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# NONPARAMETRIC ESTIMATE OF REGRESSION COEFFICIENTS

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**1. Summary and introduction.** The present investigation is a follow up of [7] to a class of multiple regression problems, and is devoted to the construction of an estimate of regression parameter vector based on suitable rank statistics. Asymptotic linearity of these rank statistics in the multiple regression set up is established and the asymptotic multi-normality of the derived estimates is deduced. There exists the choice of the score-generating function to every basic distribution so that the asymptotic distribution of the estimates is the same as that of maximal-likelihood estimates.

**2. Notation and assumptions.** The following assumptions are to be satisfied for $N = 1, 2, \ldots$.

ASSUMPTION 1. $Y_1, Y_2, \cdots, Y_N$ are independent random variables, $Y_i$ having distribution function

$$F(y-\alpha^0-\boldsymbol{\beta}^0\mathbf{c}^{(i)}) \qquad i = 1, 2, \cdots, N \tag{2.1}$$

for $F$ possesing finite Fisher's information, i.e. $\int[f'(x)/f(x)]^2 f(x)\,dx < \infty$ with $f$ being the density of the distribution.

ASSUMPTION 2. $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_K)$ is a real vector parameter.

ASSUMPTION 3. $\mathbf{C}_N = [c_{ji}]$ is a $K\times N$ matrix with rows $\mathbf{c}_{(j)}$ and columns $\mathbf{c}^{(i)}$ satisfying the conditions

ASSUMPTION 3a. $c_{ji} = c'_{ji}+c''_{ji}$, $1 \leqq j \leqq K$, $1 \leqq i \leqq N$.

ASSUMPTION 3b. The vectors $\mathbf{c}'_{(j)} = (c'_{j1}, \cdots, c'_{jN})$, $j = 1, 2, \cdots, K$ satisfy either

$$(\mathbf{c}'_{(j)}-\bar{c}_j{}')(\mathbf{c}'_{(j)}-\bar{c}_j{}')' = 0 \tag{2.2}$$

for all but a finite number of $N$, or

$$(\mathbf{c}'_{(j)}-\bar{c}_j{}')(\mathbf{c}'_{(j)}-\bar{c}_j{}')' > 0 \tag{2.3}$$

for all but a finite number of $N$; further

$$N^{-1}(\mathbf{c}'_{(j)}-\bar{c}_j{}')(\mathbf{c}'_{(j)}-\bar{c}_j{}')' \leqq M_1 \qquad \text{for } N = 1, 2, \cdots \tag{2.4}$$

and if (2.3) is satisfied, then

$$\lim_{N\to\infty} \max_{1\leqq i\leqq N}(c'_{ji}-\bar{c}_j{}')^2/[\textstyle\sum_{i=1}^N (c'_{ji}-\bar{c}_j{}')^2] = 0; \tag{2.5}$$

here $\bar{c}_j{}' = 1/N[\sum_{i=1}^N c'_{ji}]$ and $M_1 > 0$ is a constant independent of $N$. Analogous assumptions are to be satisfied for vectors $\mathbf{c}''_{(j)}$, $j = 1, 2, \cdots, K$.

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ASSUMPTION 3c. It holds for all pairs $j, l = 1, 2, \cdots, K$

$$(c'_{ji} - c'_{jk})(c'_{li} - c'_{lk}) \geqq 0$$
$$(c'_{ji} - c'_{jk})(c''_{li} - c''_{lk}) \leqq 0$$
$$(c''_{ji} - c''_{jk})(c''_{li} - c''_{lk}) \geqq 0$$

for all $i, k = 1, 2, \cdots, N$; $N = 2, 3, \cdots$.

ASSUMPTION 3d. $\lim_{N\to\infty} N^{-1}(\mathbf{c}_{(l)} - \bar{c}_l)(\mathbf{c}_{(j)} - \bar{c}_j)' = \sigma_{lj}$, $l, j = 1, 2, \cdots, K$ where $\mathbf{\Sigma} = [\sigma_{lj}]_{l,j=1}^{K} = [\boldsymbol{\sigma}^{(1)}, \cdots, \boldsymbol{\sigma}^{(K)}]$ is a positive definite matrix.

ASSUMPTION 4. $(R_1{}^{\beta}, R_2{}^{\beta}, \cdots, R_N{}^{\beta})$ is the vector of ranks corresponding to variables $Y_i - \boldsymbol{\beta}\mathbf{c}^{(i)}$, $i = 1, 2, \cdots, N$.

ASSUMPTION 5. Consider the linear rank statistics

$$S_{Nj}(Y - \boldsymbol{\beta}\mathbf{c}) = N^{-\frac{1}{2}} \sum_{i=1}^{N} (c_{ji} - \bar{c}_j) a_N(R_i{}^{\beta}), \qquad j = 1, 2, \cdots, K \tag{2.6}$$

where the scores $a_N(i)$, $i = 1, 2, \cdots, N$ are generated by a function $\varphi(u)$, $0 < u < 1$ by either of the following two ways:

$$a_N(i) = E\varphi(U_N{}^{(i)}) \tag{2.7}$$

$$a_N(i) = \varphi(i/(N+1)), \qquad i = 1, 2, \cdots, N \tag{2.8}$$

where $U_N{}^{(i)}$ denotes the $i$th order statistic in a sample of size $N$ from uniform distribution on (0, 1).

ASSUMPTION 6. The score-generating function $\varphi(u)$ is non-constant non-decreasing and square-integrable on (0, 1).

REMARK. (2.4) and analogous assumption for $c''_{(j)}$ imply that

$$N^{-1}(\mathbf{c}_{(j)} - \bar{c}_j)(\mathbf{c}_{(j)} - \bar{c}_j)' \leqq 2\{N^{-1}(\mathbf{c}'_{(j)} - \bar{c}_j{}')(\mathbf{c}'_{(j)} - \bar{c}_j{}')'$$
$$+ N^{-1}(\mathbf{c}''_{(j)} - \bar{c}_j{}'')(\mathbf{c}''_{(j)} - \bar{c}_j{}'')'\} \leqq 2(M_1 + M_2) = M \qquad \text{for } N = 1, 2, \cdots.$$

Put

$$\varphi(u, f) = -f'(F^{-1}(u))/f(F^{-1}(u)) \tag{2.9}$$

$$\gamma = \int_0^1 \varphi(u)\varphi(u, f)\, du \tag{2.10}$$

$$A^2 = \int_0^1 (\varphi(u) - \bar{\varphi})^2\, du, \qquad \text{where } \bar{\varphi} = \int_0^1 \varphi(u)\, du. \tag{2.11}$$

REMARK 1. The described regression model does not cover the whole class of regression models where the least square method succeeds; this fact is due to Assumption 3d.

