-\infty,k\neq 0,-1}^{\infty}R_0(\tau+kT)\sum_{k'=-\infty}^{\infty}\rho(\tau+k'T)\}^2\\
&\quad\leqq(2\pi)^{-2}\textstyle\sum_{\tau=0}^{T-1}\{T^{\frac{1}{2}}E^{\frac{1}{2}}[R_T(\tau)-R_0(\tau)]^2\\
&\qquad+T^{\frac{1}{2}}E^{\frac{1}{2}}[R_T(\tau-T)-R_0(\tau-T)]^2\}\textstyle\sum_{k=-\infty}^{\infty}\sup_{f\in F_1}|\rho(\tau+kT)|\\
&\qquad+0(T^{-\frac{1}{2}-\alpha_2}).
\end{aligned}$$

Since

$$\begin{aligned}
(2.26)\quad TE[R_T(\tau)-R_0(\tau)]^2&=|\tau|^2T^{-1}R_0^{\ 2}(\tau)\\
&\quad+T^{-1}\textstyle\sum_{1\leqq t,t'\leqq T-|\tau|}[R_0^{\ 2}(t-t')+R_0(t'-t+|\tau|)R_0(t'-t-|\tau|)]\\
&\leqq T^{-1}+2\textstyle\sum_{v=-\infty}^{\infty}R_0^{\ 2}(v)
\end{aligned}$$

and $\sum_{\tau=0}^{T-1}\sum_{k=-\infty}^{\infty}\sup_{f\in F_1}|\rho(\tau+kT)|<\infty$, the right-hand side of (2.25) is bounded. Since $\{X(t)\}$ is Gaussian, the higher moments of $T^{-\frac{1}{2}}\sup_{f\in F_1}\sum_{j=0}^{T-1}[\xi(2\pi j/T)-f_0(2\pi j/T)]/f(2\pi j/T)$ are also bounded. Since $\beta'>\frac{1}{2}$, Lemma 3 follows from the Markov inequality and the Borel–Cantelli lemma.

In order to complete the proof of Theorem 1, we consider

$$(2.27)\qquad I_k=(2\pi)^T\int n_r X\,|\,0,\Sigma)\,d\mu_k,$$

$$(2.28)\qquad I_k{}'=(2\pi)^T\int_{E_T}n_r(X\,|\,0,\Sigma)\,d\mu_k.$$

In the preceding lemmas, we showed that

$$\begin{aligned}
&\sup_{f\in F_1}|T^{-\beta'}(\log\det\Sigma+X'\Sigma^{-1}X-T\log 2\pi)\\
&\qquad-T^{-\beta'}\textstyle\sum_{j=0}^{T-1}\{\log f(2\pi j/T)+f_0(2\pi j/T)/f(2\pi j/T)\}|\to 0\text{ a.s.}
\end{aligned}$$

The supremum of $\log f+f_0/f$ for $f\in E_T{}'$ and the infimum of $\log f+f_0/f$ for $f\in F_1-E_T$ are attained at a point where $|f-f_0|=(K_n/T)^{\frac{1}{2}}$ and $|f-f_0|=(K_n/T^{1-\beta})^{\frac{1}{2}}$, respectively. Thus, the supremum of $\log f+f_0/f$ for $f\in E_T{}'$ is $\log f_0+1+\frac{1}{2}K_nf_0^{\ -2}T^{-1}+0(T^{-\frac{3}{2}})$ and the infimum of $\log f+f_0/f$ for $f\in F_1-E_T$ is $\log f_0+1+\frac{1}{2}K_nf_0^{\ -2}T^{-1+\beta}+0(T^{-3(1-\beta)/2})$. For each $\{X(t)\}$ outside a null set, there is a value of T such that for all larger T, we have

$$(2.29)\qquad I_k{}'>\exp\{-\tfrac{1}{2}\textstyle\sum_{j=0}^{T-1}[\log f_0(2\pi j/T)+1]-\lambda T^{\beta'}\}\mu_k(E_T{}')$$

and

$$(2.30)\qquad I_k-I_k{}'<\exp\{-\tfrac{1}{2}\textstyle\sum_{j=0}^{T-1}[\log f_0(2\pi j/T)+1]-\tfrac{1}{2}K_nf_0^{\ -2}T^{\beta}+\lambda T^{\beta'}\},$$

where λ is an arbitrary positive constant.

