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SOME ASYMPTOTIC RESULTS IN A MODEL OF POPULATION GROWTH
II. POSITIVE RECURRENT CHAINS[1]

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We treat a model describing the continued formation and growth of mutant biological populations. At each transition time of a Poisson process a new mutant population begins its evolution with a fixed number of elements and evolves according to the laws of a continuous time positive recurrent Markov Chain $Y(t)$ with stationary transition probabilities $P_{ik}(t)$, $i, k = 0, 1, 2, \cdots, t \geqq 0$. Our principal concern is the asymptotic behavior of moments and of the distribution function of the functional $S(t) = \{$number of different sizes of mutant populations at time $t\}$. When the recurrence time distribution to any state of the Markov Chain $Y(t)$ has a finite second moment, the moments of $S(t)$ and limit behavior of its distribution function are controlled by the stationary measure associated with $Y(t)$. When the second moment of the recurrence time distribution is infinite, then a local limit theorem and speed of convergence estimate for $P_{ik}(t)$ with $k = k(t) \to \infty$, $t \to \infty$ are required to establish asymptotic formulas for moments of $S(t)$.

1. Introduction. We continue the study begun in [3], [6] of a model describing the continued formation and growth of mutant biological populations. Our basic structure assumes that a new mutant population begins its evolution at each transition time of a non-decreasing integer valued stochastic process $\{n(t), t > 0\}$. Each new mutant population begins its evolution with a fixed number of elements and evolves according to the laws of a continuous time Markov chain $\{Y(t), t > 0\}$ with stationary transition probabilities

$$P_{ik}(t) \quad i, k = 0, 1, 2, \cdots; t \geqq 0.$$

We assume that all populations evolve according to the same Markov chain, independent of each other and of the process $\{n(t), t > 0\}$.

In this paper we again consider limit behavior of the special functional

$$S_{n(t)} = \text{number of different sizes of mutant populations at time } t$$

of the vector process

$$\mathbf{N}(t) = \{N_0'(t), N_1(t), \cdots\}$$

where

$$N_k(t) = \text{number of populations with exactly } k \text{ elements at time } t.$$

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Our treatment of $S_{n(t)}$ is the prototype for functionals of $\mathbf{N}(t)$ having the form

$$\mathbf{g}(\mathbf{N}(t)) = \sum_{k=0}^{\infty} g(N_k(t))$$

where $g(x)$ is the indicator function of a Borel set in $[0, \infty)$. Adopting the terminology of [6], $S_{n(t)}$ will be referred to as the "occupied states" process generated by the input process $\{n(t), t > 0\}$ and the growing process $\{Y(t), t > 0\}$. Without loss of generality we assume that all chains begin their evolution in state 0. In contrast with [6] we assume that $\{Y(t), t > 0\}$ is a general continuous time positive recurrent Markov chain with all stable states and that the input process is Poisson with parameter $\lambda = 1$. To simplify the notation we write $S(t)$ for the occupied states process associated with a Poisson input process.

Our primary purpose is to establish asymptotic formulas for the moments of $S(t)$ and to discuss its limit behavior as $t \to \infty$. The asymptotic moment formulas for $S(t)$ generated by positive recurrent growing processes are qualitatively different from those associated with null recurrent and transient processes [6]. In particular, if $\{Y(t), t > 0\}$ has a recurrence time distribution to any state, say 0, with a finite second moment ($m_{00}^{(2)} < \infty$), then asymptotic moment formulas of $S(t)$ are controlled exclusively by the stationary measure $\{p_k\}_{k=0}^{\infty}$ associated with $Y(t)$. This dependence is made precise in

THEOREM 4.1. *If the growing process associated with $\{S(t), t > 0\}$ is a positive recurrent Markov chain with all stable states and $m_{00}^{(2)} < \infty$, then*

(i)
$$\lim_{t\to\infty} \frac{ES^k(t)}{\left(\sum_{j=0}^{\infty} 1 - e^{-p_j t}\right)^k} = 1. \tag{1}$$

(ii) *If the stationary measure also satisfies the condition*

$$\alpha(x) = \max\,(k : p_k \geqq 1/x) = x^{\gamma} L(x)$$

where $L(x)$ is slowly varying and $0 < \gamma \leqq 1$, then

(a)
$$\sum_{j=0}^{\infty} (1 - e^{-p_j t}) \sim h_\gamma(t), \qquad t \to \infty \tag{2}$$

where

$$h_\gamma(t) = \Gamma(1-\gamma) t^{\gamma} L(t) \qquad \text{if } 0 < \gamma < 1,$$
$$= t \int_0^{\infty} \frac{e^{-1/y}}{y} L(ty)\,dy \qquad \text{if } \gamma = 1,$$

(b)
$$E(S(t) - ES(t))^{2k}$$
$$\sim \left[\sum_{j=0}^{\infty} \left(e^{-p_j t} - e^{-2p_j t}\right)\right]^k \frac{(2k)!}{k!2^k}$$
$$\sim \frac{(2k)!}{k!2^k} (2^{\gamma} - 1)^k h_\gamma^{\,k}(t), \qquad t \to \infty, k = 1, 2, \cdots. \tag{3}$$

(c) $$E(S(t)-ES(t))^{2k-1}$$

$$\sim \frac{(2k-1)!}{6\cdot 2^{k-2}(k-2)!}(2^\gamma-1)^{k-2}h_\gamma^{k-1}(t)\frac{(c_1-3c_2+2c_3)}{\Gamma(1-\gamma)}, \qquad 0<\gamma<1$$

$$\sim \frac{(2k-1)!t}{6\cdot 2^{k-2}(k-2)!}h_\gamma^{k-2}(t)(d_1(t)-3d_2(t)+2d_3(t)), \qquad \gamma=1 \qquad (4)$$

where

$$c_m = \int_0^\infty m y^{\gamma-2} e^{-1/y}(1-e^{-1/y})^{m-1}\,dy$$

$$d_m(t) = \int_0^\infty L(ty)\frac{me^{-1/y}}{y}(1-e^{-1/y})^{m-1}dy, \qquad m=1,2,3,\cdots.$$

Formulas (1)–(4) have an interesting interpretation in terms of an infinite urn scheme. In particular, if we assume that at each event time of a Poisson process one ball is thrown at an infinite array of cells with probability p_k of hitting the kth cell, then

$$E\hat{S}(t) = \sum_{j=0}^\infty (1-e^{-p_j t})$$

where

$$\hat{S}(t) = \text{number of occupied cells at time } t$$

and

$$\operatorname{Var}\hat{S}(t) = \sum_{j=0}^\infty (e^{-p_j t}-e^{-2p_j t}).$$

The infinite urn scheme has served as a model problem for Theorem 4.1 and the limit theorem,

THEOREM 6.1. *If the hypotheses of Theorem* 4.1 (ii) *hold, then*

(a) $$\lim_{t\to\infty} P\left(\frac{S(t)-ES(t)}{(\operatorname{Var}S(t))^{\frac{1}{2}}} \leqq x\right) = \Phi(x) \qquad (5)$$

where

$$\Phi(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^x e^{-y^2/2}\,dy.$$

(b) $$\lim_{t\to\infty}\frac{S(t)}{h_\gamma(t)} = 1 \quad \textit{with probability } 1. \qquad (6)$$

An extensive discussion of limit theorems for the infinite urn scheme can be found in [2].