REMARK 2. We do not denote explicitly the dependence of variables of the model on $N$; we hope that this simplification will not cause a confusion.

REMARK 3. We shall assume without any loss of generality that $N^{-1}(\mathbf{c}_{(j)} - \bar{c}_j)(\mathbf{c}_{(j)} - \bar{c}_j)' = 1$; this is obtained by a proper reparametrization.

We shall deal with the problem of estimating of parameter $\boldsymbol{\beta}$. In order to be in agreement with the notation of one-dimensional case in [7], we shall work in the sequel with new variables which are connected with old ones by relation

$$N^{\frac{1}{2}}\boldsymbol{\beta} = \boldsymbol{\Delta}, \qquad N^{-\frac{1}{2}}c_{ji} = x_{ji}, \qquad i = 1, 2, \cdots, N; j = 1, 2, \cdots, K. \tag{2.12}$$

We thus get the model: $Y_i$ has the distribution function

$$F(y - \alpha^0 - \boldsymbol{\Delta}^0\mathbf{x}^{(i)}), \qquad i = 1, 2, \cdots, N; \tag{2.13}$$

the properties of the variables of the model follow from Assumptions 1–6. So, we shall consider the statistics

$$S_{Nj}(Y - \boldsymbol{\Delta}\mathbf{x}) = \sum_{i=1}^N (x_{ji} - \bar{x}_j)a_N(R_i^{\Delta}), \qquad j = 1, 2, \cdots, K \tag{2.14}$$

where $R_1^{\Delta}, \cdots, R_N^{\Delta}$ is the vector of ranks corresponding to variables $Y_\iota - \boldsymbol{\Delta}\mathbf{x}^{(i)}$, $i = 1, 2, \cdots, N$; we shall consider the possibility of estimating of $\boldsymbol{\Delta}$ based on $S_{Nj}(Y - \boldsymbol{\Delta}\mathbf{x}), j = 1, 2, \cdots, K$.

**3. Asymptotic linearity of $S_{Nj}\,(Y - \boldsymbol{\Delta}\mathbf{x})$ in $\boldsymbol{\Delta}$.** The following theorem generalizes Theorem 3.1. of [7].

THEOREM 3.1. *Let $P_{\Delta^0}$ denote the probability distribution with the density $p_{\Delta^0} = \prod_{i=1}^N f(y_i - \alpha - \boldsymbol{\Delta}^0\mathbf{x}^{(i)})$; let $||\boldsymbol{\Delta} - \boldsymbol{\Delta}^0|| = [(\boldsymbol{\Delta} - \boldsymbol{\Delta}^0)(\boldsymbol{\Delta} - \boldsymbol{\Delta}^0)']^{\frac{1}{2}}$. Then under Assumption* 1 *through* 6 *and* (2.12)

$$\lim_{N\to\infty} P_{\Delta^0}\{\max_{||\Delta - \Delta^0|| \leqq C} |S_{Nj}(Y - \boldsymbol{\Delta}\mathbf{x}) - S_{Nj}(Y - \boldsymbol{\Delta}^0\mathbf{x}) + \gamma(\boldsymbol{\Delta} - \boldsymbol{\Delta}^0)\boldsymbol{\sigma}^{(j)}| \geqq \varepsilon\} = 0 \tag{3.1}$$

*holds for any $\varepsilon > 0$, $C > 0$ and $j = 1, 2, \cdots, K$.*

PROOF. Let $I = \{\mathbf{x}; ||\mathbf{x}|| \leqq C\}$. We may suppose without loss of generality that $\boldsymbol{\Delta}^0 = \mathbf{0}$. In order to be in agreement with the formulation of Theorem 3.1 of [7] we denote $\boldsymbol{\delta} = -\boldsymbol{\Delta}$.

Let $R_1^{\delta',\delta''}, \cdots, R_N^{\delta',\delta''}$ be vector of ranks for variables $Y_i + \boldsymbol{\delta}'\mathbf{x}^{(i)'} + \boldsymbol{\delta}''\mathbf{x}^{(i)''}$, $i = 1, 2, \cdots, N$.

For $j = 1, 2, \cdots, K$, consider the statistics

$$\begin{aligned} S_{Nj}(Y + \boldsymbol{\delta}'\mathbf{x}' + \boldsymbol{\delta}''\mathbf{x}'') &= \sum_{i=1}^N (x_{ji} - \bar{x}_j)a_N(R_i^{\delta',\delta''}) \\ &= \sum_{i=1}^N (x'_{ji} - \bar{x}_j')a_N(R_i^{\delta',\delta''}) + \sum_{i=1}^N (x''_{ji} - \bar{x}_j'')a_N(R_i^{\delta',\delta''}) \\ &= S'_{Nj}(Y + \boldsymbol{\delta}'\mathbf{x}' + \boldsymbol{\delta}''\mathbf{x}'') + S''_{Nj}(Y + \boldsymbol{\delta}'\mathbf{x}' + \boldsymbol{\delta}''\mathbf{x}''). \end{aligned} \tag{3.2}$$

As it follows from Theorem 2.1 of [7], statistic $S'_{Nj}$ with fixed $Y_1, \cdots, Y_N$ is non-decreasing function of $\delta_1', \cdots, \delta_K'$ and non-increasing function of $\delta_1'', \cdots, \delta_K''$. Similarly, $S''_{Nj}$ is non-increasing in $\delta_1', \cdots, \delta_K'$ and non-decreasing in $\delta_1'', \cdots, \delta_K''$.

