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CONFIDENCE INTERVALS FOR LINEAR FUNCTIONS OF THE NORMAL MEAN AND VARIANCE[1]

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1. Introduction and summary. If $Y = g(X)$ is normal (μ, σ^2), where g is a one-to-one real function and X is a random variable whose expectation exists, we may write $EX = f(\mu, \sigma^2)$. The practical importance of this observation is that we often are concerned with testing hypotheses about, and constructing confidence intervals for, known functions of both the mean and variance of a normal distribution. This may happen when we use a statistical model, such as the lognormal distribution, that is related to the normal distribution by a transformation of variables. A slightly different case occurs when a transformation of data is made before applying a statistical method, such as analysis of variance or regression analysis, that involves the assumption of normality for the transformed data. Some familiar examples in this context are

(i) $Y = X^{\frac{1}{2}}$, $EX = \mu^2 + \sigma^2$;

(ii) $Y = X^{\frac{1}{3}}$, $EX = \mu^3 + 3\mu\sigma^2$;

(iii) $Y = \arcsin(X^{\frac{1}{2}})$, $EX = \frac{1}{2}(1 - \cos(2\mu)\exp(-2\sigma^2))$;

(iv) $Y = \operatorname{arcsinh}(X^{\frac{1}{2}})$, $EX = \frac{1}{2}(\cosh(2\mu)\exp(2\sigma^2) - 1)$;

(v) $Y = \log(X)$, $EX = \exp(\mu + \frac{1}{2}\sigma^2)$.

The theory of statistical inference in terms of $\mu = EY$ alone or $\sigma^2 = \operatorname{Var} Y$ alone is not easily extended to problems of inference in terms of EX or $\operatorname{Var} X$, parametric functions of both μ and σ^2. Minimum variance unbiased estimators (MVUE's) for EX and $\operatorname{Var} X$ were obtained by Finney (1941) for the case $Y = \log X$. Solutions for a much wider class of transformations were obtained by Neyman and Scott (1960) and Hoyle (1968). However there have been no analogous achievements with respect to hypothesis tests and confidence interval estimates for EX and $\operatorname{Var} X$. The present paper, in which uniformly most accurate unbiased confidence interval procedures of level $1-\alpha$ are derived for linear functions of μ and σ^2, is an approach to these problems. The results of this paper define an optimal solution for EX when $Y = \log X$, since in this case the parametric function of interest is a monotone function of $\mu + \frac{1}{2}\sigma^2$. The results also provide a basis for approximate confidence interval solutions for other parametric functions of μ and σ^2.

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It is helpful to consider the problem in terms of confidence regions in the half-plane of points (μ, σ^2). For any transformation $Y = g(X)$ likely to be of practical significance, a confidence interval for $f(\mu, \sigma^2) = EX$ or Var X is a region in this half-plane, bounded by one or two contours of the form $f(\mu, \sigma^2) = m$.

Kanofsky (1969) has proposed a method of simultaneous confidence estimation for all functions of μ and σ^2. He constructs a trapezoidal-shaped confidence region of level $1-\alpha$ for μ and σ^2, and for an arbitrary function $h(\mu, \sigma^2)$, defines a confidence set for this function as the set of values m such that the curve $h(\mu, \sigma^2) = m$ intersects this confidence region. If one is only interested in a single function, the procedure is conservative. However, for most such functions this is the only method based on exact distribution theory, to my knowledge, that has been proposed.

The usual approach to confidence interval estimation for EX or Var X has been to rely on approximate methods. For example, a common method of confidence interval estimation for EX is to transform a level $1-\alpha$ confidence interval for $EY = E(g(X))$, say (μ_1, μ_2), by the inverse transform. Then $(g^{-1}(\mu_1), g^{-1}(\mu_2))$ would be an approximate level $1-\alpha$ confidence interval for EX if g is monotone increasing. More sophisticated versions of this method have been proposed by Patterson (1966) and Hoyle (1968).

A more direct approach is to use an estimator T of $f(\mu, \sigma^2)$ and an estimator V of the variance of T. T is then assumed to be approximately normally distributed with mean $(f(\mu, \sigma^2)$ and variance equal to the observed value of V. For example, the sample mean $\bar{X}$ is an estimate of EX, and $S_X{}^2/(n(n-1)) = \sum (X_i - \bar{X})^2/(n(n-1))$ is an estimate of the variance of $\bar{X}$ (e.g., see Aitchison and Brown (1957) Section 5.62). Hoyle (1968) has suggested letting T be the MVUE of EX, and V the MVUE of Var T, which he has given for a number of transformations.

In this paper an optimal exact confidence interval procedure is presented for linear functions of μ and σ^2. That is, the procedure gives uniformly most accurate unbiased joint confidence regions of level $1-\alpha$ for μ and σ^2, bounded by one or two contours of form $\mu + \lambda\sigma^2 = m$, for arbitrary λ. This provides an optimal confidence interval procedure for EX when $Y = \log(X)$ is normal. Also it provides the basis for a new approximate confidence interval method for EX in the general case $Y = g(X)$. That is, by a proper choice of the linear coefficient λ, it seems reasonable that a confidence region bounded by one or two contours of form $f(\mu, \sigma^2) = m$ might be approximated with some success by a confidence region bounded by contours of the form $\mu + \lambda\sigma^2 = m$. Certainly the degree of approximation possible should be better than that obtainable using only vertical bounding contours, as when a confidence interval for μ is transformed to give an approximate confidence interval for EX. Also, if the contours $f(\mu, \sigma^2) = m$ are fairly straight within a convex joint confidence region of level $1-\alpha$ for μ and σ^2, it is not unreasonable to hope that an approximate confidence region should be possible that would have a true level near $1-\alpha$, and that would be less conservative than a level $1-\alpha$ region for $f(\mu, \sigma^2)$ determined by Kanofsky's method.

The main result of the paper is the derivation in Section 2 of uniformly most powerful unbiased level α hypothesis tests for linear functions of μ and σ^2. The theoretical interest of this section is mainly in the analytic detail of how a well-known theorem applies to this somewhat unusual case. A numerical example follows, illustrating the use of the tables of critical values given in the Appendix. It is not obvious that the confidence procedures defined by these tests in Section 4 define confidence sets that are intervals, an extremely desirable property both for ease of calculation and for practical usefulness of the confidence sets. The proof in Section 5 that the one-sided tests define one-sided confidence intervals provided that ν, the number of degrees of freedom available for the estimate of σ^2, is at least two, is the second major result of the paper. In Section 6 it is shown that this property does not obtain when $\nu = 1$. The analogous result in the two-sided case is proved only for $\nu = 2$ in Section 7. However it is conjectured that, as in the one-sided case, the desired property also holds for all larger values of ν.

The final section contains a brief discussion of applications of the method to confidence interval estimation for EX when $Y = g(X)$ is normal. It is shown that essentially the only direct application is to the case where $Y = \log(X)$, and that there are no nontrivial direct applications where EX is a function of μ or σ^2 alone. The construction of normal tolerance limits involves confidence interval estimation of functions of the form $\mu+\delta\sigma$ (Owen (1958)). However it is shown here that there are essentially no transformations to normality such that EX is a function of $\mu+\delta\sigma$ for some δ. A more complete discussion of approximate applications of the method is left for a subsequent paper.