For positive recurrent chains $Y(t)$ with $m_{00}^{(2)} = \infty$, control of moments and limit distributions of $S(t)$ by the stationary measure may break down, as indicated in the examples of Section 5. Asymptotic formulas analogous to Theorem 4.1 require the use of local limit theorems for the transition probabilities of the growing process $Y(t)$. We establish such formulas for a special class of birth and death processes, deferring a complete treatment of the case $m_{00}^{(2)} = \infty$ to a separate work.

2. Outline. Section 3 contains the basic facts about positive recurrent Markov chains which are required to check the moment formulas of Theorem 4.1. In Section 4 we prepare the asymptotic relations

$$\lim_{t\to\infty}\frac{E(S^k(t))}{(ES(t))^k}=1 \qquad k=1,2,\cdots, \tag{7}$$

$$\lim_{t\to\infty}\frac{k!2^k}{(2k)!}\frac{E(S(t)-ES(t))^{2k}}{[E(S(t)-ES(t))^2]^k}=1 \qquad k=1,2,\cdots, \tag{8}$$

provided

$$\alpha(x)=x^{\gamma}L(x), \qquad 0<\gamma\leqq 1,$$

$$\lim_{t\to\infty}\frac{(k-2)!}{(2k-1)!}\frac{6\cdot 2^{k-2}E(S(t)-ES(t))^{2k-1}}{E((S(t)-ES(t))^2)^{k-2}E(S(t)-ES(t))^3}=1, \tag{9}$$

provided

$$\alpha(x)=x^{\gamma}L(x), \qquad 0<\gamma\leqq 1 \qquad k=2,3,\cdots.$$

This reduces the computation of explicit formulas for kth moments to evaluation of at most the third central moment. Relations (7)–(9) are immediate consequences of the representation

$$S(t)=\sum_{k=0}^{\infty}X_k(t)$$

where

$$X_k(t)=1 \quad \text{if} \quad N_k(t)>0,$$
$$=0 \quad \text{if} \quad N_k(t)=0.$$

$\{N_k(t)\}_{k=0}^{\infty}$ are independent Poisson random variables with parameters $\int_0^t P_{0k}(s)\,ds$, $k=0,1,2,\cdots$ respectively.

For input processes $n(t)$ other than Poisson, $S_{n(t)}$ cannot be represented as a sum of independent random variables. Formulas analogous to (1), (3), (4), and (7)–(9) require delicate estimates of the strength of dependence among the binary valued random variables

$$X_k(t)=1 \quad \text{if} \quad N_k(t)>0,$$
$$=0 \quad \text{if} \quad N_k(t)=0, \qquad \{N_k(t)\}_{k=0}^{\infty}\ \textit{dependent}$$

generating

$$S_{n(t)}=\sum_{k=0}^{\infty}X_k(t).$$

General input processes, dependence estimates, and associated limit theorems will be treated in a separate paper. Section 4 concludes with a proof of Theorem 4.1.

Section 5 contains some preliminary facts and a local limit theorem for positive recurrent birth and death processes and then indicates their role in computing asymptotic formulas for moments of $S(t)$ when $m_{00}^{(2)}=\infty$. The central limit theorem and strong law of Theorem 6.1 conclude the paper.

3. Positive recurrent chains: Inequalities and identities. The inequalities and identities in Lemmas 3.1–3.3 are essential for the proofs in Sections 4 and 5. Our terminology is that of Chung [1].

LEMMA 3.1. *For a positive recurrent continuous time Markov chain with all stable states*

$$P_{0k}(t)-p_k = {}_0P_{0k}(t)+q_0\int_0^t (r_{00}(t-s)-p_0)_0P_{0k}(s)\,ds-p_0q_0\int_t^\infty {}_0P_{0k}(s)\,ds \qquad (10)$$

where

$$p_k = \lim_{t\to\infty} P_{ik}(t), \qquad i,k = 0,1,2,\cdots,$$
$$r_{ik}(t-s) = P(X(t) = k \mid \rho_i = s),$$
$$\rho_i = \inf(t: t > 0, X(t) \neq i, X(0) = i),$$
$$= \text{first exit time from the initial state } i.$$
$${}_HP_{ik}(t) = \delta_{ik}e^{-q_it}+P(\rho_i(\omega) < t, X(s)\notin H, \rho_i(\omega) < s < t; X(t) = k \mid X(0) = i)$$

and H is an arbitrary possibly empty set.

$$q_i = \lim_{t\to 0}\frac{1-P_{ii}(t)}{t}.$$

PROOF. A standard last entrance decomposition yields the identity

$$P_{0k}(t) = {}_0P_{0k}(t)+\int_0^t q_0r_{00}(s)_0P_{0k}(t-s)\,ds. \qquad (11)$$

(See Chung [1] page 239 for a proof.)

Applying the identities (12) and (19) on page 216 in Chung [1] yields the formulas

$$p_k = \frac{p_0\int_0^\infty {}_0P_{kk}(s)\,ds}{\int_0^\infty {}_kP_{00}(s)\,ds} \qquad (12)$$
$$= \frac{p_0q_0\int_0^\infty {}_0P_{0k}(s)\,ds}{F_{0k}(\infty)}$$

where

$$F_{ik}(t) = P(\alpha_{ik} \leqq t \mid X(0) = i)$$

and

$$\alpha_{ik} = \inf(t: t > \rho_i(\omega), X(t) = k)$$
$$= \text{first entrance time to state } k \text{ after an exit from } i.$$

Since we are considering only positive recurrent Markov chains, $F_{0k}(\infty) = 1$. Combining this fact with (11) and (12) yields

$$P_{0k}(t)-p_k = {}_0P_{0k}(t)+q_0\int_0^t (r_{00}(t-s)-p_0)_0P_{0k}(s)\,ds-p_0q_0\int_t^\infty {}_0P_{0k}(s)\,ds$$

and the proof is complete.

LEMMA 3.2. *Given a positive recurrent irreducible continuous time Markov chain with all stable states and* $\varepsilon > 0$ *arbitrary* $\exists\, T(\varepsilon)$ *independent of* k *such that for* $t > T(\varepsilon)$

$$-\varepsilon p_k t - p_0 q_0 \int_0^t dw \int_w^\infty {}_0P_{0k}(s)\,ds \leqq \int_0^t (P_{0k}(s) - p_k)\,ds \leqq \varepsilon p_k t. \tag{13}$$

PROOF. Integrating the identity (10) with respect to t, deleting the last term, and bringing in (12) we obtain the inequality

$$\begin{aligned}\int_0^t (P_{0k}(s) - p_k)\,ds &\leqq \int_0^t {}_0P_{0k}(s)\,ds + \int_0^t ds \int_0^s q_0 |r_{00}(s-w) - p_0|\, {}_0P_{0k}(w)\,dw \\ &\leqq \frac{p_k}{p_0 q_0} + \frac{p_k t}{p_0}\left[\frac{1}{t}\int_0^t |r_{00}(s) - p_0|\,ds\right].\end{aligned} \tag{14}$$

By Theorem 8, page 237 in Chung [1], $\lim_{t\to\infty} r_{00}(t) = p_0$.

Thus

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t |r_{00}(s) - p_0|\,ds = 0 \tag{15}$$

and the upper bound

$$\int_0^t (P_{0k}(s) - p_k)\,ds \leqq \varepsilon p_k t$$

follows for $t >$ some $T(\varepsilon)$.

For the lower bound we again use the identity (10) to obtain

$$\begin{aligned}\int_0^t (P_{0k}(s) - p_k)\,ds \geqq -\frac{p_k t}{p_0}\left[\frac{1}{t}\int_0^t |r_{00}(s) - p_0|\,ds\right] - p_0 q_0 \int_0^t ds \int_s^\infty {}_0P_{0k}(w)\,dw.\end{aligned} \tag{16}$$

An application of (15) in the first term on the right-hand side of (16) completes the proof.