Consider a fixed $j$ such that $\sum_{i=1}^N (x'_{ji} - \bar{x}_j')^2 > 0$ for all but a finite number of $N$. We shall prove

$$P\{\max_{\delta',\delta'' \in I} |S'_{Nj}(Y + \boldsymbol{\delta}'\mathbf{x}' + \boldsymbol{\delta}''\mathbf{x}'') - S'_{Nj}(Y)$$

$$(3.3)\qquad -\gamma\sum_{l=1}^{K}\delta_l'[(\mathbf{x}'_{(l)}-\bar{x}_l')(\mathbf{x}'_{(j)}-\bar{x}_j')']-\gamma\sum_{l=1}^{K}\delta_l''[(\mathbf{x}''_{(l)}-\bar{x}_l'')(\mathbf{x}'_{(j)}-\bar{x}_j')']| \geqq \varepsilon||\mathbf{x}'_{(j)}-\bar{x}_j'||\} \to 0 \qquad \text{for } N\to\infty.$$

The weaker proposition which differs from (3.3) only in that the convergence holds for fixed point $(\boldsymbol{\delta}', \boldsymbol{\delta}'')$ and not necessarily for $\max_{\delta',\delta''\in I}$ follows from Theorem 3.1 of [7] and from the contiguity of sequences of densities

$$q_{\delta_l',N} = \prod_{i=1}^{N} f(y_i-\alpha-\delta_{l'}(x_{li}'-\bar{x}_l')), \qquad l = 1,2,\cdots,K$$

and

$$q_{\delta_l'',N} = \prod_{i=1}^{N} f(y_i-\alpha-\delta_l''(x_{li}''-\bar{x}_l'')), \qquad l = 1,2,\cdots,K$$

to densities $p_N = \prod_{i=1}^{N} f(y_i)$ (the definition of contiguity see e.g. [4]).

For proving the stronger proposition (3.3), let us consider a partition of $[-C, C]$

$$-C = \delta^{(0)} < \delta^{(1)} < \cdots < \delta^{(r)} = C$$

such that

$$(3.4)\qquad |(\delta^{(k)}-\delta^{(k-1)})\gamma| \leqq \varepsilon/(2KM_1^{\frac{1}{2}}), \qquad k = 1,2,\cdots,r.$$

($M_1 > 0$ is the constant from (2.4).)

Then by using (3.4) and the monotonicity of $S'_{Nj}$ in the components of $\boldsymbol{\delta}'$ and $\boldsymbol{\delta}''$, we get

$$\begin{aligned}
&\max_{\delta',\delta''\in I}|S'_{Nj}(Y+\boldsymbol{\delta}'\mathbf{x}'+\boldsymbol{\delta}''\mathbf{x}'')-S'_{Nj}(Y)-\gamma\sum_{l=1}^{K}\delta_l'[(\mathbf{x}'_{(l)}-\bar{x}_l')(\mathbf{x}'_{(j)}-\bar{x}_j')'] \\
&\quad -\gamma\sum_{l=1}^{K}\delta_l''[(\mathbf{x}''_{(l)}-\bar{x}_l'')(\mathbf{x}'_{(j)}-\bar{x}_j')']| \\
(3.5)\qquad &\leqq 2\max_{q_1',\cdots,q_K',q_1'',\cdots,q_K''}|S'_{Nj}(Y_i+\sum_{l=1}^{K}\delta^{(q_l')}x_{li}'+\sum_{l=1}^{K}\delta^{(q_l'')}x_{li}'')-S'_{Nj}(Y) \\
&\quad -\gamma\sum_{l=1}^{K}\delta^{(q_l')}[(\mathbf{x}'_{(l)}-\bar{x}_l')(\mathbf{x}'_{(j)}-\bar{x}_j')'] \\
&\quad -\gamma\sum_{l=1}^{K}\delta^{(q_l'')}[(\mathbf{x}''_{(l)}-\bar{x}_l'')(\mathbf{x}'_{(j)}-\bar{x}_j')']|+\varepsilon/2||\mathbf{x}'_{(j)}-\bar{x}_j'||
\end{aligned}$$

where maximum is taken over the set of all groups $q_1', \cdots, q_K', q_1'', \cdots, q_K''$ where each $q_l', q_l''$ runs through $0, 1, \cdots, r$. This together with the noted weaker proposition proves (3.3).

Analogous proposition may be proved for $S''_{Nj}(Y+\boldsymbol{\delta}'\mathbf{x}'+\boldsymbol{\delta}''\mathbf{x}'')$ with such $j$ that $\sum_{i=1}^{N}(x''_{ji}-\bar{x}_j'')^2 > 0$ for all but a finite number of $N$.

Let us distinguish two cases:

(i) $(\mathbf{x}''_{(j)}-\bar{x}_j'')(\mathbf{x}''_{(j)}-\bar{x}_j'')' = 0$ for $N > N_0$. Then for $N > N_0$ it is $\mathbf{x}_{(j)}-\bar{x}_j = \mathbf{x}'_{(j)}-\bar{x}_j'$, $(\mathbf{x}'_{(j)}-\bar{x}_j')(\mathbf{x}'_{(j)}-\bar{x}_j')' = (\mathbf{x}_{(j)}-\bar{x}_j)(\mathbf{x}_{(j)}-\bar{x}_j)' = 1$ by Assumption 3a, Remark 3 and (2.12); $S_{Nj}(Y+\boldsymbol{\delta}'\mathbf{x}'+\boldsymbol{\delta}''\mathbf{x}'') = S'_{Nj}(Y+\boldsymbol{\delta}'\mathbf{x}'+\boldsymbol{\delta}''\mathbf{x}'')$ satisfies (3.3). This together with (2.12) and Assumption 3d means that (3.1) is valid. Analogous result we get in the case $(\mathbf{x}'_{(j)}-\bar{x}_j')(\mathbf{x}'_{(j)}-\bar{x}_j')' = 0$ for $N > N_0$.

(ii) $(\mathbf{x}'_{(j)}-\bar{x}_j')(\mathbf{x}'_{(j)}-\bar{x}_j')' > 0$, $(\mathbf{x}''_{(j)}-\bar{x}_j'')(\mathbf{x}''_{(j)}-\bar{x}_j'')' > 0$ for $N > N_1$. Then (3.3) is satisfied by $S'_{Nj}$ and $S''_{Nj}$ and this implies that

$$\begin{aligned}
&P\{\max_{\delta',\delta''\in I}|S_{Nj}(Y+\boldsymbol{\delta}'\mathbf{x}'+\boldsymbol{\delta}''\mathbf{x}'')-S_{Nj}(Y)-\gamma\sum_{l=1}^{K}\delta_l'[(\mathbf{x}_l'-\bar{x}_l')(\mathbf{x}_{(j)}-\bar{x}_j)] \\
&\quad -\gamma\sum_{l=1}^{K}\delta_l''[(\mathbf{x}''_{(l)}-\bar{x}_l'')(\mathbf{x}_{(j)}-\bar{x}_j)']| \geqq \varepsilon\} \to 0 \qquad \text{for } N\to\infty
\end{aligned}$$

holds for any $\varepsilon > 0$ and $C > 0$. This together with (2.12) and Assumption 3 implies that (3.1) is valid in this case too.