2. Hypothesis tests involving $\mu+\lambda\sigma^2$. Let $\mathbf{W}$ be a spherical normal random vector of dimension n, with covariance matrix $\sigma^2 I$. Let $E\mathbf{W}$ be restricted to a subspace with orthonormal basis $\{\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_k\}$, which we extend to an orthonormal basis $\{\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_n\}$ for the vector space of $\mathbf{W}$. Let $\mathbf{U} = (U_1, \cdots, U_n)'$ be the canonical form of $\mathbf{W}$ with respect to this basis (Scheffé (1959) Section 1.6). The probability-density function of $\mathbf{U}$ is

$$f(\mathbf{u}; \boldsymbol{\beta}, \sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}n} \exp\{-\tfrac{1}{2}(\mathbf{u}'\mathbf{u} - 2\mathbf{u}'\boldsymbol{\beta} + \boldsymbol{\beta}'\boldsymbol{\beta})/\sigma^2\},$$

where $\boldsymbol{\beta} = E\mathbf{U} = (\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_k, 0, \cdots, 0)'$. This density can also be written in its exponential form, with parameters $\zeta = -1/(2\sigma^2)$ and $\eta_i = \boldsymbol{\beta}_i/\sigma^2$, $i = 1, \cdots, k$. In this form, the density function is

$$f^*(\mathbf{u}; \boldsymbol{\eta}, \zeta) = c(\boldsymbol{\eta}, \zeta) \exp\{v\zeta + \textstyle\sum_{i=1}^k u_i\eta_i\}, \tag{2.1}$$

where $v = \mathbf{u}'\mathbf{u}$, for a suitable function $c(\boldsymbol{\eta}, \zeta)$.

Let $\mu = \mathbf{a}'E\mathbf{W}$, where $\mathbf{a}$ is an arbitrary n-vector. Without loss of generality, we can assume that $\boldsymbol{\alpha}_1 = r^{\frac{1}{2}}\mathbf{a}$, for some positive number r. Then $\mu = r^{-\frac{1}{2}}\beta_1$, and $Y = r^{-\frac{1}{2}}U_1$ is the Gauss–Markov estimator of μ. Setting $\theta = \mu/\sigma^2$, we can rewrite (2.1) as

$$f^*(\mathbf{u}; \theta, \boldsymbol{\eta}, \zeta) = c(\boldsymbol{\eta}, \zeta) \exp\{ry\theta + v\zeta + \textstyle\sum_{i=2}^k u_i\eta_i\}\,.$$

Let λ be an arbitrary number, and consider the problem of testing $H(m)$: $\mu+\lambda\sigma^2 = m$ against any one of the alternatives $A_1(m): \mu+\lambda\sigma^2 < m$, $A_2(m): \mu+\lambda\sigma^2 > m$, and $A_3(m): \mu+\lambda\sigma^2 \neq m$. By a translation of $\mathbf{W}$ the problem reduces to the case $m = 0$. For this case, we may rewrite these hypotheses in terms of $\theta = \mu/\sigma^2$, as $H: \theta = -\lambda$, $A_1: \theta < -\lambda$, $A_2: \theta > -\lambda$, and $A_3: \theta \neq -\lambda$.

A theorem that applies to this problem, concerning the existence of uniformly most powerful unbiased (UMPU) tests, is to be found in Lehmann ((1959) Section 4.4). The tests are defined in terms of the conditional distribution of Y given $U_2, \cdots, U_k$, and $V = \mathbf{U}'\mathbf{U}$. The transformation

$$T = r^{\frac{1}{2}} Y/(S^2/v)^{\frac{1}{2}}, \tag{2.2}$$

where

$$S^2 = V - \sum_{i=1}^{k} U_i^2 = \sum_{i=k+1}^{n} U_i^2,$$

and $v = n-k$, is monotone in Y for fixed $U_2, \cdots, U_k$ and V. Also, unlike Y, which is bounded by the inequality $rY^2 \leqq V$, T has an unbounded conditional distribution. S^2/v, of course, is the usual estimator of σ^2, and Y has variance σ^2/r. According to the theorem, the UMP unbiased level α tests of H against A_1, A_2, and A_3 respectively are given by the rules R_1: "Reject H in favor of A_1 if $T < t(\alpha)$," R_2: "Reject H in favor of A_2 if $T > t(1-\alpha)$," and R_3: "Reject H in favor of A_3 if $T < t_1(\alpha)$ or $T > t_2(\alpha)$," where the functions t, t_1, and t_2 depend on $U_2, \cdots, U_k$, and V. If $f(\tau; -\lambda)$ denotes the conditional density, under H, of T given $U_2, \cdots, U_k$ and V, $t(\alpha)$ is defined by

$$\int_{-\infty}^{t(\alpha)} f(\tau; -\lambda)\, d\tau = \alpha, \tag{2.3}$$

and $t_1(\alpha)$ and $t_2(\alpha)$ are defined by the pair of equations

$$\int_{t_1}^{t_2} f(\tau; -\lambda)\, d\tau = 1-\alpha, \tag{2.4}$$

$$\int_{t_1}^{t_2} \tau(v+\tau^2)^{-\frac{1}{2}} f(\tau: -\lambda)\, d\tau = (1-\alpha) \int_{-\infty}^{\infty} \tau(v+\tau^2)^{-\frac{1}{2}} f(\tau; -\lambda)\, d\tau. \tag{2.5}$$

The joint density of $Y, U_2, \cdots, U_k$, and V may be obtained from the joint distribution of S^2 and $U_1, \cdots, U_k$. Since these statistics are mutually independent, with S^2/σ^2 a chi-square variate with v degrees of freedom and U_i normal (β_i, σ^2), $i = 1, \cdots k$, we have

$$f(y, u_2, \cdots, u_k, v) \propto (s^2)^{\frac{1}{2}v-1} \exp\{v\zeta + ry\theta + \sum_{i=2}^{k} u_i\eta_i\} \tag{2.6}$$

where $s^2 = v - ry^2 - \sum_{i=2}^{k} u_i^2$, for $v > 0$ and $ry^2 + \sum_{i=2}^{k} u_i^2 < v$. This gives the conditional density for Y,

$$f(y \mid u_2, \cdots, u_k, v) \propto (v^* - ry^2)^{\frac{1}{2}v-1} \exp\{r\theta y\} \tag{2.7}$$

where $ry^2 \leqq v^* = v - \sum_{i=2}^{k} u_i^2$. This conditional density depends only on r, θ, v, and v^*. The conditional density for the transformed statistic T, accordingly, is (from (2.2) and (2.7))

$$f(\tau \mid v^*; \theta, v) \propto (v+\tau^2)^{-\frac{1}{2}(v+1)} \exp\{\theta(rv^*)^{\frac{1}{2}}\tau/(v+\tau^2)^{\frac{1}{2}}\}, \tag{2.8}$$

for $-\infty < \tau < \infty$.