LEMMA 3.3. *For an irreducible continuous time Markov chain with all stable states*

$$\sum_{k=1}^\infty {}_0P_{0k}(s) = 1 - F_{00}(s), \qquad \forall s > 0. \tag{17}$$

PROOF. Let

$$\alpha_{00}(\omega) = \inf(t : t > \rho_0(\omega), X(t,\omega) = 0)$$

and

$$\Delta_0 = (\omega : X(0,\omega) = 0).$$

Then

$$\begin{aligned}&(\omega : \alpha_{00}(\omega) > t) \cap \Delta_0 \\ &\quad = (\omega : X(s,\omega) \neq 0, \min[t, \rho_0(\omega)] < s \leqq t) \cap \Delta_0 \\ &\quad = \textstyle\bigcup_{k=1}^\infty (\omega : X(s,\omega) \neq 0, \min[t, \rho_0(\omega)] < s < t, X(t,\omega) = k) \cap \Delta_0.\end{aligned}$$

Recalling the definition of taboo probabilities ${}_0P_{0k}(t)$ as given in the statement of Lemma 3.1, we have

$$\begin{aligned} P(\alpha_{00}(\omega) > t \mid \Delta_0)P(\Delta_0) \\ = (1-F_{00}(t))P(\Delta_0) \\ = \textstyle\sum_{k=1}^{\infty} {}_0P_{0k}(t)P(\Delta_0). \end{aligned}$$

4. Asymptotic moment formulas when $m_{00}^{(2)} < \infty$. We prepare the

LEMMA 4.1. *If* $\{S(t),\ t > 0\}$ *is an occupied states process generated by an* arbitrary *irreducible continuous time Markov chain with all stable states and a Poisson input process with parameter* $\lambda = 1$, *then*

(i) $$E(S^k(t)) = (ES(t))^k + O((ES(t))^{k-1}). \tag{18}$$

(ii) $$E(S(t)-ES(t))^{2k} = \left(\sum_{j=0}^{\infty} EY_j^2(t)\right)^k \frac{(2k)!}{k!2^k} + O\left(\left(\sum_{j=0}^{\infty} EX_j(t)\right)^{k-1}\right) \tag{19}$$

where

$$\begin{aligned} X_j(t) &= 1 \qquad \text{if} \quad N_j(t) > 0, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and $Y_j(t) = X_j(t) - EX_j(t)$.

(iii) $$\begin{aligned} E(S(t)-ES(t))^{2k-1} = \left(\sum_{j=0}^{\infty} EY_j^2(t)\right)^{k-2} \left(\sum_{j=0}^{\infty} EY_j^3(t)\right) \frac{(2k-1)!}{6\cdot 2^{k-2}(k-2)!} \\ + O\left(\sum_{j=0}^{\infty} EX_j(t)\right)^{k-2}. \end{aligned} \tag{20}$$

PROOF. (i) Expand $S^k(t)$ as

$$\textstyle\sum_{j_1,\cdots,j_k} \prod_{i=1}^k X_{j_i}^{(t)} = \sum_{m=1}^k \sum_{j_1 \neq j_2 \neq \cdots \neq j_m,\ \sum_{i=1}^m l_i = k} \prod_{i=1}^m X_{j_i}^{l_i}(t). \tag{21}$$

Since $\{X_j(t)\}_{j=0}^{\infty}$ are independent random variables for each $t > 0$

$$ES^k(t) = \textstyle\sum_{m=1,\ \sum_{i=1}^m l_i = k}^k \sum_{j_1 \neq \cdots \neq j_m} \prod_{i=1}^m EX_{j_i}^{l_i}(t).$$

Adding and subtracting

$$\textstyle\sum_{m=1}^{k-1} \sum_{j_1 \neq \cdots \neq j_m,\ \sum_{i=1}^m l_i = k} \prod_{i=1}^m (EX_{j_i})^{l_i}$$

to the above expression yields the representation

$$ES^k(t) = (ES(t))^k + \textstyle\sum_{m=1}^{k-1} \sum_{j_1 \neq \cdots \neq j_m,\ \sum_{i=1}^m l_i = k} \left(\prod_{i=1}^m E(X_{j_i}^{l_i}) - \prod_{i=1}^m (EX_{j_i})^{l_i}\right). \tag{22}$$

Using $X_j^l(t) = X_j(t)$. $l = 1, 2, \cdots$ and

$$(EX_j(t))^l \leqq EX_j(t) = 1 - \exp\left(-\textstyle\int_0^t P_{0j}(s)\,ds\right), \qquad l = 1, 2, \cdots$$

in (22), it suffices to check by induction on m that

$$\sum_{j_1\neq\cdots\neq j_m}\prod_{i=1}^m EX_{j_i}(t) = O(ES(t))^m). \tag{23}$$

For $m = 2$

$$\begin{aligned}\sum_{j_1\neq j_2}\prod_{i=1}^2 EX_{j_i}(t) &= \sum_{j_1,j_2}\prod_{i=1}^2 EX_{j_i}(t)-\sum_{j=0}^\infty (EX_j(t))^2\\ &= (ES(t))^2+O(ES(t)),\\ &= O((ES(t))^2),\end{aligned} \tag{24}$$

since

$$\sum_{j=0}^\infty (EX_j(t))^l \leqq ES(t)), \qquad l = 1,2,\cdots \tag{24a}$$

and for $j < k$

$$\frac{(ES(t))^j}{(ES(t))^k}\to 0, \qquad t\to\infty. \tag{24b}$$

The limit (24b) is immediate from $ES(t)\uparrow\infty$, as $t\to\infty$.

Now assume that (23) holds for $m = 3, 4, \cdots, r$.

Then

$$\begin{aligned}&\sum_{j_1\neq\cdots\neq j_{r+1}}\prod_{i=1}^{r+1} EX_{j_i}(t)\\ &\quad= (ES(t))^{r+1}-\sum_{m=1}^r\sum_{j_1\neq\cdots\neq j_m,\ \sum_{i=1}^m l_i=r+1}(EX_{j_i}(t))^{l_i}\\ &\quad= (ES(t))^{r+1}+O((ES(t))^r)\\ &\quad= O((ES(t))^{r+1})\end{aligned}$$

by (24a), (24b), and the induction hypothesis.

(ii) Because $\{Y_j(t)\}_{j=0}^\infty$ are independent for each $t > 0$ and $EY_j(t) = 0$,

$$\begin{aligned}E\Big(\sum_{j=0}^\infty Y_j(t)\Big)^{2k} &= \sum_{m=1}^k\sum_{j_1\neq\cdots\neq j_m,\ \sum_{i=1}^m l_i=2k}\prod_{i=1}^m (EY_{j_i}^{l_i})\\ &= \frac{\binom{2k}{2,\cdots,2}\binom{k}{1,\cdots,1}}{\binom{k}{1,1,\cdots 1}}\sum_{1\leqq j_i<\cdots<j_k}\prod_{i=1}^k(EY_{j_i}^2)+O((ES(t))^{k-1},\\ &= \frac{\binom{2k}{2,\cdots,2}}{\binom{k}{1,\cdots,1}}\left[\Big(\sum_{j=0}^\infty EY_j^2\Big)^k-\sum_{m=1}^{k-1}\sum_{j_1\neq\cdots\neq j_m,\ \sum_{i=1}^m l_i=k}\prod_{i=1}^m(EY_{j_i}^2)^{l_i}\right]+O((ES(t))^{k-1})\\ &= \frac{(2k)!}{k!2^k}[E(S(t)-ES(t))^2]^k+O((ES(t))^{k-1}\end{aligned} \tag{24c}$$

where

$$\binom{m}{j_1,\cdots,j_r} = \frac{m!}{j_1!j_2!\cdots j_r!}, \qquad \sum_{i=1}^r j_i = m.$$

The last three equalities in (24c) require an application of the properties

(a) $EY_j^l(t) = O(EX_j(t))$.