We can now finish the proof. Using (2.1), (2.29), and (2.30), we see that

$$|P(k \mid X) - \frac{q_k I_k'}{\sum_l q_l I_l'}| \leqq \left|\frac{q_k(I_k - I_k')}{\sum_l q_l I_l}\right|$$
$$+\left|\frac{q_k I_k'}{\sum_l q_l I_l'}\left\{\frac{\sum_l q_l(I_l - I_l')}{\sum q_l I_l}\right\}\right| \leqq \frac{2}{\exp\{\frac{1}{4}K_n f_0^{-2}T^\beta - 2\lambda T^{\beta'}\}\sum_l q_l \mu_l(E_T')} \to 0$$

for $\{X(t)\}$ outside a null set and so

$$P(k \mid X) - q_k I_k'/\textstyle\sum_l q_l I_l' \to 0 \text{ a.s.} \tag{2.31}$$

In Lemmas 1 and 2, we showed that

$$\sup_{f \in E_T} |\log\det\Sigma + X'\Sigma^{-1}X - T\log 2\pi - S(f, T)| \to 0 \text{ a.s.}$$

and so

$$\frac{q_k I_k'}{\sum_l q_l I_l'} - \frac{q_k \int_{E_T} \exp\{-\frac{1}{2}S(f, T)\}\, d\mu_k}{\sum_l q_l \int_{E_T} \exp\{-\frac{1}{2}S(f, T)\}\, d\mu_l} \to 0 \text{ a.s.}$$

Using the same argument that led to (2.31), we see that

$$\frac{q_k \int_{E_T} \exp\{-\frac{1}{2}S(f, T)\}\, d\mu_k}{\sum_l q_l \int_{E_T} \exp\{-\frac{1}{2}S(f, T)\}\, d\mu_l} - \frac{q_k \int \exp\{-\frac{1}{2}S(f, T)\}\, d\mu_k}{\sum_l q_l \int \exp\{-\frac{1}{2}S(f, T)\}\, d\mu_l} \to 0 \text{ a.s.}$$

This concludes the proof of Theorem 1.

3. Extension to general spectral estimates. In this section, we extend Theorem 1 to a broad class of spectral estimates and then demonstrate that the expected cost of the test based on $P(k \mid \hat{f})$ approaches the expected cost of the test based on $P(k \mid X)$.

THEOREM 2. *Let p be a fixed positive integer. Let $\{\hat{f}(\omega_m),\ \omega_m = \pi(m+\frac{1}{2})/M_T,\ m = 0, 1, \cdots, M_T - 1\}$ be spectral estimates of the type defined in* (1.5)–(1.7) *and let F_p and μ_l be defined as in Section* 1. *Let $\{X(t)\}$ be a real, zero mean, stationary, Gaussian time series with spectral density function f_0, where $f_0 \in F_p$. For some β in the open interval $(\frac{1}{2}, \min\{1 - 2\alpha_1^{-1}(1-\delta p)(p+1+\alpha_1), 1\})$, let $\lim_{T\to\infty} \exp\{\gamma T^\beta\}$ $\Sigma_l q_l \mu_l(E_T' \cap F_p) = \infty$. Then*

$$P(k \mid X) - P(k \mid \hat{f}) \to 0 \text{ a.s.},$$

where $P(k \mid \hat{f})$ is defined in (1.3) *and* (1.4) *with $n_m = \frac{1}{2}T/M_T$.*

The proof is similar to the proof of Theorem 1.

LEMMA 4.

$$\textstyle\sum_{j=0}^{T-1} \log f(2\pi j/T) - (T/M_T)\sum_{m=0}^{M_T-1} \log f(\omega_m) \to 0$$

uniformly for $f \in F_p$.