Since $V^* = S^2 + rY^2$ is a random variable whose mean, for small μ, is roughly proportional to $v+1$, and since $r = \sigma^2/\mathrm{Var}\,(Y)$ can be expected to be roughly proportional to $v+1$, it is convenient to define

(2.9) $$Z = (rV^*)^{\frac{1}{2}}/(v+1) = r^{\frac{1}{2}}(S^2+rY^2)^{\frac{1}{2}}/(v+1)$$

as a variate whose magnitude should not depend much on the sample size. In what follows, we shall speak of "the conditional distribution of T given $Z = z$." Assuming $H:\theta = -\lambda$, and defining a "noncentrality parameter" $\xi = -\lambda z$, this conditional distribution may be referred to as "the conditional t distribution with v degrees of freedom and noncentrality ξ," with density

(2.10) $$f_v(\tau \mid \xi) \propto (v+\tau^2)^{-\frac{1}{2}(v+1)} \exp\{(v+1)\xi\tau/(v+\tau^2)^{\frac{1}{2}}\},$$

for $-\infty < \tau < \infty$, $-\infty < \xi < \infty$, and $v = 1, 2, \cdots$.

Substituting the density (2.10) into the integral equations (2.3) to (2.5), the critical values may be determined. The critical value obtained from (2.3) is denoted by $t(v, \xi, \alpha)$, and the critical values obtained from (2.4) and (2.5) are denoted by $t_1(v, \xi, \alpha)$ and $t_2(v, \xi, \alpha)$. It is apparent from (2.10) that $f_v(\tau \mid \xi) = f_v(-\tau \mid -\xi)$, giving the symmetry relations for the critical values,

(2.11) $$t(v, \xi, \alpha) = -t(v, -\xi, 1-\alpha),$$

(2.12) $$t_1(v, \xi, \alpha) = -t_2(v, -\xi, \alpha).$$

Tables of critical values for one and two-sided tests have been computed for selected values of v, ξ, and α. The complete tables and the method of their construction are given elsewhere (Land (1969)). A brief subcollection of the one and two-sided tables is given in Tables 1 and 2. Briefly, the one-sided tables were computed by the step-by-step Runge-Kutta-Gill numerical integration of a system of differential equations based on the conditional density (2.10), closely following the method used by Johnson *et al* ((1963) Section 4.1) to compute quantiles for certain Pearson distributions. The two-sided tables were computed only for even values of v, by a cruder method in which the integrals in equations (2.4) and (2.5) were computed directly. The integration limits were adjusted until the ratios of left-hand to right-hand sides in both equations were arbitrarily close to one. Numerical problems prevented the calculation of critical values for some combinations of large v and small ξ.

An examination of (2.10) shows that the conditional t distribution with v degrees of freedom and noncentrality ξ is asymptotically normal as $v \to \infty$. More specifically $T-(v+1)\xi v^{-\frac{1}{2}}$ is asymptotically normal with zero mean and unit variance. The critical value tables, however, indicate that this convergence is slow for large ξ. Nevertheless, the Cornish-Fisher approximations of order $1/v$ and $1/v^2$ for the quantiles $t(v, \xi, \alpha)$ have been found to work reasonably well for v above 25 and ξ less than 100 (Land (1968)).

3. A numerical example. Let $y = 3.7$ and $s^2/10 = 10.0$ be the sample mean and variance of a random sample of size 11 from a normal (μ, σ^2) distribution. In order to test the null hypothesis that $\mu+\sigma^2/2$ is equal to 6.0 against the alternative that it is larger, we first compute $t_{6.0} = (3.7-6.0)/(10.0/11)^{\frac{1}{2}} = -2.412$, and $z_{6.0} = ((10/11)(10.0)+(3.7-6.0)^2)^{\frac{1}{2}} = 3.728$. Since $\xi_{6.0} = (-1/2)z_{6.0} = -1.864$, we

TABLE 1

Critical values $t(\nu, \xi, \alpha)$ for one-sided tests, for selected values of ν, ξ, and α

	α					
ξ	.01	.05	.10	.90	.95	.99
$\nu = 2$						
0	−6.965	−2.920	−1.886	1.886	2.920	6.965
.1	−5.929	−2.446	−1.540	2.245	3.416	8.053
.2	−4.967	−2.004	−1.217	2.610	3.922	9.169
.5	−2.668	−0.939	−0.426	3.680	5.415	12.474
1.	−0.737	.024	.349	5.202	7.559	17.255
2.	.338	.818	1.107	7.447	10.746	24.403
5.	1.359	1.887	2.249	11.869	17.057	38.613
10.	2.249	2.922	3.398	16.830	24.153	54.621
20.	3.398	4.306	4.957	23.832	34.180	77.256
50.	5.575	6.970	7.978	37.712	54.063	122.161
100.	7.978	9.932	11.349	53.347	76.467	172.766
$\nu = 5$						
0	−3.365	−2.015	−1.476	1.476	2.015	3.365
.1	−2.886	−1.658	−1.158	1.798	2.377	3.852
.2	−2.427	−1.314	−0.851	2.116	2.736	4.337
.5	−1.258	−0.434	−0.059	3.003	3.745	5.711
1.	−0.079	.515	.830	4.214	5.138	7.639
2.	1.029	1.557	1.872	6.008	7.227	10.575
5.	2.577	3.208	3.609	9.583	11.428	16.548
10.	4.059	4.874	5.404	13.599	16.170	23.331
20.	6.026	7.129	7.852	19.267	22.877	32.946
50.	9.795	11.493	12.612	30.497	36.180	52.048
100.	13.977	16.358	17.929	43.146	51.171	73.587
$\nu = 10$						
0	−2.764	−1.812	−1.372	1.372	1.812	2.764
.1	−2.305	−1.414	−0.995	1.750	2.213	3.225
.2	−1.862	−1.029	−0.631	2.121	2.606	3.680
.5	−0.715	−0.025	.325	3.138	3.689	4.945
1.	.546	1.130	1.451	4.502	5.159	6.689
2.	1.953	2.528	2.867	6.509	7.347	9.329
5.	4.173	4.892	5.331	10.482	11.721	14.683
10.	6.397	7.336	7.917	14.927	16.639	20.747
20.	9.393	10.669	11.464	21.186	23.579	29.332
50.	15.178	17.148	18.379	33.573	37.329	46.371
100.	21.618	24.382	26.112	47.514	52.813	65.576