(b) $0\leqq (EX_j(t))^r\leqq EX_j(t)\leqq 1$.

(c) Binomial expansion:

$$E(Y_j^l(t)) = (-EX_j(t))^l+\sum_{r=1}^l\binom{l}{r}EX_j^{\,l-r+1}(t)(-1)^{l-r}.$$

A similar verification yields (iii) of Lemma 4.1.

PROOF OF THEOREM 4.1 (i). Because of $ES(t) \uparrow \infty$, $t \to \infty$ and part (i) of Lemma 4.1, it suffices to check

$$\lim_{t\to\infty} \frac{ES(t)}{E\hat{S}(t)} = 1.$$

Step 1. *An upper bound.* Using the upper bound in Lemma 3.2 we have

$$\begin{aligned} ES(t) - E\hat{S}(t) &= \textstyle\sum_{j=0}^{\infty} \exp(-p_j t)(1 - \exp(-\int_0^t (P_{0j}(s) - p_j)\, ds)) \\ &\leqq \textstyle\sum_{j=0}^{\infty} \exp(-p_j t)(1 - \exp(-\varepsilon p_j t)) \\ &\leqq \varepsilon \textstyle\sum_{j=0}^{\infty} p_j t \exp(-p_j t) \\ &\leqq \varepsilon E\hat{S}(t) \end{aligned} \tag{25}$$

for $\varepsilon > 0$ arbitrary and $t >$ some $T(\varepsilon)$.

REMARK. Notice that the condition $m_{00}^{(2)} < \infty$ is not required for (25). Hence $ES(t)$ can never be an order of magnitude larger than $E\hat{S}(t)$,

$$\left(\text{i.e., } \limsup_{t\to\infty} \frac{ES(t)}{E\hat{S}(t)} \leqq 1\right).$$

Furthermore, $\lim_{t\to\infty}(E\hat{S}(t))/t = 0$, (see e.g. [2]) which means that $\lim_{t\to\infty}(ES(t))/t = 0$. This is in sharp contrast to $\{S(t), t > 0\}$ generated by null recurrent and transient Markov chains where it is possible to have $\lim_{t\to\infty}(ES(t))/t = 1$. See [6] for examples of this behavior.

Step 2. *A lower bound.* Use the lower bound in Lemma 3.2 together with addition and subtraction of $\exp(-(1-\varepsilon)p_k t)$ to the kth term in the series for $ES(t) - E\hat{S}(t)$ to obtain the inequality

$$\begin{aligned} ES(t) - E\hat{S}(t) &\geqq \textstyle\sum_{j=0}^{\infty} \exp(-(1-\varepsilon)p_j t)[\exp(-\varepsilon p_j t) - 1] \\ &\quad - \textstyle\sum_{j=0}^{\infty} \exp(-(1-\varepsilon)p_j t)[\exp(p_0 q_0 \int_0^t dw \int_w^{\infty} {}_0P_{0j}(s)\, ds) - 1] \\ &= I_1(t) - I_2(t). \end{aligned} \tag{26}$$

A trivial estimate gives

$$I_1(t) \geqq -\varepsilon \textstyle\sum_{j=0}^{\infty} p_j t \exp(-(1-\varepsilon)p_j t) \geqq -\frac{\varepsilon}{1-\varepsilon} E\hat{S}(t) \tag{27}$$

for $\varepsilon > 0$ arbitrary and $t >$ some $T(\varepsilon)$.

Since $m_{00}^{(2)} < \infty$, an application of the mean value theorem, Tonelli's Theorem, and Lemma 3.3 in $I_2(t)$ yield

$$\begin{aligned} I_2(t) &\leqq \textstyle\sum_{j=0}^{\infty} (p_0 q_0 \int_0^t dw \int_w^{\infty} {}_0P_{0j}(s)\, ds) \cdot e^{c_j} \\ &\leqq p_0 q_0 m_{00}^{(2)} \exp(p_0 q_0 m_{00}^{(2)}) \\ &\equiv C^* < \infty \end{aligned} \tag{28}$$

where

$$0 < c_j < \frac{m_{00}^{(2)}}{2} p_0 q_0, \qquad j = 1, 2, \cdots,$$

and

$$m_{00}^{(2)} = 2 \int_0^\infty s(1 - F_{00}(s))\, ds.$$

Combining (26)–(28) we obtain

$$\frac{ES(t)}{E\hat{S}(t)} - 1 \geqq -\frac{\varepsilon}{1-\varepsilon} - \frac{C^*}{E\hat{S}(t)}. \tag{29}$$

Letting $t \to \infty$ and recalling that $E\hat{S}(t) \uparrow \infty$, $t \to \infty$, we have

$$\liminf_{t \to \infty} \frac{ES(t)}{E\hat{S}(t)} \geqq 1$$

and the proof is complete.

Before proving parts (ii) and (iii) of the theorem, we remark that the assumption $\alpha(x) = x^\gamma L(x)$, $0 < \gamma \leqq 1$ is introduced to ensure $\lim_{t \to \infty} E(S(t) - ES(t))^2 = \infty$. This also implies that growth behavior of Var $S(t)$ is determined by that of $ES(t)$. Asymptotic formulas for bounded variances require more delicate calculations as indicated by the examples in the infinite urn case [2]. A sufficient condition for $\limsup_{t \to \infty}$ Var $S(t) < \infty$ with $m_{00}^{(2)} < \infty$ is given by

$$\limsup_{k \to \infty} \frac{p_{k+1}}{p_k} < 1.$$

This follows from a verbatim imitation of the proof of Remark 3 in [2] using the estimates of our Lemma 3.2. Of greater significance is the fact that the Central Limit Theorem 6.1 does not hold for bounded variances. A more extensive discussion of this case is presently in preparation.

PROOF OF THEOREM 4.1 (ii). Rewrite Var $S(t)$ as

$$\begin{aligned} \operatorname{Var} S(t) &= \textstyle\sum_{j=0}^\infty \left(1 - \exp\left(-2 \int_0^t P_{0j}(s)\, ds\right)\right) \\ &\quad - \textstyle\sum_{j=0}^\infty \left(1 - \exp\left(-\int_0^t P_{0j}(s)\, ds\right)\right) \\ &= I_3(t) - I_4(t). \end{aligned}$$

By part (i),

$$\lim_{t \to \infty} \frac{I_3(t)}{E\hat{S}(2t)} = 1$$

and

$$\lim_{t \to \infty} \frac{I_4(t)}{E\hat{S}(t)} = 1.$$

Now we invoke Theorem 1 in [2] to assert that $E\hat{S}(t) \sim h_\gamma(t)$, $t \to \infty$.

Since $\gamma > 0$, we may subtract asymptotic formulas to obtain

$$\operatorname{Var} S(t) \sim (2^\gamma - 1)h_\gamma(t), \qquad t \to \infty.$$

Finally an application of Lemma 4.1, (ii) completes the proof.