PROOF. We have

$$(T/M_T)\textstyle\sum_{m=0}^{M_T-1} \log f(\omega_m) = (2\pi)^{-1} T \sum_{k=-\infty}^{\infty} (-1)^k r(2kM_T)$$
$$= (2\pi)^{-1} T \textstyle\int_{-\pi}^{\pi} \log f(\omega)\, d\omega + 0(T^{1-\delta(p+\alpha_2)}),$$

where $r(n)$ is defined in (2.8). Since α_2 can be chosen close enough to α_1 so that $\delta(p+\alpha_2) > 1$ and since (2.10) holds, the proof is complete.

LEMMA 5. *For* $\frac{1}{2} < \beta' < \beta$, *we have*

$$(3.1) \quad T^{-\beta'} \sup_{f \in Fp} \left|\sum_{j=0}^{T-1} \xi(2\pi j/T)/f(2\pi j/T) - (T/M_T)\sum_{m=0}^{M_T-1} \hat{f}(\omega_m)/f(\omega_m)\right| \to 0 \text{ a.s.}$$

and

$$(3.2) \quad \sup_{f \in E_T \cap Fp} \left|\sum_{j=0}^{T-1} \xi(2\pi j/T)[f^{-1}(2\pi j/T) - f_0^{-1}(2\pi j/T)] - (T/M_T)\sum_{m=0}^{M_T-1} \hat{f}(\omega_m)[f^{-1}(\omega_m) - f_0^{-1}(\omega_m)]\right| \to 0 \text{ a.s.}$$

PROOF. Applying the procedure used in Lemma 3, we prove that

$$T^{-\beta'} \sup_{f \in Fp} \left|(T/M_T)\sum_{m=0}^{M_T-1} [\hat{f}(\omega_m) - f_0(\omega_m)]/f(\omega_m)\right| \to 0 \text{ a.s.}$$

For this it is sufficient to note that from (1.1), (1.6), (2.9), and (2.26), we obtain

$$\begin{aligned}
&E^{\frac{1}{2}}\{(T^{\frac{1}{2}}/M_T) \sup_{f \in Fp} |\textstyle\sum_{m=0}^{M_T-1} [\hat{f}(\omega_m) - f_0(\omega_m)]/f(\omega_m)|\}^2 \\
&\leqq (2\pi)^{-2} \textstyle\sum_{\tau=-M_T}^{M_T-1} \{T^{\frac{1}{2}} |h(\tau/M_T)| E^{\frac{1}{2}}[R_T(\tau) - R_0(\tau)]^2 \\
&\quad + T^{\frac{1}{2}} |1 - h(\tau/M_T)|\, |R_0(\tau)| + T^{\frac{1}{2}} \textstyle\sum_{k=-\infty, k\neq 0}^{\infty} |R_0(\tau + 2kM_T)|\} \\
&\quad \cdot \textstyle\sum_{k'=-\infty}^{\infty} \sup_{f \in Fp} |\rho(\tau + 2kM_T)| \\
&\quad \leqq (2\pi)^{-2}\{[T^{-\frac{1}{2}} + 2\textstyle\sum_{v=-\infty}^{\infty} R_0^2(v)]^{\frac{1}{2}} \sup_u |h(u)| \\
&\quad + K_1 K_h T^{\frac{1}{2} - \delta p} + 2\pi^{1+\alpha_2} K_2 T^{\frac{1}{2} - \delta(p+\alpha_2)} \textstyle\sum_{k=1}^{\infty} (2k-1)^{-p-\alpha_2}\} \\
&\quad \cdot \textstyle\sum_{\tau=-\infty}^{\infty} \sup_{f \in F_1} |\rho(\tau)|,
\end{aligned}$$

where by assumption $\delta p > \frac{1}{2}$.

Since

$$\sum_{j=0}^{T-1} f_0(2\pi j/T)/f(2\pi j/T) - (T/M_T)\sum_{m=0}^{M_T-1} f_0(\omega_m)/f(\omega_m) \to 0$$

uniformly for $f \in F_p$ by the reasoning that led to Lemma 4 and since Lemma 3 holds, we have proved (3.1).