COROLLARY 3.1. *Under Assumptions* 1 *through* 6 *and* (2.12)

$$(3.6)\quad \lim_{N\to\infty} P_{\Delta^0}\{\max_{\Delta-\Delta^0\in I} |\sum_{j=1}^K |S_{Nj}(Y-\Delta\mathbf{x})| - \sum_{j=1}^K |S_{Nj}(Y-\Delta^0\mathbf{x}) - \gamma(\Delta-\Delta^0)\sigma^{(j)}|| \geqq \varepsilon\} = 0$$

*holds for any* $C > 0$ *and* $\varepsilon > 0$.

**4. Estimation of parameters** $\Delta_1 \cdots, \Delta_K$. It is known that $E[S_{Nj}(Y-\Delta^0\mathbf{x})] = 0$ for $j = 1, 2, \cdots, N$. Moreover, as it follows easily from Theorem V.1.5 of [4], the random vector

$$\mathbf{S}_N(Y-\Delta^0\mathbf{x}) = (S_{N1}(Y-\Delta^0\mathbf{x}), \cdots, S_{NK}(Y-\Delta^0\mathbf{x}))$$

is asymptotically normal

$$(4.1)\qquad (0, A^2\mathbf{\Sigma})$$

where $\mathbf{\Sigma}$ is the matrix from Assumption 3. So we are led to representing the unknown parameter by $\Delta$ for which $\mathbf{S}_N(Y-\Delta\mathbf{x})$ is as near to zero as possible. One possibility is to represent the unknown parameter by points of set

$$(4.2)\qquad D_N = \{\Delta = (\Delta_1, \Delta_2, \cdots, \Delta_K); \sum_{j=1}^K |S_{Nj}(Y-\Delta\mathbf{x})| = \min\}.$$

The set $D_N \subset E^K$ is not empty for all $Y_1, \cdots, Y_N$, for $S_{Nj}(Y-\Delta\mathbf{x})$ ($j = 1, 2, \cdots, K$) as a function of $\Delta_1, \cdots, \Delta_K$ with fixed $Y_1, \cdots, Y_N$ takes on a finite number of different values.

We shall deal with the asymptotic properties of points of $D_N$ in the sequel. All points of $D_N$ are asymptotically equivalent in the sense that they all have the same asymptotic distribution.

We take the definition.

DEFINITION. We say that each point of set $D_N$ is asymptotically normal ($d$, $\mathbf{A}$), if there exists a sequence of random vectors $\{\hat{\Delta}_N\}_{N=1}^\infty$ asymptotically normal ($d$, $\mathbf{A}$) such that

$$(4.3)\qquad \lim_{N\to\infty} P\{\sup_{\Delta\in D_N} ||\Delta - \hat{\Delta}_N|| \geqq \varepsilon\} = 0$$

holds for any $\varepsilon > 0$.

We shall prove the theorem

THEOREM 4.1. *Under Assumptions* 1 *through* 6 *and* (2.12), *each point of set* $D_N$ *is asymptotically normal*

$$(4.4)\qquad (\Delta^0, \gamma^{-2}A^2\Sigma^{-1}).$$

PROOF. We shall prove the Theorem 4.1 by help of several lemmas; some of them having importance of their own.

One of the problems is that of boundedness of $D_N$. Lemma 4.4 solves this problem.

First of all, consider the following function of $\Delta$ for any fixed $Y_1, \cdots, Y_N$:

$$h_Y(\Delta) = \sum_{j=1}^K |S_{Nj}(Y-\Delta^0\mathbf{x})-\gamma(\Delta-\Delta^0)\sigma^{(j)}|.$$

Theorem 2.1 and 3.1 of [7], regarding the Assumption 5 of monotonicity of $\varphi$ imply that $\gamma \geqq 0$; if we may suppose that $\varphi$ is such that $\int_0^1\varphi(u)\varphi(u,f)\,du \neq 0$ then $\gamma > 0$ (this is the case when both $\varphi(u)$ and $\varphi(u,f)$ relate to strongly unimodal distributions). $h_Y(\Delta)$ is then easily seen to be a convex function of $\Delta$ which has a unique minimum. The point of minimum is equal to the solution of the system of equations

$$S_{Nj}(Y-\Delta^0\mathbf{x}) = \gamma(\Delta-\Delta^0)\sigma^{(j)}, \qquad j=1,2,\cdots,K. \tag{4.5}$$

Let us denote the solution of (4.5), which exists and is unique for any fixed $Y_1, \cdots, Y_N$ for which the left-hand sides are well defined, as $\hat{\Delta}_N$.

The asymptotic normality of $\mathbf{S}_N(Y-\Delta^0\mathbf{x})$ implies that the sequence $\{\hat{\Delta}_N\}$ is asymptotically normal

$$(\Delta^0, \gamma^{-2}A^2\Sigma^{-1}) \tag{4.6}$$

which is equal to (4.4).

LEMMA 4.1. *Under assumptions* 1 *through* 6, (2.12),

$$\lim_{N\to\infty} P_{\Delta^0}\{\sum_{j=1}^K |S_{Nj}(Y-\hat{\Delta}_N\mathbf{x})| \geqq \varepsilon\} = 0 \tag{4.7}$$

*holds for any* $\varepsilon > 0$.

PROOF. The Lemma is a direct consequence of Corollary 3.1 and of asymptotic normality (4.6).