TABLE 1—*continued*

ξ	α .01	.05	.10	.90	.95	.99
$\nu = 20$						
0	−2.528	−1.725	−1.325	1.325	1.725	2.528
.1	−1.992	−1.224	−0.838	1.813	2.226	3.066
.2	−1.472	−0.739	−0.366	2.290	2.716	3.592
.5	−0.113	.537	.881	3.584	4.054	5.039
1.	1.462	2.062	2.394	5.300	5.843	7.003
2.	3.383	4.010	4.372	7.800	8.475	9.939
5.	6.629	7.436	7.913	12.706	13.685	15.831
10.	9.976	11.039	11.673	18.166	19.509	22.463
20.	14.531	15.982	16.851	25.835	27.705	31.826
50.	23.377	25.622	26.969	40.988	43.917	50.380
100.	33.248	36.401	38.294	58.032	62.161	71.277
$\nu = 50$						
0	−2.403	−1.676	−1.299	1.299	1.676	2.403
.1	−1.645	−0.938	−0.569	2.029	2.414	3.162
.2	−0.910	−0.222	.140	2.740	3.133	3.902
.5	1.029	1.672	2.019	4.660	5.080	5.915
1.	3.371	4.002	4.351	7.175	7.647	8.598
2.	6.409	7.099	7.488	10.797	11.370	12.538
5.	11.801	12.713	13.235	17.829	18.646	20.324
10.	17.491	18.703	19.400	25.605	26.718	29.012
20.	25.307	26.967	27.923	36.495	38.040	41.228
50.	40.556	43.131	44.616	57.979	60.394	65.382
100.	57.610	61.229	63.316	82.125	85.527	92.556
$\nu = 100$						
0	−2.364	−1.660	−1.290	1.290	1.660	2.364
.1	−1.332	−0.642	−0.277	2.304	2.679	3.397
.2	−0.329	.348	.708	3.290	3.671	4.404
.5	2.325	2.975	3.326	5.947	6.348	7.128
1.	5.592	6.246	6.605	9.404	9.848	10.720
2.	9.939	10.669	11.075	14.346	14.879	15.937
5.	17.823	18.802	19.351	23.881	24.633	26.135
10.	26.232	27.537	28.272	34.384	35.407	37.452
20.	37.830	39.622	40.632	49.072	50.489	53.325
50.	69.513	63.296	64.866	78.018	80.230	84.663
100.	85.908	89.820	92.028	110.537	113.653	119.897

TABLE 1—*continued*

ξ	α .01	.05	.10	.90	.95	.99
$\nu = 200$						
0	−2.345	−1.653	−1.286	1.286	1.653	2.345
.1	−0.912	−0.229	.134	2.706	3.077	3.779
.2	.481	1.155	1.514	4.088	4.462	5.175
.5	4.178	4.836	5.190	7.802	8.192	8.940
1.	8.783	9.458	9.826	12.612	13.039	13.865
2.	15.015	15.778	16.197	19.449	19.958	20.948
5.	26.474	27.505	28.076	32.574	33.287	34.683
10.	38.781	40.161	40.926	46.993	47.960	49.854
20.	55.805	57.703	58.756	67.130	68.468	71.092
50.	89.152	92.102	93.741	106.788	108.875	112.973
100.	126.514	130.662	132.966	151.327	154.266	160.036
$\nu = 500$						
0	−2.334	−1.648	−1.283	1.283	1.648	2.334
.1	−0.092	.588	.950	3.517	3.884	4.576
.2	2.086	2.760	3.120	5.689	6.058	6.757
.5	7.884	8.552	8.911	11.517	11.898	12.623
1.	15.182	15.877	16.253	19.032	19.445	20.235
2.	25.189	25.985	26.417	29.658	30.147	31.084
5.	43.802	44.886	45.478	49.958	50.639	51.951
10.	63.912	65.366	66.162	72.201	73.123	74.899
20.	91.797	93.800	94.895	103.230	104.504	106.962
50.	146.494	149.609	151.314	164.299	166.286	170.119
100.	207.813	212.194	214.592	232.865	235.662	241.059
$\nu = 1000$						
0	−2.330	−1.646	−1.282	1.282	1.646	2.330
.1	.828	1.508	1.870	4.435	4.801	5.489
.2	3.898	4.573	4.934	7.501	7.868	8.561
.5	12.080	12.754	13.116	15.720	16.097	16.811
1.	22.430	23.136	23.517	26.293	26.700	27.474
2.	36.713	37.526	37.965	41.202	41.681	42.595
5.	63.423	64.536	65.139	69.612	70.279	71.552
10.	92.363	93.858	94.669	100.700	101.600	103.322
20.	132.541	134.601	135.719	144.041	145.285	147.665
50.	211.404	214.608	216.349	229.314	231.254	234.965
100.	299.840	304.348	306.797	325.041	327.771	332.995

TABLE 2

Critical values $t_1(\nu, \xi, \alpha)$ and $t_2(\nu, \xi, \alpha)$ for two-sided tests, for selected values of ν, ξ, and α

	α					
ξ	.010		.050		.100	
$\nu = 2$						
0	−9.925	9.925	−4.303	4.303	−2.920	2.920
.1	−8.924	10.925	−3.851	4.755	−2.596	3.244
.2	−7.937	11.916	−3.404	5.202	−2.276	3.565
.5	−5.202	14.758	−2.168	6.493	−1.389	4.495
1.	−2.063	18.977	−0.754	8.461	−0.361	5.933
2.	−0.150	26.189	.298	11.839	.519	8.373
5.	.949	41.422	1.320	18.777	1.546	13.322
10.	1.755	58.593	2.200	26.583	2.483	18.879
20.	2.750	82.872	3.333	37.614	3.712	26.728
50.	4.593	131.041	5.476	59.492	6.054	42.287
100.	6.608	185.324	7.840	84.143	8.649	59.815
$\nu = 4$						
0	−4.604	4.604	−2.776	2.776	−2.132	2.132
.1	−4.103	5.105	−2.435	3.118	−1.840	2.424
.2	−3.611	5.600	−2.099	3.455	−1.553	2.712
.5	−2.267	7.015	−1.184	4.424	−0.770	3.540
1.	−0.782	9.118	−0.143	5.865	.144	4.768
2.	.434	12.534	.873	8.163	1.111	6.701
5.	1.803	19.590	2.265	12.853	2.539	10.615
10.	3.008	27.612	3.585	18.162	3.937	15.030
20.	4.564	38.987	5.333	25.677	5.807	21.271
50.	7.504	61.588	8.676	40.593	9.405	33.649
100.	10.745	87.071	12.385	57.404	13.405	47.594
$\nu = 10$						
0	−3.169	3.169	−2.228	2.228	−1.812	1.812
.5	−1.121	5.293	−0.456	4.058	−0.140	3.533
1.	.197	7.057	.726	5.556	.998	4.929
2.	1.624	9.786	2.114	7.839	2.383	7.035
5.	3.780	15.362	4.367	12.445	4.704	11.252
10.	5.892	21.689	6.649	17.640	7.089	15.989
20.	8.711	30.652	9.734	24.979	10.331	22.669
50.	14.129	48.447	15.703	39.527	16.625	35.900
100.	20.149	68.506	22.353	55.916	23.648	50.798