PROOF OF THEOREM 4.1 (iii). The proof of Var $\hat{S}(t) \sim (2^\gamma - 1)h_\gamma(t)$ in [2] is readily adapted to check that

$$E(\hat{S}(t) - E\hat{S}(t))^3 \sim \frac{(c_1 - 3c_2 + 2c_3)}{\Gamma(1-\gamma)} h_\gamma(t), \qquad 0 < \gamma < 1 \tag{30}$$

$$\sim (d_1(t) - 3d_2(t) + 2d_3(t))t, \qquad \gamma = 1$$

where

$$c_m = \int_0^\infty m y^{\gamma-2} e^{-1/y}(1 - e^{-1/y})^{m-1}\, dy,$$

$$d_m(t) = \int_0^\infty L(ty) m\,(e^{-1/y}/y)(1 - e^{-1/y})^{m-1}\, dy \qquad m = 1, 2, 3.$$

Thus it remains for us to verify that

$$\lim_{t \to \infty} \frac{E(S(t) - ES(t))^3}{E(\hat{S}(t) - E\hat{S}(t))^3} = 1.$$

Notice that

$$E\left(\sum_{j=0}^\infty Y_j(t)\right)^3 = \sum_{j=0}^\infty E(Y_j^3(t)),$$

since $\{Y_j(t)\}_{j=0}^\infty$ are independent and $EY_j(t) = 0$.

But

$$\sum_{j=0}^\infty E(Y_j^3(t)) = \sum_{j=0}^\infty [EX_j(t) - 3(EX_j(t))^2 + 2(EX_j(t))^3].$$

Hence, our problem reduces to checking

$$\lim_{t \to \infty} \frac{\sum_{j=0}^\infty (EX_j(t))^m}{\sum_{j=0}^\infty (E\hat{X}_j(t))^m} = 1, \qquad m = 1, 2, \cdots \tag{31}$$

and observing that for $\gamma > 0$ we may add and subtract asymptotic formulas based on

$$\sum_{j=0}^\infty (E\hat{X}_j(t))^m \sim c_m t^\gamma L(t), \qquad 0 < \gamma < 1 \tag{32}$$

$$\sim d_m(t)t, \qquad \gamma = 1.$$

Given $\varepsilon_1, \varepsilon_2 > 0$ arbitrary and independent of each other, the upper bound of Lemma 3.2 and formula (32) imply the existence of $T_1(\varepsilon_1\ \varepsilon_2)$ such that for $t > T_1$

$$\frac{\sum_{j=0}^\infty (EX_j(t))^m}{\sum_{j=0}^\infty (E\hat{X}_j(t))^n} \leqq \frac{\sum_{j=0}^\infty (E\hat{X}_j((1+\varepsilon_1)t))^m}{\sum_{j=0}^\infty E\hat{X}_j(t))^m} \leqq (1+\varepsilon_1)^\gamma(1+\varepsilon_2). \tag{33}$$

For a lower bound, we again use Lemma 3.2 and the mean value theorem to write

$$(1 - \exp(-\int_0^t P_{0j}(s)\, ds)) \geqq 1 - \exp(-p_j t(1-\varepsilon) + c_j(t)) \tag{34}$$

$$\geqq 1 - \exp(-p_j t(1-\varepsilon)) - K_0 c_j(t),$$

where

$$c_j(t) = p_0 q_0 \int_0^t dw \int_w^\infty {}_0P_{0j}(s)\, ds$$

$$K_0 = \exp\left(p_0 q_0 \frac{m_{00}^{(2)}}{2}\right).$$

Then make the identification

$$a_j = 1 - \exp(-p_j t(1-\varepsilon)), \qquad b_j = K_0 c_j(t),$$

and use the binomial expansion to write

$$(a_j - b_j)^m \geqq a_j^{\,m} - \sum_{k=0}^{m-1} \binom{m}{k} b_j^{\,m-k}. \tag{35}$$

When $t \geqq T_1(\varepsilon_1, \varepsilon_2)$, (34) yields

$$\frac{\sum_{j=0}^\infty (EX_j(t))^m}{\sum_{j=0}^\infty (E\hat{X}_j(t))^m} \geqq (1-\varepsilon_1)^\gamma (1-\varepsilon_2) - \frac{\sum_{k=0}^{m-1} \sum_{j=0}^\infty \binom{m}{k} K_0 (c_j(\infty))^{m-k}}{(\sum_{j=0}^\infty E\hat{X}_j(t))^m}. \tag{36}$$

Because

$$\sum_{j=0}^\infty c_j(\infty) = p_0 q_0 \frac{m_{00}^{(2)}}{2} < \infty \quad \text{and} \quad c_j(\infty) \geqq 0,$$

the numerator of the second term is bounded. Thus letting $t \to \infty$ in (36) and combining this lower bound with (33) completes the proof.

5. Examples $m_{00}^{(2)} = \infty$. The question of control of moments of $S(t)$ by the stationary measure of the growing process $\{Y(t), t > 0\}$ when $m_{00}^{(2)} = \infty$ is very delicate. Asymptotic formulas require an estimate of the speed of convergence of the transition probabilities $P_{0,k}(t)$ to the stationary measure p_k when $k = k(t) \uparrow \infty$ as $t \to \infty$. If the stationary measure has a monotone decreasing density we may view p_k as the restriction to nonnegative integers of a monotone decreasing continuous function $p(x)$, $x \in [0, \infty)$. If, in addition, $p(x)$ has an inverse function p^{-1} such that $p^{-1}(1/t) = t^a L(t)$, $0 \leqq a < 1$, $L(t)$ slowly varying, then a sufficient condition for

$$\lim_{t\to\infty} \frac{ES(t)}{E\hat{S}(t)} = 1 \tag{37}$$

is that

$$\lim_{t\to\infty} t\left|P_{0,[yp^{-1}(1/t)]}(tu) - p([yp^{-1}(1/t)])\right| = 0 \tag{38}$$

where $y, u > 0$ are independent of t, and $[x]$ denotes the integer part of x.

We will exhibit two positive recurrent birth and death processes having the same stationary measure $\{p_k\}_{k=0}^\infty$, $m_{00}^{(2)} = \infty$ and only one of which satisfies (37). The process whose limit behavior is not controlled exclusively by $\{p_k\}_{k=0}^\infty$ has an absorbing barrier component which converges to a Bessel diffusion. The transition density of the diffusion appears explicitly in the asymptotic formulas for $ES(t)$ and Var $S(t)$.

We will require the following facts about birth and death processes on the nonnegative integers.

5.1. The transition probability matrix satisfies the conditions as $t \to 0$.

$$\begin{aligned} P_{i,k}(t) &= \lambda_i t + o(t) && \text{if} \quad k = i+1, \\ &= \mu_i t + o(t) && \text{if} \quad k = i-1, \\ &= 1-(\lambda_i+\mu_i)t + o(t) && \text{if} \quad k = i, \end{aligned}$$

where $\lambda_i > 0$ for $i \geqq 0$, $\mu_i > 0$ for $i \geqq 1$ and $\mu_0 \geqq 0$. If $\mu_0 > 0$ then -1 is appended to the nonnegative integers as a permanent absorbing state.