COROLLARY 4.1. *If* $\varepsilon$ *is any positive number, then*

$$\lim_{N\to\infty} P_{\Delta^0}\{\min_{\Delta\in E^K}\sum_{j=1}^K |S_{Nj}(Y-\Delta\mathbf{x})| \geqq \varepsilon\} = 0. \tag{4.8}$$

LEMMA 4.2. *For any* $C > 0$, *denote as* $0_C$ *the set* $0_C = \{\Delta \in E^K; ||\Delta-\Delta^0|| \leqq C\}$. *Then under Assumptions* 1 *through* 6 *and* (2.12),

$$\lim_{N\to\infty} P_{\Delta^0}\{\sup_{\Delta\in D_N\cap 0_C} ||\Delta-\hat{\Delta}_N|| \geqq \varepsilon, D_N\cap 0_C \neq \varnothing\} = 0 \tag{4.9}$$

*holds for any* $C > 0$ *and* $\varepsilon > 0$.

PROOF. The continuity of the operator $\mathbf{\Sigma}^{-1}$ implies existence of $\delta > 0$ to given $\varepsilon > 0$ such that

$$\sum_{j=1}^K |S_{Nj}(Y-\Delta^0\mathbf{x})-\gamma(\Delta-\Delta^0)\sigma^{(j)}| \geqq \delta$$

holds for any $\Delta \in E^K$ satisfying $||\Delta-\hat{\Delta}_N|| \geqq \varepsilon$. Then

$$\begin{aligned} &P_{\Delta^0}\{\sup_{\Delta\in D_N\cap 0_C} ||\Delta-\hat{\Delta}_N|| \geqq \varepsilon, D_N\cap 0_C \neq \varnothing\} \\ &\quad \leqq P_{\Delta^0}\{\sup_{\Delta\in D_N\cap 0_C}\sum_{j=1}^K |S_{Nj}(Y-\Delta^0\mathbf{x})-\gamma(\Delta-\Delta^0)\sigma^{(j)}| \geqq \delta, D_N\cap 0_C \neq \varnothing\} \\ &\quad \leqq P_{\Delta^0}\{\sup_{\Delta\in D_N\cap 0_C}\sum_{j=1}^K |S_{Nj}(Y-\Delta^0\mathbf{x})-\gamma(\Delta-\Delta^0)\sigma^{(j)}| \geqq \delta, D_N\cap 0_C \neq \varnothing, \end{aligned}$$

$$\sup_{\Delta \in D_N \cap 0_C} \sum_{j=1}^K |S_{Nj}(Y - \Delta \mathbf{x})| < \delta/2\}$$
$$+ P_{\Delta^0}\{D_N \cap 0_C \neq \emptyset, \sup_{\Delta \in D_N \cap 0_C} \sum_{j=1}^K |S_{Nj}(Y - \Delta \mathbf{x})| \geqq \delta/2\}$$
$$\leqq P_{\Delta^0}\{\sup_{\Delta \in 0_C} |\sum_{j=1}^K |S_{Nj}(Y - \Delta^0 \mathbf{x}) - \gamma(\Delta - \Delta^0)\sigma^{(j)}| - \sum_{j=1}^K |S_{Nj}(Y - \Delta \mathbf{x})|| \geqq \delta/2\}$$
$$+ P_{\Delta^0}\{\min_{\Delta \in EK} \sum_{j=1}^K |S_{Nj}(Y - \Delta \mathbf{x})| \geqq \delta/2\}$$
$$\to 0 \qquad \text{for } N \to \infty,$$

for the first summand tends to zero in view of Corollary 3.1 and the second one in view of Corollary 4.1. □

LEMMA 4.3. *Let Assumptions* 1–6, (2.12) *be satisfied. Then there exist* $C^* > 0$, $\delta > 0$ *and index* $N_0$ *corresponding to any* $\varepsilon > 0$ *such that*

$$P_{\Delta^0}\{\min_{||\Delta - \Delta^0|| \geqq C^*} \sum_{j=1}^K |S_{Nj}(Y - \Delta \mathbf{x})| < \delta\} < \varepsilon \tag{4.10}$$

*holds for any* $N > N_0$.

PROOF. Let $M > 0$ be such that $\Phi(-M) < \varepsilon/8K$ where $\Phi$ is the standard normal cdf. Let $C^*$ and $\delta$ be any numbers satisfying the inequalities

$$C^* \geqq 2M \cdot (K \cdot A^2)^{\frac{1}{2}} \cdot (\lambda_0 \gamma)^{-1}; \qquad \delta \leqq M/2(K \cdot A^2)^{\frac{1}{2}} \tag{4.11}$$

where $\lambda_0$ is the minimal eigenvalue of the matrix $\Sigma$.

We shall prove at first the existence of $N_0$ such that

$$P_{\Delta^0}\{\min_{||\Delta - \Delta^0|| = C^*} \sum_{j=1}^K (\Delta_j^0 - \Delta_j) S_{Nj}(Y - \Delta \mathbf{x}) < \delta^*\} < \varepsilon \tag{4.12}$$

holds for all $N > N_0$ with $\delta^* = \delta C^*$.

The left-hand side of (4.12) is bounded from above by the sum

$$\begin{aligned} &P_{\Delta^0}\{\min_{||\Delta - \Delta^0|| = C^*} \sum_{j=1}^K (\Delta_j^0 - \Delta_j) S_{Nj}(Y - \Delta \mathbf{x}) < \delta^*, \\ &\quad \min_{||\Delta - \Delta^0|| = C^*} \sum_{j=1}^K (\Delta_j^0 - \Delta_j)[S_{Nj}(Y - \Delta^0 \mathbf{x}) + \gamma(\Delta^0 - \Delta)\sigma^{(j)}] \geqq 2\delta^*\} \\ &+ P_{\Delta^0}\{\min_{||\Delta - \Delta^0|| = C^*} \sum_{j=1}^K (\Delta_j^0 - \Delta_j)[S_{Nj}(Y - \Delta^0 \mathbf{x}) + \gamma(\Delta^0 - \Delta)\sigma^{(j)}] < 2\delta^*. \end{aligned} \tag{4.13}$$

The first term of (4.13) may be bounded from above by probability

$$\begin{aligned} &P_{\Delta^0}\{\max_{||\Delta - \Delta^0|| = C^*} \sum_{j=1}^K (\Delta_j^0 - \Delta_j)[S_{Nj}(Y - \Delta^0 \mathbf{x}) + \gamma(\Delta^0 - \Delta)\sigma^{(j)} \\ &\quad - S_{Nj}(Y - \Delta \mathbf{x})] \geqq \delta^*\} \\ &\leqq P_{\Delta^0}\{\max_{||\Delta - \Delta^0|| = C^*} \sum_{j=1}^K |S_{Nj}(Y - \Delta^0 \mathbf{x}) + \gamma(\Delta^0 - \Delta)\sigma^{(j)} \\ &\quad - S_{Nj}(Y - \Delta \mathbf{x})| \geqq \delta^*/C^*\}. \end{aligned} \tag{4.14}$$