TABLE 2—*continued*

ξ	α = .010		.050		.100	
$\nu = 20$						
0	−2.845	2.845	−2.086	2.086	−1.725	1.725
1.	1.156	7.318	1.689	6.208	1.968	5.705
2.	3.069	10.327	3.609	8.921	3.903	8.293
5.	6.233	16.398	6.913	14.332	7.293	13.416
10.	9.459	23.245	10.348	20.397	10.849	19.138
20.	13.829	32.918	15.039	28.943	15.723	27.188
50.	22.294	52.094	24.161	45.858	25.220	43.108
100.	31.729	73.694	34.349	64.899	35.836	61.021
$\nu = 50$						
0	−2.678	2.678	−2.009	2.009	−1.676	1.676
2.	6.101	12.888	6.704	11.791	7.027	11.271
5.	11.398	20.825	12.187	19.245	12.616	18.501
10.	16.958	29.698	18.002	27.536	18.573	26.520
20.	24.578	42.184	26.006	39.177	26.788	37.765
50.	39.426	66.878	41.640	62.171	42.853	59.963
100.	56.024	94.665	59.133	88.031	60.838	84.921
$\nu = 100$						
0	−2.626	2.626	−1.984	1.984	−1.660	1.660
2.	9.637	16.278	10.277	15.293	10.618	14.814
5.	17.413	26.613	18.271	25.215	18.730	24.538
10.	25.687	38.103	26.828	36.197	27.441	35.277
20.	37.083	54.228	38.648	51.584	39.489	50.309
50.	59.353	86.075	61.782	81.942	63.089	79.949
100.	84.278	121.886	87.692	116.064	89.529	113.256

reject the null hypothesis at level α if $t_{6.0}$ is greater than $t(10, -1.864, 1-\alpha)$. Using the symmetry relation (2.11) we obtain from Table 1 the critical values $t(10, -0.5, .95) = 0.025$, $t(10, -1.0, .95) = -1.130$, $t(10, -2.0, .95) = -2.528$, and $t(10, -5.0, .95) = -4.982$. Four-point Lagrangian interpolation on these numbers with respect to $\log \xi$ gives the approximate value $t(10, -1.864, .95) = -2.372$. (This interpolation method, applied to Table 1, gives interpolated values accurate to within three significant digits, according to comparisons using more detailed tables. For $\nu = 10$ and ξ around ± 1.75, the accuracy is within .003.) Similarly, we have $t(10, -0.5, .90) = -0.325$, $t(10, -1.0, .90) = 1.451$, $t(10, -2.0, .90) = -2.867$, and $t(10, -5.0, .90) = -5.331$, from which we obtain the interpolated value $t(10, -1.864, .90) = -2.707$. Thus we have $t(10, -1.864, .90) < t_{6.0} < t(10, -1.864, .95)$, so that the null hypothesis is rejected at level .10, but not at level .05.

If we vary the above procedure, computing $t_m = (3.7-m)/(10.0/11)^{\frac{1}{2}}$ and $z_m = [(10/11)(10.0)+(3.7-m)^2]^{\frac{1}{2}}$ for different values of m, we find that the hypothesis that $\mu+\sigma^2/2 = m$ is rejected at level .05 in favor of the alternative that $\mu+\sigma^2/2 > m$ for $m < 5.62$, and accepted for $m > 5.62$.

4. Confidence intervals for $\mu+\lambda\sigma^2$. The interval $[5.62, \infty)$ obtained in the above example as the set of values m for which $t_m < t(14, -\frac{1}{2}z_m, .95)$ is a level .95 confidence interval for $\mu+\frac{1}{2}\sigma^2$, corresponding to the given sample mean and variance. More generally, if we define

$$T_m = (r)^{\frac{1}{2}}(Y-m)/(S^2/v)^{\frac{1}{2}} \tag{4.1}$$

and

$$Z_m = (rS^2+r^2(Y-m)^2)^{\frac{1}{2}}/(v+1), \tag{4.2}$$

and define $P_i(Y, S^2)$ as the set of values m such that the rule $R_i (i = 1, 2, 3)$ applied to T_m and Z_m at level α does not reject the null hypothesis that $\mu+\lambda\sigma^2 = m$, then $P_i(Y, S^2)$ is a level $1-\alpha$ confidence procedure for $\mu+\lambda\sigma^2$. Because the rules R_i define UMPU level α tests, the confidence procedures $P_i(Y, S^2)$ are uniformly most accurate unbiased level $1-\alpha$, in the sense used by Lehmann (1959).

It remains to show that the confidence sets $P_i(y, s^2)$ are intervals. Although these confidence sets are defined in terms of UMPU tests of hypotheses of form $H(-\lambda):\theta = -\lambda$ against one and two-sided alternatives, where θ is one of the parameters of a multiparameter exponential family, the result does not follow from this fact. In particular, the argument in Lehmann ((1959) pages 179–80), showing that in such cases the UMPU tests, do define confidence intervals for θ, does not apply in the present case since θ is not the parameter of interest.

Given two functions $g(m)$ and $h(m)$, the sets $\{m:g(m) < h(m)\}$, $\{m:g(m) = h(m)\}$, and $\{m:g(m) > h(m)\}$ are, respectively, a right-infinite interval, a point, and a left-infinite interval if and only if

$$g(m') \leqq h(m') \Rightarrow g(m'') < h(m'') \text{ for } m' < m''. \tag{4.3}$$

Setting $g(m) = t_m$, the confidence sets $P_i(y, s^2)(i = 1, 2)$ are intervals if and only if (4.3) holds for $h(m) = t(v, -\lambda z_m, \alpha)$, and the sets $P_3(y, s^2)$ are intervals if and only if (4.3) holds for $h(m) = t_i(v, -\lambda z_m, \alpha)$ for both $i = 1$ and $i = 2$.

Introducing the monotone transformation, given z,

$$w(t;z) = ((v+1)/r)zt/(v+t^2)^{\frac{1}{2}}, \tag{4.4}$$

we have, by (4.1), $w(t_m; z_m) = y-m$, so that it is necessary to show that the derivatives with respect to m of the transformed critical values, considered as functions of m, are greater than -1. Since, by (4.2),

$$\frac{\partial}{\partial m} z_m = -(r/(v+1))r(y-m)/(rs^2+r^2(y-m)^2)^{\frac{1}{2}},$$

which has absolute value less than $r/(v+1)$, it is enough to show that the derivatives with respect to z of

$$p(z) = w(t(\nu, -\lambda z, \alpha); z)r/(\nu+1) \tag{4.5}$$

and

$$p_i(z) = w(t_i(\nu, -\lambda z, \alpha); z)r/(\nu+1), \qquad i = 1, 2, \tag{4.6}$$

are between -1 and $+1$.

Section 5 contains a proof that $|p'(z)| \leqq 1$ for $\nu \geqq 2$, and in Section 6 it is shown that this is not true in general for $\nu = 1$. The two-sided case is discussed in Section 7.

5. The one-sided case, $\nu \geqq 2$. Let $p = p(z)$, $\beta = -(\nu+1)\lambda$, and $k = (\nu/2)-1$. From (4.4) and (4.5) we have

$$p = zt(\nu, -\lambda z, \alpha)/(\nu+t^2(\nu, -\lambda z, \alpha))^{\frac{1}{2}}. \tag{5.1}$$

Note that while $-\infty < t(\nu, -\lambda z, \alpha) < \infty$, we have $-z < p < z$, and that $t(\nu, -\lambda z, \alpha) = \nu^{\frac{1}{2}}p/(z^2-p^2)^{\frac{1}{2}}$. It follows, therefore, from (2.3) that

$$\textstyle\int_{-z}^{p} g(u)\, du = \alpha, \tag{5.2}$$

where $g(u)$ is an appropriate probability density function obtained by a change of variable formula from the conditional density of T given $Z = z$. From (5.1) and (2.10) we obtain this density as $g(u) = g_k(u)/G_k(z)$, where

$$g_k(u) = (z^2-u^2)^k \exp\{\beta u\} \tag{5.3}$$

and

$$G_k(u) = \textstyle\int_{-z}^{u} g_k(v)\, dv, \tag{5.4}$$

for $-z < u < z$. Therefore we can write (5.2) in the form

$$G_k(p) = \alpha G_k(z). \tag{5.5}$$

For $\nu = 2$, (5.5) reduces to

$$p = \log(\alpha R+(1-\alpha)R^{-1})/\beta$$

where $R = \exp[\beta z]$. Therefore

$$p' = (\alpha R-(1-\alpha)R^{-1})/(\alpha R+(1-\alpha)R^{-1}),$$

which has absolute value not greater than one, for $0 < \alpha < 1$.