In order to prove (3.2), we observe that

$$\begin{aligned}
&\sup_{f \in E_T \cap Fp} |\textstyle\sum_{j=0}^{T-1} \xi(2\pi j/T)[f^{-1}(2\pi j/T) - f_0^{-1}(2\pi j/T)] \\
&\quad - (T/M_T)\textstyle\sum_{m=0}^{M_T-1} \hat{f}(\omega_m)[f^{-1}(\omega_m) - f_0^{-1}(\omega_m)]| \\
(3.3) \quad &\leqq \{(2\pi)^{-2} T^{1-\theta} \sup_\tau (T - |\tau|)^{-1} |R_T(\tau)|\} \\
&\quad \cdot \{T^\theta \sup_{f \in E_T \cap Fp} \textstyle\sum_{\tau=-(T-1)}^{T-1} (T - |\tau|) |1 - h(\tau/M_T)|\, |\rho(\tau) - \rho_0(\tau)| \\
&\quad + (T - |\tau|) \textstyle\sum_{k=-\infty, k\neq 0}^{\infty} |\rho(\tau + kT) - \rho_0(\tau + kT)| \\
&\quad + (T - |\tau|) |h(\tau/M_T)| \textstyle\sum_{k'=-\infty, k'\neq 0}^{\infty} |\rho(\tau + 2k'M_T) - \rho_0(\tau + 2k'M_T)|\},
\end{aligned}$$

where

$$(3.4) \quad 0 < \theta < \min\{\delta p - 1 + \tfrac{1}{2}\alpha_1(1-\beta)/(p+1+\alpha_1), \tfrac{1}{2}(1-\beta)(p-1+\alpha_1)/(p+1+\alpha_1)\}.$$

We will show that the second factor on the right-hand side of (3.3) is bounded. Since $\sup_\tau T(T-|\tau|)^{-1}|R_T(\tau)| \leqq \sup_{1\leqq t\leqq T}|X(t)|^2$, we can then apply (2.16) to prove (3.2).

By (1.6), we have

$$(3.5)\quad T^\theta \sup_{f\in E_T\cap F_p} \sum_{\tau=-(T-1)}^{T-1}(T-|\tau|)\,|1-h(\tau/M_T)|\,|\rho(\tau)-\rho_0(\tau)| \leqq T^{(1+\theta-\delta p)}K_h \sup_{f\in E_T\cap F_p}\sum_{\tau=-(T-1)}^{T-1}|\tau|^p|\rho(\tau)-\rho_0|(\tau).$$

Since $[d^p(f^{-1})/d\omega^p - f^{(p)}f^{-2}]$ has a derivative which satisfies a Lipschitz condition, we have

$$|\tau|^p|\rho(\tau)| = \left|\int_{-\pi}^{\pi} e^{i\omega\tau}[d^p(f^{-1})/d\omega^p]\,d\omega\right| \leqq (2\pi)^{-1}\sum_{v=-\infty}^{\infty}|q(v)|\,|\tau-v|^p|R(\tau-v)| + 0(|\tau|^{-1-\alpha_2}),$$

where $0 < \alpha_2 < \alpha_1$ and

$$q(v) = \int_{-\pi}^{\pi} e^{i\omega v} f^{-2}(\omega)\,d\omega.$$

Letting $\varepsilon = \frac{1}{2}(1-\beta)/(p+1+\alpha_1)$ and $\zeta < \varepsilon$, we obtain

$$(3.6)\quad \sum_\tau |\tau|^p|\rho(\tau)-\rho_0(\tau)| \leqq \sum_{|\tau|\leqq T^\varepsilon}|\tau|^p|\rho(\tau)-\rho_0(\tau)| + (\pi)^{-1}\sum_{|v|\leqq T^\zeta}|q(v)|\sum_{|\tau|>T^\varepsilon}|\tau-v|^p|R(\tau-v)| + (\pi)^{-1}\sum_{|v|>T^\zeta}|q(v)|\sum_{|\tau|>T^\varepsilon}|\tau-v|^p|R(\tau-v)| + 0(T^{-\varepsilon\alpha_1}).$$