The last inequality is valid in view of inequality

$$\begin{aligned} &\sum_{j=1}^K (\Delta_j^0 - \Delta_j)[S_{Nj}(Y - \Delta^0 \mathbf{x}) + \gamma(\Delta^0 - \Delta)\sigma^{(j)} - S_{Nj}(Y - \Delta \mathbf{x})] \\ &\quad \leqq \sum_{j=1}^K |S_{Nj}(Y - \Delta^0 \mathbf{x}) + \gamma(\Delta^0 - \Delta)\sigma^{(j)} - S_{Nj}(Y - \Delta \mathbf{x})| \cdot (\max_{1 \leqq j \leqq K} |\Delta_j^0 - \Delta_j|). \end{aligned}$$

The right-hand side of (4.14) tends to zero by Theorem 3.1 and this means that the first term of (4.13) tends to zero.

The second term of (4.13) is not greater than

$$P_{\Delta^0}\{\min_{||\Delta-\Delta^0||=C^*}\textstyle\sum_{j=1}^K(\Delta_j{}^0-\Delta_j)S_{Nj}(Y-\Delta^0\mathbf{x})+(C^*)^2\lambda_0\gamma<2\delta^*\}$$
$$\leqq P_{\Delta^0}\{-C^*[\textstyle\sum_{j=1}^K S_{Nj}^2(Y-\Delta^0\mathbf{x})]^{\frac{1}{2}}<2\delta^*-(C^*)^2\lambda_0\gamma\}$$
$$\leqq \textstyle\sum_{j=1}^K P_{\Delta^0}\{|S_{Nj}(Y-\Delta^0\mathbf{x})|>C_0/K^{\frac{1}{2}}\}$$

where $C_0 = -2\delta+C^*\lambda_0\gamma$. The first inequality follows from Schwarz's inequality.

It is known that $S_{Nj}(Y-\Delta^0\mathbf{x})$ are asymptotically normal $(0, A^2)$, so that there exists $N_1$ such that $\sum_{j=1}^K P_{\Delta^0}\{|S_{Nj}(Y-\Delta^0\mathbf{x})|>C_0/K^{\frac{1}{2}}\}<2K\cdot\Phi(-C_0/(KA^2)^{\frac{1}{2}}+\varepsilon/4\leqq 2K\Phi(-M)+\varepsilon/4<\varepsilon/2$ holds for all $N>N_1$, so that the second term of (4.13) also tends to zero. (4.12) is proved.

If $\Delta^1$ is any point such that $||\Delta^1-\Delta^0|| = C^*$ and we denote as

$$x_i{}^* = \textstyle\sum_{j=1}^K(\Delta_j{}^0-\Delta_j{}^1)x_{ji}, \qquad Y_i{}^* = Y_i-\Delta^0\mathbf{x}^{(i)}, \qquad i=1,2,\cdots,N$$

then we may consider the statistic $S(\tau) = \sum_{i=1}^N(x_i{}^*-\bar{x}^*)a_N(R_i{}^\tau)$ where $(R_1{}^\tau,\cdots,R_N{}^\tau)$ is the vector of ranks for variables $Y_i{}^*+\tau x_i{}^*$, $i = 1, 2, \cdots, N$. $S(\tau)$ is non-decreasing function of $\tau$ for any fixed $Y_1,\cdots,Y_N$, as it follows from Theorem 2.1 of [7], so that

$$\text{(4.15)}\quad \textstyle\sum_{j=1}^K(\Delta_j{}^0-\Delta_j{}^1)S_{Nj}(Y-[\Delta^0+\tau(\Delta^1-\Delta^0)]\mathbf{x}) = S(\tau)\geqq S(1)$$
$$=\textstyle\sum_{j=1}^K(\Delta_j{}^0-\Delta_j{}^1)S_{Nj}(Y-\Delta^1\mathbf{x}) \qquad \text{for any } \tau\geqq 1.$$

If $||\Delta-\Delta^0||\geqq C^*$ then $||\Delta^1-\Delta^0|| = C^*$ for

$$\text{(4.16)}\qquad \Delta^1 = \Delta^0+[C^*/||\Delta-\Delta^0||](\Delta-\Delta^0)$$

and $\Delta = \Delta^0+\tau(\Delta^1-\Delta^0)$ for

$$\text{(4.17)}\qquad \tau = ||\Delta-\Delta^0||/C^*\geqq 1.$$

(4.15), (4.16) and (4.17) then imply

$$\text{(4.18)}\quad \textstyle\sum_{j=1}^K(\Delta_j{}^0-\Delta_j{}^1)S_{Nj}(Y-\Delta\mathbf{x})\geqq\sum_{j=1}^K(\Delta_j{}^0-\Delta_j{}^1)S_{Nj}(Y-\Delta^1\mathbf{x})$$
$$-\min_{\{\Delta^1;||\Delta^1-\Delta^0||=C^*\}}\textstyle\sum_{j=1}^K(\Delta_j{}^0-\Delta_j{}^1)S_{Nj}(Y-\Delta^1\mathbf{x})$$

and this further implies the inequalities

$$P_{\Delta^0}\{\min_{||\Delta-\Delta^0||\geqq C^*}\textstyle\sum_{j=1}^K|S_{Nj}(Y-\Delta\mathbf{x})|<\delta\}$$
$$\leqq P_{\Delta^0}\{\min_{||\Delta-\Delta^0||\geqq C^*}(C^*/||\Delta^0-\Delta||)[\textstyle\sum_{j=1}^K(\Delta_j{}^0-\Delta_j)S_{Nj}(Y-\Delta\mathbf{x})]<\delta C^*\}$$
$$\leqq P_{\Delta^0}\{\min_{\{\Delta^1;||\Delta^1-\Delta^0||=C^*\}}\textstyle\sum_{j=1}^K(\Delta_j{}^0-\Delta_j{}^1)S_{Nj}(Y-\Delta^1\mathbf{x})<\delta C^*\}<\varepsilon$$
$$\text{for } N>N_0.$$

The first inequality is valid in view of the inequality

$$|\textstyle\sum_{j=1}^K(\Delta_j{}^0-\Delta_j{}^1)S_{Nj}(Y-\Delta\mathbf{x})|\leqq\sum_{j=1}^K|S_{Nj}(Y-\Delta\mathbf{x})|\cdot(\max_{1\leqq j\leqq K}|\Delta_j{}^0-\Delta_j{}^1|).$$

The second inequality follows from (4.16), (4.17) and (4.18) and the third one from (4.12). The proof is complete.