For $\nu > 2$, we can differentiate both sides of (5.5) with respect to z to obtain

$$p'g_k(p)+2kzG_{k-1}(p) = 2kzG_{k-1}(z)\alpha.$$

By (5.5) this reduces to

$$p'g_k(p) = -2kz(G_{k-1}(p)-G_k(p)M), \tag{5.6}$$

where $M = G_{k-1}(z)/G_k(z)$.

For $0 \leqq b \leqq 2z$, let $C(b)$ be the partial derivative with respect to z of $G_k(b-z)/G_k(z)$. Then we have

$$C(b)G_k(z) = -g_k(b-z)+2kz(G_{k-1}(b-z)-MG_k(b-z)). \tag{5.7}$$

For any fixed value of z, if b is chosen so that $b-z = p$, (5.6) and (5.7) imply that

$$C(b)G_k(z) = -(1+p')g_k(b-z).$$

Since $g_k(u)$ is positive for $-z < u < z$, it follows that for any fixed z, $p' \geqq -1$ if and only if $C(b) \leqq 0$ for $b = z+p$. Remembering that (5.1) defines p as a function of α and λ as well as z, it follows from the symmetry relation (2.11) that if $p'(z) = p' \geqq -1$ for $0 < \alpha < 1$, $z > 0$, and $-\infty < \lambda < \infty$, then $p'(z) \leqq 1$ for the same range of α, z, and λ. Therefore it is enough to show that $C(b) \leqq 0$ for $0 < b < 2z, z > 0$, and $-\infty < \beta < \infty$.

From (5.7), $C(b)$ is a differentiable function of b, and $C(0) = C(2z) = 0$. Thus there must exist a zero of $C'(b)$ in the interval $0 < b < 2z$. Differentiating (5.7) and simplifying, we get

$$C'(b)G_k(z) = g_{k-1}(b-z)b(2k-(2z-b)(\beta+2kzM)). \tag{5.8}$$

Thus the only possible zero of $C'(b)$ is at $b^* = 2(z-k/(\beta+2kzM))$. Since $b^* > 0$, it follows that $k-z(\beta+2kzM) < 0$. But by (5.8), $C'(b^*/2)G_k(z) = g(b^*/2-z)$ $(b^*/2)(k-z(\beta+2kzM))$, and thus $C'(b^*/2) < 0$, which implies that $C(b^*) < 0$. Therefore $C(b) < 0$ for $0 < b < 2z$.

6. The one-sided case, $\nu = 1$. For $\nu = 1$, equation (5.5) may be rewritten as

$$H(p/z) = \alpha H(1) \tag{6.1}$$

where $H(u) = \int_{-1}^{u} h(v)\,dv$ and $h(u) = (1-u^2)^{-\frac{1}{2}} \exp(\beta z u)$ for $-1 < u < 1$, where $\beta = -2\lambda$. Differentiating (6.1) with respect to z gives

$$(p'-p/z)h(p/z)/z+\beta H^*(p/z) = \alpha\beta H^*(1),$$

where $H^*(u) = \int_{-1}^{u} vh(v)\,dv$.
By (6.1) this reduces to

$$p'h(p/z) = (p/z)h(p/z)+\beta z(H(p/z)A-H^*(p/z)), \tag{6.2}$$

where $A = H^*(1)/H(1)$.

As in Section 5, $|p'| \leqq 1$ for $0 < \alpha < 1$, $0 < z$, and $-\infty < \beta < \infty$ if and only if $p' \geqq -1$ for the same values of α, z, and β. Again as in Section 5, let b be any fixed number between zero and $2z$. Defining $D(b)$ as the derivative with respect to z of $H(b/z-1)/(H(1)$, we have

$$D(b)H(1) = -(b/z^2)h(b/z-1)+\beta \int_{-1}^{b/z-1} (u-A)h(u)\,du. \tag{6.3}$$

For fixed z, if b is chosen so that $p(z) = b-z$, we have, by (6.2) and (6.3) $D(b)H(1) = -(1+p')h(p/z)/z$. Since $h(u)$ is positive for $-1 < u < 1$, it follows that $p' \geqq -1$ for $0 < \alpha < 1$, $z > 0$, and $-\infty < \beta < \infty$ if and only if $D(b)$ is negative for $0 < b < 2z$.

By (6.3) it is easy to show that $D(b) \to 0$ as $b \to 0$, and that $D(b) \to -\infty$ as $b \to 2z$. By differentiating (6.3) with respect to b and simplifying we obtain

(6.4) $$D'(b)H(1) = h(b/z-1)[-1-\beta(2z-b)(1+A)]/(2z^2-bz).$$

The only possible zero of $D'(b)$ is at $b^* = 2z+1/(\beta(1+A))$, assuming $\beta \neq 0$ (if $\beta = 0$, $D'(b) < 0$ for $0 < b < 2z$). If $b^* > 2z$, or $b^* < 0$, then $D(b) < 0$ for $0 < b < 2z$. However, if $0 < b^* < 2z$ then either b^* represents an inflexion point of $D(b)$, which is negative for $0 < b < 2z$, or $D(b) > 0$ at least for $0 < b < b^*$.

In order that $0 < b^* < 2z$, it is necessary and sufficient that $-2z < \beta(1+A))^{-1} < 0$. $A = A(\beta z)$ can be expressed as the ratio of two Bessel functions, $A(\beta z) = I_1(\beta z)/I_0(\beta z)$ (since $I_0'(v) = I_1(v)$), where, for $m > -\frac{1}{2}$ and real v,

(6.5) $$I_m(v) = (v/2)^m \pi^{-\frac{1}{2}}(1/\Gamma(m+\tfrac{1}{2})) \int_{-1}^{1} (1-u^2)^{m-\frac{1}{2}} \exp(vu)\, du.$$

Since z is always positive, it is clear that $A(\beta z)$ has the same sign as β. Therefore $(\beta(1+A))^{-1}$ is positive for positive β, which means that $b^* > 2z$. If $\beta < 0$, $A(\beta z)$ is negative, and $\beta(1+A(\beta z)) < 0$ if and only if $A(\beta z) > -1$. Also, $-2z < (\beta(1+A(\beta z)))^{-1}$ if and only if $A(\beta z) > -1-(2z\beta)^{-1}$ or $A(\beta z) < -1$. Therefore $0 < b^* < 2z$ if and only if $A(\beta z) > -1-(2z\beta)^{-1}$. From (6.5) and the preceding discussion we obtain the relation $A(-v) = -A(v)$, from which it follows that $0 < b^* < 2z$ for some negative β if there exists a positive number v such that $A(v) = I_1(v)/I_0(v) < 1-(2v)^{-1}$. An examination of tables of $I_0(v)$ and $I_1(v)$ (Abramovitz and Stegun (1964) Table 9.8) shows that this inequality obtains for many values of v. For example, $I_1(1)/I_0(1) = .4464$, which is less than $1-\frac{1}{2}$. Thus for $\beta\xi = -1$, we must have $0 < b^* < 2z$.