The supremum over $E_T \cap F_p$ of the first term on the right-hand side of (3.6) is $0(T^{\varepsilon(p+1)-(1-\beta)/2})$. By (1.1), the second term is $0(T^{-\varepsilon\alpha_1})$. By (2.6), the third term is $0(T^{-\zeta\alpha_2})$. By choosing α_2 and ζ close enough to α_1 and ε, respectively, we see that the right side of (3.5) is bounded. Using the reasoning applied to (3.6), we obtain

$$(3.7)\quad T^\theta\sum_{\tau=-(T-1)}^{T-1}(T-|\tau|)\sum_{k=-\infty,k\neq 0}^{\infty}|\rho(\tau+kT)-\rho_0(\tau+kT)| \leqq T^\theta\sum_{\tau=-\infty}^{\infty}|\tau|\,|\rho(\tau)-\rho_0(\tau)| = 0(T^{\theta-\zeta(p-1+\alpha_2)})$$

and

$$(3.8)\quad T^\theta\sum_{\tau=-(T-1)}^{T-1}(T-|\tau|)\sum_{k'=-\infty,k'\neq 0}^{\infty}|\rho(\tau+2k'M_T)-\rho_0(\tau+2k'M_T)| \leqq 2T^{1+\theta}\sum_{\tau=M_T}^{\infty}|\rho(\tau)-\rho_0(\tau)| = 0(T^{1+\theta-\delta(p+\alpha_2)}).$$

Applying (3.4), we see that (3.7) and (3.8) are bounded. This completes the proof of Lemma 5.

Using Lemmas 4 and 5 and the argument that led from Lemmas 1–3 to the proof of Theorem 1, we obtain

$$(3.9)\quad P(k\,|\hat{f}) - q_k\int\exp\{-\tfrac{1}{2}S(f,T)\}\,d\mu_k\Big/\sum_{l=0}^{L}q_l\int\exp\{-\tfrac{1}{2}S(f,T)\}\,d\mu_l \to 0 \text{ a.s.}$$

Theorem 2 then follows from Theorem 1 and (3.9).

COROLLARY. *Let Theorem 2 apply to all $f_0 \in F_p$ except for a set that is null in the measure $\Sigma_l q_l\mu_l$. The expected cost of using the approximate posterior probabilities* *considered in Theorem 2 in the Bayes test approaches the expected cost of using the exact posterior probabilities.*

PROOF. Let R_k, $k = 0, 1, \cdots, L$ be disjoint regions in the space of $\{X(t)\}$ chosen so that for $\{X(t)\} \in R_k$

$$\min_{0 \leqq j \leqq L} \sum_{l=0}^{L} C(j \mid l) P(l \mid \hat{f}) = \sum_{l=0}^{L} C(k \mid l) P(l \mid \hat{f}).$$

The cost of our approximate test is $\sum_{j=0}^{L} \int_{R_j} \sum_{l=0}^{L} C(j \mid l) P(l \mid x) p(x)\, dx$, where

$$p(x) = \sum_{l=0}^{L} q_l \int n_r(x \mid 0, \Sigma)\, d\mu_l.$$

We have

$$\begin{aligned}(3.10)\quad \sum_{j=0}^{L} \int_{R_j} \sum_{l=0}^{L} C(j \mid l) P(l \mid x) p(x)\, dx &= \int \min_{0 \leqq j \leqq L} \sum_{l=0}^{L} C(j \mid l) P(l \mid x) p(x)\, dx \\ &\quad + \sum_{j=0}^{L} \int_{R_j} \sum_{l=0}^{L} C(j \mid l) [P(l \mid x) - P(l \mid \hat{f})] p(x)\, dx \\ &\quad + \int [\min_{0 \leqq j \leqq L} \sum_{l=0}^{L} C(j \mid l) P(l \mid \hat{f}) \\ &\quad - \min_{0 \leqq j \leqq L} \sum_{l=0}^{L} C(j \mid l) P(l \mid x)] p(x)\, dx.\end{aligned}$$

The first term on the right-hand side of (3.10) is the Bayes cost and the second and third terms approach zero by Theorem 2 and the Dominated Convergence Theorem.

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