LEMMA 4.4. *Under Assumptions* 1 *through* 6, (2.12), *if* $\bar{0}_C = \{\boldsymbol{\Delta} \in E^K;\ ||\boldsymbol{\Delta}-\boldsymbol{\Delta}^0|| \geqq C\}$ *for any* $C > 0$, *then, corresponding to any* $\varepsilon > 0$, *there exist* $C^* > 0$ *and a positive integer* $N_0$ *such that*

$$P_{\Delta^0}\{D_N \cap \bar{0}_{C^*} \neq \varnothing\} < \varepsilon \tag{4.19}$$

*holds for any* $N > N_0$.

PROOF. If $\varepsilon > 0$ is given, then Lemma 4.3 guarantees the existence of $C^* > 0$, $\delta > 0$ and $N_1$ such that

$$P_{\Delta^0}\{\min_{||\Delta-\Delta^0|| \geqq C^*} \sum_{j=1}^K |S_{Nj}(Y-\boldsymbol{\Delta}\mathbf{x})| < \delta\} < \varepsilon/2 \tag{4.20}$$

holds for any $N > N_1$.

Corollary 4.1 further implies the existence of $N_2$ such that

$$P_{\Delta^0}\{\min_{\Delta \in E^K} \sum_{j=1}^K |S_{Nj}(Y-\boldsymbol{\Delta}\mathbf{x})| \geqq \delta\} < \varepsilon/2 \tag{4.21}$$

holds for all $N > N_2$.

Then

$$\begin{aligned} &P_{\Delta^0}\{D_N \cap \bar{0}_{C^*} \neq \varnothing\} \\ &\quad = P_{\Delta^0}\{D_N \cap \bar{0}_{C^*} \neq \varnothing, \min_{||\Delta-\Delta^0|| \geqq C^*} \sum_{j=1}^K |S_{Nj}(Y-\boldsymbol{\Delta}\mathbf{x})| < \delta\} \\ &\qquad + P_{\Delta^0}\{D_N \cap \bar{0}_{C^*} \neq \varnothing, \min_{||\Delta-\Delta^0|| \geqq C^*} \sum_{j=1}^K |S_{Nj}(Y-\boldsymbol{\Delta}\mathbf{x})| \geqq \delta\} \\ &\quad \leqq P_{\Delta^0}\{\min_{||\Delta-\Delta^0|| \geqq C^*} \sum_{j=1}^K |S_{Nj}(Y-\boldsymbol{\Delta}\mathbf{x})| < \delta\} \\ &\qquad + P_{\Delta^0}\{\min_{\Delta \in E^K} \sum_{j=1}^K S_{Nj}(Y-\boldsymbol{\Delta}\mathbf{x})| \geqq \delta\} < \varepsilon \end{aligned}$$

for all $N > N_0 = \max(N_1, N_2)$. □

LEMMA 4.5. *Under Assumptions* 1 *through* 6, (2.12)

$$\lim_{N\to\infty} P_{\Delta^0}\{\sup_{\Delta \in D_N} ||\boldsymbol{\Delta}-\hat{\boldsymbol{\Delta}}_N|| \geqq \varepsilon\} = 0 \tag{4.22}$$

*holds for any* $\varepsilon > 0$.

PROOF. Let $\varepsilon > 0$ and $\eta > 0$ be given. Lemma 4.4 guarantees existence of $C^* > 0$ and $N_1$ such that $P_{\Delta^0}\{D_N \cap \bar{0}_{C^*} \neq \varnothing\} < \eta/4$ for all $N > N_1$. By Lemma 4.2 there exists $N_2$ such that $P_{\Delta^0}\{\sup_{\Delta \in D_N \cap 0_{C^*}} ||\boldsymbol{\Delta}-\hat{\boldsymbol{\Delta}}_N|| \geqq \varepsilon,\ D_N \cap 0_{C^*} \neq \varnothing\} < \eta/2$ for all $N > N_2$. We then have for all $N > \max(N_1, N_2)$

$$\begin{aligned} &P_{\Delta^0}\{\sup_{\Delta \in D_N} ||\boldsymbol{\Delta}-\hat{\boldsymbol{\Delta}}_N|| \geqq \varepsilon\} \\ &\quad \leqq P_{\Delta^0}\{\sup_{\Delta \in D_N} ||\boldsymbol{\Delta}-\hat{\boldsymbol{\Delta}}_N|| \geqq \varepsilon, D_N \cap 0_{C^*} \neq \varnothing\} + P_{\Delta^0}\{D_N \cap \bar{0}_{C^*} \neq \varnothing\} \\ &\quad \leqq P_{\Delta^0}\{\sup_{\Delta \in D_N \cap 0_{C^*}} ||\boldsymbol{\Delta}-\hat{\boldsymbol{\Delta}}_N|| \geqq \varepsilon, D_N \cap 0_{C^*} \neq \varnothing\} \\ &\qquad + P_{\Delta^0}\{\sup_{\Delta \in D_N} ||\boldsymbol{\Delta}-\hat{\boldsymbol{\Delta}}_N|| \geqq \varepsilon, \sup_{\Delta \in D_N \cap 0_{C^*}} ||\boldsymbol{\Delta}-\hat{\boldsymbol{\Delta}}_N|| < \varepsilon, D_N \cap 0_{C^*} \neq \varnothing\} \\ &\qquad + P_{\Delta^0}\{D_N \cap \bar{0}_{C^*} \neq \varnothing\} \\ &\quad \leqq P_{\Delta^0}\{\sup_{\Delta \in D_N \cap 0_{C^*}} ||\boldsymbol{\Delta}-\hat{\boldsymbol{\Delta}}_N|| \geqq \varepsilon, D_N \cap 0_{C^*} \neq \varnothing\} \\ &\qquad + 2P_{\Delta^0}\{D_N \cap \bar{0}_{C^*} \neq \varnothing\}. \quad □ \end{aligned}$$

Theorem 4.1 is then an immediate consequence of Lemma 4.5 and of the asymptotic normality (4.6) of $\hat{\mathbf{\Delta}}_N$.