An examination of (6.4) readily shows that $D'(b^*+\delta)$ is positive for $\delta < 0$ and negative for $\delta > 0$, which implies that $D(b)$ has a maximum, which must be positive, at $b = b^*$. Thus there exist combinations of α, λ, y, s^2 such that the confidence sets $P_1(y, s^2)$ and $P_2(y, s^2)$ for $\mu+\lambda\sigma^2$ are not intervals.

7. The two-sided case. Let $g_k(u)$ and $G_k(u)$ be defined as in Section 5 for $-z < u < z$, and let $H_k(u) = \int_{-z}^{u} v g_k(v)\, dv$. Let p_1 and p_2 be defined in terms of the critical values $t_1(\nu, \xi, \alpha)$ and $t_2(\nu, \xi, \alpha)$, respectively, according to (5.1). Under this transformation of variables, equations (2.4) and (2.5) may be rewritten as

(7.1) $$G_k(p_2)-G_k(p_1) = (1-\alpha)G_k(z),$$

(7.2) $$H_k(p_2)-H_k(p_1) = (1-\alpha)H_k(z).$$

For $\nu = 2 (k = 0)$, equations (7.1) and (7.2) may be evaluated by direct integration to give, after simplification,

(7.3) $$e^{\beta p_2}-e^{\beta p_1} = (1-\alpha)(e^{\beta z}-e^{-\beta z})$$

and $e^{\beta p_2}(\beta p_2-1)-e^{\beta p_1}(\beta p_1-1) = (1-\alpha)(e^{\beta z}(\beta z-1)+e^{-\beta z}(\beta z+1))$, which reduces by (7.3), to

(7.4) $$p_2 e^{\beta p_2}-p_1 e^{\beta p_1} = z(1-\alpha)(e^{\beta z}+e^{-\beta z}).$$

If we differentiate both sides of (7.3) with respect to z we obtain

$$p_2' \, e^{\beta p_2} - p_1' \, e^{\beta p_1} = (1-\alpha)(e^{\beta z} + e^{-\beta z}). \tag{7.5}$$

By (7.4) this reduces to

$$(p_2' - p_2/z) \, e^{\beta p_2} = (p_1' - p_1/z) \, e^{\beta p_1}. \tag{7.6}$$

Similarly, differentiation of (7.4) gives

$$p_2'(1+\beta p_2) \, e^{\beta p_2} - p_1'(1+\beta p_1) \, e^{\beta p_1} = (1-\alpha)(1+\beta z) \, e^{\beta z} + (1-\beta z) \, e^{-\beta z},$$

which reduces, by (7.3) and (7.5), to

$$(p_2' p_2 - z) \, e^{\beta p_2} = (p_1' p_1 - z) \, e^{\beta p_1}. \tag{7.7}$$

Equations (7.6) and (7.7) may be used to solve for p_1' and p_2' in terms of p_1, p_2, and z. If we multiply both sides of (7.6) by p_2 and subtract from (7.7), we obtain, after simplification,

$$p_1' = (p_1 p_2 - z^2 + (z^2 - p_2{}^2) \, e^{\beta(p_2 - p_1)})/(z(p_2 - p_1)). \tag{7.8}$$

If (7.6) is multiplied by p_1 and subtracted from (7.7), we get

$$p_2' = (p_1 p_2 - z^2 + (z^2 - p_1{}^2) \, e^{\beta(p_1 - p_2)})/(z(p_2 - p_1)). \tag{7.9}$$

In particular, it follows from (7.8) and (7.9) that $|p_i'| \leqq 1$, $i = 1, 2$, for $\nu = 2$, if and only if

$$\frac{z+p_1}{z+p_2} \leqq e^{\beta(p_2 - p_1)} \leqq \frac{z-p_1}{z-p_2}. \tag{7.10}$$

If both sides of (7.3) are multiplied by z and added to the corresponding sides of (7.4), we obtain

$$(p_2 + z) \, e^{\beta p_2} - (p_1 + z) \, e^{\beta p_1} = 2z(1-\alpha) \, e^{\beta z},$$

while subtraction gives

$$(p_2 - z) \, e^{\beta p_2} - (p_1 - z) \, e^{\beta p_1} = 2z(1-\alpha) \, e^{-\beta z}.$$

These two equations can be restated, by dividing through by $\exp(\beta p_1)$ and $(p_2 + z)$ or $(p_2 - z)$, in the forms

$$e^{\beta(p_2 - p_1)} - \frac{p_1 + z}{p_2 + z} = \frac{2z(1-\alpha)}{p_2 + z} \, e^{\beta(z - p_1)} \tag{7.11}$$

and

$$e^{\beta(p_2 - p_1)} - \frac{p_1 - z}{p_2 - z} = \frac{2z(1-\alpha)}{p_2 - z} \, e^{-\beta(z + p_1)}). \tag{7.12}$$

Since z is positive and p_2 is less than z, the right-hand side of (7.11) is positive, while that of (7.12) is negative. Thus for $\nu = 2$, (7.10) holds for any α and any β, which means that the two-sided tests define confidence intervals for $\mu + \lambda\sigma^2$, for any λ.

For $\nu > 2$, a development analogous to that of equations (7.3) through (7.10) may be obtained, using integration by parts to express $H_k(u)$ in terms of $g_k(u)$, $G_k(u)$, and $G_{k-1}(u)$ (a device not possible in the case $\nu = 2$ since $G_{-1}(u)$ is not defined). As in the case $\nu = 2$, it is possible to solve for p_1' and p_2' in terms of p_1, p_2, and z. From this we obtain the result that $|p_i'| \leqq 1$, for $i = 1, 2$, if and only if

$$\frac{z+p_1}{z+p_2} \leqq \frac{g_k(p_2)}{g_k(p_1)} \leqq \frac{z-p_1}{z-p_2}, \tag{7.13}$$

obviously the general form of (7.10). However, at this writing no proof of (7.13) has been obtained for $k > 0$. It should be noted that if the conditional density of $f_\nu(t \mid \xi)$ were symmetric about its mean, we would have $t_1(\nu, \xi, \alpha) = t(\nu, \xi, \alpha/2)$ and $t_2(\nu, \xi, \alpha) = t(\nu, \xi, 1-\alpha/2)$, and by the results of Section 5, the two-sided tests would define confidence intervals. As ν increases, $f_\nu(t \mid \xi)$ becomes more nearly symmetric, in fact $f_\nu(t \mid \xi)$ approaches a normal density as $\nu \to \infty$. It has been shown above that confidence intervals are defined by the two-sided tests when $\nu = 2$, for all α and all ξ. A plausible conjecture is that the same is true for $\nu > 2$, when $f_\nu(t \mid \xi)$ is more nearly symmetric than in the case $\nu = 2$. This conjecture is supported by the observation that in all cases where two-sided critical values have been calculated for $\nu > 2$, no example has been found where the inequalities (7.13) do not hold.