5.2. The transition probabilities have an integral representation [4]

$$P_{ik}(t) = \pi_k \int_0^\infty e^{-xt} Q_i(x) Q_k(x)\, d\psi(x) \tag{39}$$

in terms of a system of polynomials $\{Q_n(x)\}_{n=0}^\infty$ orthogonal on $[0, \infty)$ with respect to the positive regular measure ψ, and

$$\pi_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k}.$$

5.3. When $\mu_0 = 0$, the representation (39) yields the finer decomposition [5]

$$P_{0j}(t) = \pi_j + \pi_j \int_0^\infty e^{-xt}\, d\psi^*(x) - \pi_j \sum_{k=0}^{j-1} P_{0k}^*(t) \tag{40}$$

where $P_{0k}^*(t)$ is the transition probability of an absorbing barrier birth and death process having infinitesimal parameters.

$$\pi_n^* = \frac{\lambda_0}{\lambda_n \pi_n}, \quad \frac{1}{\lambda_n^* \pi_n^*} = \frac{\pi_{n+1}}{\lambda_0}, \qquad \psi^*(x) = \psi(x) - \psi(0),$$

and

$$\begin{aligned} a_0 &= P\,(\text{eventual absorption at } -1 \text{ for the *-process starting from state } 0) \\ &= \frac{\mu_0^* \sum_{n=0}^\infty 1/\lambda_n^* \pi_n^*}{1 + \mu_0^* \sum_{n=0}^\infty 1/\lambda_n^* \pi_n^*}. \end{aligned}$$

5.4. The orthogonal polynomials $\{H_n(x)\}_{n=0}^\infty$ and spectral measure $\theta(x)$ which yield an integral representation like (39) for the (*)-process are related to the corresponding quantities in the original process with $\mu_0 = 0$ by

$$H_n(x) = \frac{\lambda_n \pi_n (Q_{n+1}(x) - Q_n(x))}{-x}, \qquad n = 0, 1, 2, \cdots,$$

$$d\theta(x) = \frac{x\, d\psi^*(x)}{\lambda_0} \qquad \text{[see [5]).}$$

5.5.[7] For an absorbing barrier birth and death process with

$$\pi_n^* \sim D_1 n^{\gamma-1}, \qquad 1/\lambda_n^* \pi_n^* \sim D_2 n^{\beta-1}, \qquad n \to \infty \tag{41}$$

$$D_1, D_2, \gamma, \beta+\gamma > 0, \qquad \beta \leqq 0$$

and $t^j P_{00}(t)$ monotone for t sufficiently large, $-j < \beta/(\beta+\gamma) \leqq -(j-1)$, we have

$$\lim_{t\to\infty} tP_{0,[tw]}(t^{\beta+\gamma}u) = (1-a_0)p(0,w;u) \tag{42}$$

where

$$p(0,w;u) = D_3 \frac{w^{\gamma-1}}{u^{(\gamma/\beta+\gamma)}} \exp\left(-D_4 \frac{w^{\beta+\gamma}}{u}\right),$$

$$D_3 = D_2 K_0 \Gamma\left(\frac{\gamma}{\beta+\gamma}\right),$$

$$D_4 = \frac{D_1 D_2}{(\beta+\gamma)^2},$$

$$K_0 = (-1)^j \left(\Gamma\left(\frac{\gamma}{\beta+\gamma}\right)\Gamma\left(j+\frac{\beta}{\beta+\gamma}\right)\right)^{-1} \cdot D_2(\beta+\gamma)^{-1}$$

$$\cdot \int_0^\infty y^{j+(\beta/\beta+\gamma)-1} \frac{d^j}{dy^j}(I^{-2}(y))\,dy,$$

$$I(s) = \Gamma\left(\frac{\gamma}{\beta+\gamma}\right)[D_1 D_2 s(\beta+\gamma)^{-2}]^{\alpha/2} I_{-(\beta/\beta+\gamma)}(2(D_1 D_2 s)^{\frac{1}{2}}(\beta+\gamma)^{-1}).$$

$I_{-\alpha}(x)$ is the usual modified Bessel function of order $-a$ and $\alpha = \beta/(\beta+\gamma)$.

5.6.[7] For an absorbing barrier process satisfying (41), the spectral measure $\theta(x)$ satisfies

$$\theta(x) \sim Hx^{(\gamma/(\beta+\gamma))}, \qquad x \to 0$$

where

$$H = (1-a_0)^2 K_0 \left[\Gamma\left(\frac{\gamma}{\beta+\gamma}\right)\Gamma\left(2-\frac{\beta}{\beta+\gamma}\right)\right]^{-1}.$$

EXAMPLE 1. Suppose

$$p_n \equiv \pi_n = c/(n+1)^\alpha, \qquad \alpha > 1 \tag{43}$$

$$c = \frac{1}{\sum_{n=0}^\infty 1/(n+1)^\alpha}$$

and

$$\frac{1}{\lambda_n \pi_n} = D(n+1)^{2(\alpha-1)}, \qquad D > 0. \tag{44}$$

A short calculation using

$$m_{00}^{(2)} = \frac{2}{c\lambda_0^{\,2}} + \frac{2}{\lambda_0}\left[\sum_{n=0}^\infty \frac{1}{\lambda_n \pi_n}\left(\sum_{j=n}^\infty \pi_{j+1}\right)^2\right] \tag{45}$$

verifies $m_{00}^{(2)} = \infty$ for this process.

For an infinite urn scheme with probabilities of hitting the kth cell given by (43)

$$E\hat{S}(t) \sim \Gamma\left(\frac{\alpha-1}{\alpha}\right) c^{1/\alpha} t^{1/\alpha} \tag{46}$$

$$\equiv c_2 t^{1/\alpha}, \qquad t \to \infty.$$

Nevertheless, we have

THEOREM 5.1. *If* $\{S(t), t > 0\}$ *is generated by a birth and death process with infinitesimal parameters satisfying* (43) *and* (44),

$$ES(t) \sim c_1 t^{1/\alpha} \tag{47}$$

where

$$c_1 = \int_0^\infty [1 - \exp(-(c/w^\alpha)(1 - \int_0^w ds \int_0^1 (1-a_0) p(0, s; u)\, du))]\, dw$$

$p(0, s; u)$ *is the transition density of Proposition* 5.5 *with*

$$\gamma = 2\alpha - 1, \qquad \beta = 1 - \alpha,$$

$$D_1 = \lambda_0 D, \qquad D_2 = c/\lambda_0$$

and $a_0 = \frac{1}{2}$.

REMARK. To see that $c_1 < c_2$, notice that

$$c_2 \equiv \Gamma\left(\frac{\alpha-1}{\alpha}\right) c^{1/\alpha} = \int_0^\infty \left(1 - \exp\left(-\frac{c}{w^\alpha}\right)\right) dw.$$

EXAMPLE 2. Consider a birth and death process with

$$p_n \equiv \pi_n = \frac{c}{(n+1)^\alpha}, \qquad 1 < \alpha \leqq 2 \tag{48}$$

$$c = \frac{1}{\sum_{n=0}^\infty 1/(n+1)^\alpha}$$

$$\frac{1}{\lambda_n \pi_n} = D(n+1)^{\alpha-1}. \tag{49}$$

Using formula (45) we may check $m_{00}^{(2)} = \infty$. However,

$$ES(t) \sim E\hat{S}(t)$$

$$\sim \Gamma\left(\frac{\alpha-1}{\alpha}\right) c^{1/\alpha} t^{1/\alpha}, \qquad t \to \infty. \tag{50}$$

We prepare three technical lemmas which are required in the verification of formulas (47) and (50).