Usual choice of $\varphi(u)$ if $\varphi(u) = \varphi(u, f_0)$ where $f_0$ is a supposed density possessing finite Fisher's information. $\varphi(u, f_0)$ is then and only then non-decreasing if $f_0$ is strongly unimodal.

So we see that if the supposed density is equal to the actual one then our estimates have the same asymptotic distribution as the maximal-likelihood estimates. However, the theory of maximal-likelihood estimates has not been worked out under such weak conditions up to this time (i.e. the monotonicity of $\varphi(u)$ and finiteness of Fisher's information). The behavior of maximal-likelihood estimates has not also been treated in case that the supposed density differs from the actual one.

**5. Estimation of parameter $\alpha$.** Suppose that we are given by some specific rule of choice of unique point from $D_N$, e.g. the center of gravity of $D_N$; let $\bar{\mathbf{\Delta}}_N$ be such point.

We add to the Assumptions 1–6 and (2.12) of Section 2 the assumption of symmetry of $f(y)$ about zero and of $\sum_{i=1}^N x_{ji} = 0$, $j = 1, 2, \cdots, K$; $N = 2, 3, \cdots$. Put

(5.1) $$\varphi^+(u,f) = -f'[F^{-1}((u+1)/2)]/f(F^{-1}((u+1)/2)), \qquad 0 < u < 1.$$

We propose as an estimate of $\alpha$

(5.2) $$\hat{\alpha}_N = \tfrac{1}{2}(\alpha_N{}^* + \alpha_N{}^{**})$$

where

(5.3) $$\alpha_N{}^* = \sup\{\alpha; S_N{}^+(Y-\alpha-\bar{\mathbf{\Delta}}_N\mathbf{x}) > 0\}$$
$$\alpha_N{}^{**} = \inf\{\alpha; S_N{}^+(Y-\alpha-\bar{\mathbf{\Delta}}_N x) < 0\}$$

(see [1] for one-dimensional $\Delta$). Here $S_N{}^+(Y-\alpha-\bar{\mathbf{\Delta}}\mathbf{x})$ denotes the rank statistic

(5.4) $$S_N{}^+(Y-\alpha-\mathbf{\Delta}\mathbf{x}) = \textstyle\sum_{i=1}^N \operatorname{sgn}(Y_i-\alpha-\mathbf{\Delta}\mathbf{x}^{(i)})a_N{}^+(\mathscr{R}_i{}^{\Delta,\alpha})$$

where $\mathscr{R}_1{}^{\Delta,\alpha}, \cdots, \mathscr{R}_N{}^{\Delta,\alpha}$ is the vector of ranks for variables $|Y_i-\alpha-\mathbf{\Delta}\mathbf{x}^{(i)}|$, $i = 1, 2, \cdots, N$; the scores $a_N{}^+$ are generated by a nonnegative non-decreasing square-integrable function $\varphi^+(u)$, $0 < u < 1$, about either (2.7) or (2.8).

The study of the asymptotic normality of $\hat{\alpha}_N$ is based on results of Hodges–Lehmann [5] (Theorem 1 and Theorem 4) and on a result of [8] (Theorem 1 and Corollary 1). First of all, we see that $\bar{\mathbf{\Delta}}_N$ has the invariance property $\bar{\mathbf{\Delta}}_N(Y+a+\mathbf{b}\mathbf{x}) = \bar{\mathbf{\Delta}}_N(Y)+\mathbf{b}$ for any $a$, $\mathbf{b}$ and that the distribution of $\hat{\alpha}_N$ is continuous. We then have the inequalities analogous to (9.1) of [5].

(5.5) $$P\{S_N{}^+(Y-a-\bar{\mathbf{\Delta}}_N\mathbf{x}) < 0\} \leqq P\{\hat{\alpha}_N < a\} \leqq P\{S_N{}^+(Y-a-\bar{\mathbf{\Delta}}_N\mathbf{x}) \leqq 0\}$$

which lead to the conclusion analogous to that of Theorem 4 of [5]:

(5.6) $$\lim_{N\to\infty} P_{\alpha,\Delta^0}\{N^{\frac{1}{2}}(\hat{\alpha}_N-\alpha) \leqq a\} = \lim_{N\to\infty} P_{0,0}\{S_N{}^+(Y-a/N^{\frac{1}{2}}-\bar{\mathbf{\Delta}}_N\mathbf{x}) \leqq 0\}.$$

Theorem 1 and Corollary 1 of [8] imply

$$\lim_{N\to\infty} P_{0,0}\{|S_N^+(Y-\bar{\Delta}_N\mathbf{x})-S_N^+(Y)| \geqq \varepsilon N^{\frac{1}{2}}\} = 0 \quad \text{for any } \varepsilon > 0$$

and this together with the contiguity of $q_N = \prod_{i=1}^N f(y+a/N^{\frac{1}{2}})$ to $p_N = \prod_{i=1}^N f(y$ implies

$$\lim_{N\to\infty} P_{0,0}\{|S_N^+(Y-a/N^{\frac{1}{2}}-\bar{\Delta}_N\mathbf{x})-S_N^+(Y-a/N^{\frac{1}{2}})| \geqq \varepsilon N^{\frac{1}{2}}\} = 0 \tag{5.7}$$

for any $\varepsilon > 0$. (5.7) together with Theorem VI.2.5 of [4] shows that

$$N^{-\frac{1}{2}}S_N^+\,(Y-a/N^{\frac{1}{2}}-\Delta_N\mathbf{x})$$

is asymptotically normal $(-a\int_0^1\varphi^+(u)\varphi^+(u,f)\,du,\ \int_0^1[\varphi^+(u)]^2\,du)$. Regarding (5.6), the conclusion is that $\hat{\alpha}_N$ is asymptotically normal

$$(\alpha, N^{-1}\int_0^1[\varphi^+(u)]^2\,du\cdot[\int_0^1\varphi^+(u)\varphi^+(u,f)\,du]^{-2}). \tag{5.8}$$

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