8. Applications. Exact applications of the method developed in this paper to problems involving transformations to normality are essentially limited to transformations of the form $Y = \log(X)$. To see this, suppose that $X = h(Y)$, where Y is normal (μ, σ^2). Suppose also that h is twice-differentiable, and that $E(h(Y))$ and $E(h'(Y))$ both exist. The method can be applied to confidence interval estimation of EX only if EX depends on μ and σ^2 as a function of $a\mu + b\sigma^2$ for some a and b, that is, only if

$$f(a\mu+b\sigma^2) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} h(\mu+\sigma w) \exp\left(-\tfrac{1}{2}w^2\right) dw. \tag{8.1}$$

By letting σ^2 approach zero it is clear that we must have $f(a\mu) = h(\mu)$, and that a cannot be zero. Therefore there is no loss of generality in writing $f(a\mu+b\sigma^2) = h(\mu+\lambda\sigma^2)$, where $\lambda = b/a$. With this substitution, differentiation of (8.1) with respect to σ yields

$$2\lambda\sigma h'(\mu+\lambda\sigma^2) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} wh'(\mu+\sigma w) \exp\left(-\tfrac{1}{2}w^2\right) dw. \tag{8.2}$$

Differentiation of (8.1) with respect to μ gives

$$h'(\mu+\lambda\sigma^2) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} h'(\mu+\sigma w) \exp\left(-\tfrac{1}{2}w^2\right) dw,$$

which may be rewritten as

$$h'(\mu+\lambda\sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} h'(y) \exp\left[-\tfrac{1}{2}(y-\mu)^2/\sigma^2\right] dy. \tag{8.3}$$

By differentiating (8.3) with respect to μ we obtain

$$h''(\mu+\lambda\sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (y-\mu)\sigma^{-2}h'(y) \exp\left[-\tfrac{1}{2}(y-\mu)^2/\sigma^2\right] dy,$$

which may be rewritten as

(8.4) $$\sigma h''(\mu+\lambda\sigma^2) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} wh'(\mu+\sigma w) \exp(-\tfrac{1}{2}w^2)\, dw.$$

Together, (8.2) and (8.4) imply that $h''(\mu+\lambda\sigma^2) = 2\lambda h'(\mu+\lambda\sigma^2)$ for all μ and σ^2, which implies that $h''(y) = 2\lambda h'(y)$ for all y. If $\lambda = 0$, this means that $h(y)$ is linear, in other words a transformation of a normal variate to another normal variate. If $\lambda \neq 0$, it means that $h(y) = \alpha+\beta \exp(2\lambda y)$ for some numbers α and β, so that the normalizing transform is of the form $Y = (2\lambda)^{-1} \log((X-\alpha)/\beta)$.

The results may be summarized as the following theorem.

THEOREM. *If Y is a normal (μ, σ^2) random variable, and h a twice-differentiable real function such that the expected values of $h(Y)$ and $h'(Y)$ exist for $-\infty < \mu < \infty$ and $\sigma^2 \geqq 0$, then*

(i) *$E(h(Y))$ depends nontrivially on both μ and σ^2 unless h is a constant or a linear function of y, and*

(ii) *$E(h(Y))$ depends on μ and σ^2 as a function of $\mu+\lambda\sigma^2$, for some $\lambda \neq 0$, if and only if $h(y) = \alpha+\beta \exp(2\lambda y)$ for some α and β.*

COROLLARY. *$E(h(Y))$ cannot be a function of $\mu+\delta\sigma$.*
(Take $\lambda = \delta/\sigma$. The function h cannot depend on σ^2.)

For non-logarithmic transformations to normality the countours $EX = f(\mu, \sigma^2) = \theta$ are not parallel lines, to which the method of this paper may be directly applied. However, it is easy to imagine ways in which an approximate confidence interval for EX could be obtained from a confidence interval for $\mu+\lambda\sigma^2$, for some λ. We are familiar with the idea of using a confidence region for a vector-valued parameter to define approximate confidence intervals for real functions of the parameter (e.g., Scheffé (1961), Halperin and Mantel (1963), Halperin (1964 and 1965), and Kanofsky (1969)). The kind of confidence region for the vector-valued parameter (μ, σ^2) that is discussed in this paper is one that defines exactly a confidence interval for $\mu+\lambda\sigma^2$.

The details of a method for using the confidence regions defined in this paper to obtain an approximate confidence interval for EX are the choice of the value of λ in the function $\mu+\lambda\sigma^2$, and the rule by which values of EX are associated with the confidence region defined by a confidence interval for $\mu+\lambda\sigma^2$. For a particular transformation, and thus a particular function $EX = f(\mu, \sigma^2)$, these questions might be decided by examining a plot of the contours $f(\mu, \sigma^2) = p$ in a neighborhood of the point at which μ and σ^2 are equal to their estimated values $\hat{\mu}$ and $\hat{\sigma}^2$, respectively. For example, λ might be chosen as the value characterizing the line tangent to the contour $f(\mu, \sigma^2) = f(\hat{\mu}, \hat{\sigma}^2)$ at the point $\mu = \hat{\mu}, \sigma^2 = \hat{\sigma}^2$. That is, we would have $\lambda = f_{\mu}/f_{\sigma^2}$, where f_{μ} and f_{σ^2} represent the partial derivatives of $f(\mu, \sigma^2)$ with respect to μ and σ^2, respectively, evaluated at $\mu = \hat{\mu}, \sigma^2 = \hat{\sigma}^2$.

The rule for defining an approximate confidence interval for $f(\mu, \sigma^2)$ in terms of an exact one for $\mu+\lambda\sigma^2$ might be expressed as a rule associating a line $\mu+\lambda\sigma^2 = m(p)$ with each contour $f(\mu, \sigma^2) = p$. The set of values p for which $m(p)$ is contained in the confidence interval for $\mu+\lambda\sigma^2$ would be an approximate confidence interval for $f(\mu, \sigma^2)$, provided that the function $m(p)$ is monotone. A possible rule would be to choose $m(p)$ so that the line $\mu+\lambda\sigma^2 = m(p)$ simultaneously intersects the curve $f(\mu, \sigma^2) = p$ and the perpendicular line $\mu-\sigma^2/\lambda = \hat{\mu}-\hat{\sigma}^2/\lambda$.

If $\log(X)$ is normal, approximate confidence interval methods for EX can be evaluated by comparing approximate confidence limits with the corresponding optimal exact limits, computed from the same data. This is not possible for other normalizing transformations, for which no optimal exact methods are available. However, comparisons can be made with conservative exact confidence limits computed according to Kanofsky's method, and estimates of the coverage probabilities for given values of μ and σ^2 can be obtained by Monte Carlo simulation. A subsequent paper is planned in which approximate methods of the kind discussed in this section are compared with other approximate confidence interval estimation procedures.

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