Given $\varepsilon_0 > 0$ arbitrary, choose M_0 sufficiently large and ε_1 sufficiently small that

$$\int_0^{c_1 \varepsilon_1} + \int_{M_0}^\infty h(w)\, dw < \varepsilon_0$$

where

$$h(w) = 1-\exp\left(-\frac{c}{w^\alpha}\left(1-\int_0^w dx\int_0^1 (1-a_0)p(0,x;u)\,du\right)\right).$$

Then let

$$M = \max\left(M_0, \left[\frac{c(1/p_0+1/p_0q_0)}{c_1(\alpha-1)\varepsilon_0}\right]^{1/(\alpha-1)}\right). \tag{51}$$

LEMMA 5.1. *For the birth and death processes of* Examples 1 *and* 2, *we have* $\forall t > 0$

(i) $$\frac{1}{c_i t^{1/\alpha}} \sum_{j<[c_i\varepsilon_1 t^{1/\alpha}]} \left(1-\exp\left(-\int_0^t P_{0j}(s)\,ds\right)\right) < \varepsilon_1, \tag{52}$$

(ii) $$\frac{1}{c_i t^{1/\alpha}} \sum_{j>[Mt^{1/\alpha}]} \left(1-\exp\left(-\int_0^t P_{0j}(s)\,ds\right)\right) < \varepsilon_0. \tag{53}$$

c_i is used for example i, $i = 1, 2$.

PROOF. (i) follows immediately from the crude bound $1-e^{-x} \leqq 1$ for $x \geqq 0$. Now bring in the identities (11) and (12) to obtain the bound

$$P_{0k}(t) \leqq {}_0P_{0k}(t)+q_0\int_0^\infty {}_0P_{0k}(s)\,ds = {}_0P_{0k}(t)+\frac{p_k}{p_0}. \tag{54}$$

Then

$$\begin{aligned}
&\textstyle\sum_{j>[Mt^{1/\alpha}]}(1-\exp(-\int_0^t P_{0j}(s)\,ds))\\
&\leqq \sum_{j>[Mt^{1/\alpha}]}\int_0^t\left({}_0P_{0j}(s)+\frac{p_j}{p_0}\right)ds\\
&\leqq \sum_{j>[Mt^{1/\alpha}]}\left(\frac{p_j}{p_0q_0}+\frac{p_jt}{p_0}\right)\\
&\leqq \frac{ct}{p_0}\left(1+\frac{1}{q_0}\right)\int_{Mt^{1/\alpha}}^\infty \frac{dx}{x^\alpha} \qquad \text{using (43) or (48)}\\
&= \frac{c}{(\alpha-1)p_0}\left(1+\frac{1}{q_0}\right)\cdot\frac{1}{M^{\alpha-1}}\cdot t^{1/\alpha}\\
&< \varepsilon_0 t^{1/\alpha}
\end{aligned}$$

and the proof of (ii) is complete.

LEMMA 5.2. *For the birth and death process satisfying* (41),

$$\lim_{t\to\infty}\frac{1}{t}\int_0^\infty \frac{1-e^{-xt}}{x}\,d\psi^*(x) = 0.$$

NOTE. In example 1, $\gamma = 2\alpha-1$, $\beta = 1-\alpha$, $\alpha > 1$.

In example 2, $\gamma = \alpha$, $\beta = 1-\alpha$, $1 < \alpha \leqq 2$.

PROOF. Define the spectral measure for the related absorbing barrier process by

$$d\theta(x) = \frac{1}{\lambda_0} x \, d\psi^*(x)$$

and introduce the decomposition

$$\int_0^\infty \frac{1-e^{-xt}}{x} d\psi^*(x) = \lambda_0 \int_0^{\delta_1} + \int_{\delta_1}^\infty \frac{1-e^{-xt}}{x^2} d\theta(x). \tag{55}$$

Given $\varepsilon_2 > 0$ arbitrary, choose δ_1 such that

$$\frac{4K\lambda_0 \delta_1^{(\gamma/(\beta+\gamma))-1)}}{\min(1, \gamma/(\beta+\gamma)-1)} < \frac{\varepsilon_2}{2}$$

where K is a constant independent of x satisfying, by Proposition 5.6,

$$\theta(x) \leqq Kx^{\gamma/(\beta+\gamma)} \quad \text{for} \quad 0 \leqq x \leqq 1. \tag{56}$$

Integration by parts and an application of (56) in the first term of (55) yields the estimate

$$\begin{aligned} &\lambda_0 \int_0^{\delta_1} \frac{1-e^{-xt}}{x^2} d\theta(x) \\ &\leqq \lambda_0 \left[\frac{t\theta(x)}{x}\bigg|_0^{\delta_1} + \int_0^{\delta_1} \frac{3t}{x^2}\theta(x)\,dx \right] \\ &< \frac{\varepsilon_2}{2} t. \end{aligned} \tag{57}$$

Now choose $t > 2M_1\lambda_0/(\delta_1^2\varepsilon_2)$ where $M_1 = \int_0^\infty d\theta(x)$.

The trivial estimate

$$\frac{\lambda_0}{t}\int_{\delta_1}^\infty \frac{1-e^{-xt}}{x^2} d\theta(x) \leqq \frac{\lambda_0 M_1}{\delta_1^2} < \frac{\varepsilon_2}{2}$$

completes the proof.

Now recall the condition (38) and notice that for Examples 1 and 2

$$p^{-1}\left(\frac{1}{t}\right) = (ct)^{1/\alpha}, \qquad p\left(\left[yp^{-1}\left(\frac{1}{t}\right)\right]\right) = \frac{c}{[y(ct)^{1/\alpha}]^\alpha}.$$

Then we have

LEMMA 5.3. (i) *For the process of* Example 1

$$\lim_{t\to\infty} t\left|P_{0,[yp^{-1}(1/t)]}(tu) - p([yp^{-1}(1/t)])(1 - \textstyle\int_0^{c^{1/\alpha}y}(1-a_0)p(0,x;u)\,dx)\right| = 0 \tag{58}$$

for $y, u > 0$ *where* $p(0, x; u)$ *is the transition of density of* Proposition 5.5.

(ii) *For the process of* Example 2

$$\lim_{t\to\infty} t\left|P_{0,[yp^{-1}(1/t)]}(tu) - p([yp^{-1}(1/t)])\right| = 0. \tag{59}$$

PROOF. We begin with the decomposition (40) and rewrite it as

$$P_{0,j}(tu) - p_j(1 - \Pr(X^*(tu) \leqq j-1 \mid X^*(0) = 0)) = p_j \int_0^\infty e^{-xt}\, d\psi^*(X)$$

with $j = [yp^{-1}(1/t)]$, $y > 0$ independent of t.
But

$$p\left(\left[yp^{-1}\left(\frac{1}{t}\right)\right]\right)\cdot t = \frac{1}{y^\alpha - o(1)}, \qquad t \to \infty$$

and by the integral form of Proposition 5.5

$$\begin{aligned}\lim_{t\to\infty} &\Pr(X^*(tu) < [y(ct)^{1/\alpha}] \mid X^*(0) = 0)\\ &= (1-a_0)\int_0^{yc^{1/\alpha}} p(0,x;u)\,dx \qquad \text{for Example 1,}\\ &= 0 \qquad \text{for Example 2.}\end{aligned}$$

Since

$$\lim_{t\to\infty} \int_0^\infty e^{-xt}\, d\psi^*(x) = \lim_{t\to\infty}\left(\frac{P_{00}(t) - \pi_0\psi(0)}{\pi_0}\right) = 0$$

for all birth and death processes, the proof is complete.

We restate part (i) of Lemma 5.3 in a form which is adapted to our proof of Theorem 5.1.

LEMMA 5.3′. *For* $\varepsilon_3 > 0$ *arbitrary and* $\delta_2, \delta_3 > 0$ *sufficiently small* $\exists T_1(\varepsilon_i, 0 \leqq i \leqq 3, \delta_2, \delta_3)$ *such that*

$$t > T_1 \Rightarrow \left|\int_0^t P_{0j}(s)\,ds - p_j t\left(1 - \int_{\delta_2 j/t^{1/\alpha}}^{j/t^{1/\alpha}} dx \int_{\delta_3}^1 (1-a_0)p(0,x;u)\,du\right)\right| < \varepsilon_3 p_j t \tag{60}$$

for $j \in [c_1\varepsilon_0 t^{1/\alpha}, Mt^{1/\alpha}]$, ε_2 *is the arbitrary positive number in the proof of* Lemma 5.2.

In addition to Proposition 5.5, Lemma 5.2 and a standard Riemann sum approximation are used in the verification of (60).

PROOF OF THEOREM 5.1.

Step 1. *Selection of* ε*'s and* δ*'s.* First choose ε_0, ε_1 and M as in Lemma 5.1. Then choose $\varepsilon_2 > 0$ arbitrary and notice that for $t >$ some $T(\varepsilon_2)$

$$0 \leqq \frac{1}{t}\int_0^\infty \frac{1 - e^{-xt'}}{x}\, d\psi^*(x) < \varepsilon_2.$$

This choice of ε_2 only affects the size of T_1 in Lemma 5.3′. Now select $\varepsilon_1 > 0$ arbitrary and independent of ε_0, ε_1, and ε_2. Then choose δ_2, δ_3 such that

$$\delta_2\delta_3 < \frac{\varepsilon_3(c_1\varepsilon_1)^\alpha}{cM}.$$

With this choice of δ_2 and δ_3, we have $\left|\int_{c_1\varepsilon_1}^{M} h(w)\,dw-\hat{c}_1\right| < \varepsilon_3$ where

$$\hat{c}_1 = \int_{c_1\varepsilon_1}^{M}\left\{1-\exp\left(-\frac{c}{w^\alpha}\left(1-\int_{\delta_2 w}^{w} dx\int_{\delta_3}^{1}(1-a_0)p(0,x;u)\,du\right)\right)\right\}dw$$

and $h(w)$ is the integrand in the definition of c_1 (see also (47) and (51)).

Step 2. *Final estimates.* Introduce the decomposition

$$\begin{aligned} ES(t) &= \textstyle\sum_{j<[c_1\varepsilon_1 t^{1/\alpha}]}+\sum_{j=[c_1\varepsilon_1 t^{1/\alpha}]}^{[Mt^{1/\alpha}]}+\sum_{j>[Mt^{1/\alpha}]}\left(1-\exp\left(-\int_0^t P_{0j}(s)\,ds\right)\right) \\ &= I_1+I_2+I_3. \end{aligned}$$

By Lemma 5.1:

$$\left|\frac{ES(t)}{c_1 t^{1/\alpha}}-1\right| \leqq \left|\frac{I_2(t)}{c_1 t^{1/\alpha}}-1\right|+\varepsilon_0+\varepsilon_1. \tag{61}$$

Now let

$$R_1(t) = \textstyle\sum_{j=[c_1\varepsilon_1 t^{1/\alpha}]}^{[Mt^{1/\alpha}]}\left\{1-\exp\left(-p_j t\left(1-\int_{\delta_2 j/t^{1/\alpha}}^{j/t^{1/\alpha}} dx\int_{\delta_3}^{1}(1-a_0)p(0,x;u)\,du\right)\right)\right\}.$$

Lemma 5.3′ immediately yields

$$\begin{aligned} |I_2(t)-R_1(t)| &< ES(\varepsilon_3 t) \quad \text{for} \quad t > T_1 \\ &\sim c_2(\varepsilon_3 t)^{1/\alpha} \end{aligned} \tag{62}$$

and a standard Riemann sum approximation verifies

$$\lim_{t\to\infty}\frac{R_1(t)}{t^{1/\alpha}} = \hat{c}_1. \tag{63}$$

Combining (62) and (63) in the right-hand side of (61) we obtain the inequality

$$\begin{aligned} &\limsup_{t\to\infty}\left|\frac{I_2(t)}{c_1 t^{1/\alpha}}-1\right| \\ &\leqq \limsup_{t\to\infty}\left(\left|\frac{I_2(t)-R_1(t)}{c_1 t^{1/\alpha}}\right|+\left|\frac{R_1(t)}{t^{1/\alpha}}\left(\frac{1}{c_1}-\frac{1}{\hat{c}_1}\right)\right|+\left|\frac{R_2(t)}{\hat{c}_1 t^{1/\alpha}}-1\right|\right) \\ &\leqq \frac{c_2}{c_1}\varepsilon_3^{1/\alpha}+\frac{(\varepsilon_0+\varepsilon_3)}{c_1} \end{aligned} \tag{64}$$

and the proof is complete.

To verify formula (50), use step 2 of the proof of Theorem 5.1, replacing $R_1(t)$ by

$$R_2(t) = \textstyle\sum_{j=[c_2\varepsilon_1 t^{1/\alpha}]}^{[Mt^{1/\alpha}]}\left(1-\exp\left(-p_j t\right)\right),$$

and observe, by (59) in Lemma 5.3, that

$$|I_2(t) - R_2(t)| < E\hat{S}(\varepsilon_3 t) \quad \text{for } t > \text{some } T_1 \sim c_2(\varepsilon_3 t)^{1/\alpha}$$

and

$$\lim_{t\to\infty}\frac{R_2(t)}{t^{1/\alpha}} = \hat{c}_2 = \int_{c_2\varepsilon_1}^{M}\left(1-\exp\left(-\frac{c}{w^\alpha}\right)\right)dw.$$

6. Limit theorems. The Central Limit Theorem (part (a) of Theorem 6.1) follows from an imitation of the proof of the Central Limit Theorem for finite arrays of independent random variables using the Lindeberg conditions. For the occupied states processes of this paper we have infinite arrays of uniformly bounded independent random variables $X_k(t)$, $k = 0, 1, \cdots, t > 0$. We also require that

$$S(t) = \sum_{k=0}^{\infty} X_k(t) < \infty \quad \text{with probability} \quad 1 \tag{65}$$

for all finite t. Condition (65) is satisfied for all occupied states processes of this paper since

$$S(t) \leqq n(t) = \text{number of jumps in a Poisson process up to time } t.$$

The hypothesis $\alpha(x) = \max(k : p_k \geqq 1/x) = x^\gamma L(x)$, $0 < \gamma \leqq 1$, $L(x)$ slowly varying ensures $\operatorname{Var} S(t) \to \infty$ as $t \to \infty$. Notice that the Central Limit Theorem also holds for the birth and death processes of Section 5.

To verify the strong law (6), choose k sufficiently large that $k\gamma > 1$ and apply the Markov inequality and Theorem 4.1 to obtain

$$P\left(\left|\frac{S(t)}{ES(t)} - 1\right| > \varepsilon\right) \leqq \frac{E(S(t) - ES(t))^{2k}}{\varepsilon^{2k}(ES(t))^{2k}} \tag{66}$$

$$\leqq \frac{c_k}{\varepsilon^{2k} t^{k\gamma} L^k(t)}$$

where c_k is a constant independent of t, and $\varepsilon > 0$ is arbitrary. An application of the Borel–Cantelli Lemma, the asymptotic relation

$$\lim_{t \to \infty} \frac{ES(t)}{h_\gamma(t)} = 1$$

and a standard separability argument complete the proof.

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