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FACTORIAL ASSOCIATION SCHEMES WITH APPLICATIONS TO THE CONSTRUCTION OF MULTIDIMENSIONAL PARTIALLY BALANCED DESIGNS

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0. Summary. In this paper, some new multidimensional partially balanced (MDPB) association schemes are defined, and the various parameters of the scheme are obtained. Using such schemes, we discuss procedures for the construction of multidimensional partially balanced designs. The theory so developed is illustrated with actual examples of construction of MDPB designs.

1. Introduction. The MDPB association scheme and the MDPB designs were introduced in Srivastava (1961) and Bose and Srivastava (1964). Here, for later use, we recall their definitions in brief. Let $S_1, S_2, \cdots, S_m$ be m sets of objects, where the number of objects in the set S_i is $|S_i|$ $(= s_i$, say); $i = 1, 2, \cdots, m$. The objects of S_i shall be denoted by $x_{i1}, x_{i2}, \cdots, x_{is_i}$.

DEFINITION 1.1. The class D of sets $S_1, S_2, \cdots, S_m$ is said to have a MDPB association if the following conditions are satisfied.

(i) Given any object $x_{ia} \in S_i$, the objects of S_j, $(j = 1, 2, \cdots, m)$, can be partitioned into n_{ij} disjoint subsets where each element of the αth subset is an αth associate of x_{ia}. The number n_{ij} of such subsets, and the number n_{ij}^{α} of objects in the αth subset are independent of x_{ia}, so long as $x_{ia} \in S_i$.

(ii) The relation of association is symmetric, that is, if $x_{jb} \in S_j$ is an αth associate of $x_{ia} \in S_i$, then x_{ia} is an αth associate of x_{jb}.

(iii) Let S_i, S_j, and S_k be any three sets in D, where i, j, and k are not necessarily distinct. Let $x_{jb} \in S_j$ be an αth associate of $x_{ia} \in S_i$. Then the number of βth associates of x_{ia} in S_k which are also γth associates of x_{jb}, is a constant $p(i, j, \alpha; k, \beta, \gamma)$ dependent only on i, j, k, α, β, and γ and independent of the pair x_{ia} and x_{jb} so long as x_{ia} and x_{jb} are mutually αth associates.

Let Ω be the (set-theoretic) product of the sets S_i, i.e. $\Omega = S_1 \times S_2 \times \cdots \times S_m$, and let the ordered m-tuples of Ω be called assemblies. Now, consider a factorial (or multidimensional) design with m factors (dimensions), such that S_i denotes the set of levels of the ith factor. An "m-dimensional design" T is then a collection of assemblies of Ω, where an assembly of Ω may be included any number of times; in particular, it may be included zero times.

DEFINITION 1.2. An m-dimensional design T is said to be a MDPB design if

(i) The sets $S_1, S_2, \cdots, S_m$ have a MDPB association scheme defined over them.

(ii) The number of times the rth factor $(r = 1, 2, \cdots, m)$ occurs at level $j (\in S_r)$ equals μ_r, where μ_r depends only on r and is independent of j, so long as $j \in S_r$.

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(iii) Consider two distinct factors r and t, and suppose $j \in S_r$, $j' \in S_t$, and j and j' are αth associates of each other. Then the number of assemblies in T in which rth and tth factors occur respectively at levels j and j' equals d_{rt}^{α}, $(r \neq t = 1, 2, \cdots, m)$, where d_{rt}^{α} depends on r, t, and α, but is independent of j and j' so long as $j \in S_r, j' \in S_t$, and j and j' are αth associates.

DEFINITION 1.3. If T_1 and T_2 are two m-dimensional designs, let $T = T_1 \oplus T_2$ denote the m-dimensional design in which an assembly of Ω is included $u_1 + u_2$ times, provided it is included u_1 times in T_1 and u_2 times in T_2. T is said to be the "sum" of T_1 and T_2. It is immediate from the definition that if T_1 and T_2 are MDPB designs then T is also an MDPB design.

Some necessary existence conditions on the parameters, and also the connectedness of MDPB designs were studied in Srivastava and Anderson (1970). One purpose of this paper is to consider the construction of such designs, particularly those involving a small number of observations.

The reader interested in the previous work should look into the (illustrative) bibliography presented at the end.

2. The factorial subassembly association scheme. Let Ω_m denote the set of all 2^m assemblies for a 2^m factorial experiment, say

$$\Omega_m = \{a_1^{j_1} a_2^{j_2} \cdots a_m^{j_m} \mid j_r = 0 \quad \text{or} \quad 1; r = 1, 2, \cdots, m\}. \tag{2.1}$$

Let $a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{ik}^{j_k}$ denote the set of 2^{m-k} assemblies in Ω_m in which the factors i_1, $i_2, \cdots, i_k$ are at fixed levels $j_1, j_2, \cdots, j_k$, respectively, and the remaining $(m-k)$ at level either zero or one.

DEFINITION 2.1. The set of assemblies $a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{ik}^{j_k}$ is said to be a subassembly of order k; $k = 0, 1, \cdots, m$. The set of all subassemblies of order k in a 2^m factorial experiment is denoted by S_{mk}. Note that for the case $k = 0$, there is only one subassembly, namely Ω_m itself. (This follows, since for $k = 0$, we must take all assemblies in Ω_m without any restriction to the levels at which any factor appears).

DEFINITION 2.2. If ω is a subassembly of order k, denote by $\bar{\omega}$ the subassembly of order k obtained from ω by interchanging the levels one and zero. Denote by S_{mk}^* the set of distinct unordered pairs $(\omega, \bar{\omega})$. For example,

$$S_{32} = \{a_1^0 a_2^0, a_1^1 a_2^1, a_1^0 a_3^0, a_1^1 a_3^1, a_2^0 a_3^0, a_2^1 a_3^1, a_1^0 a_2^1, a_1^1 a_2^0, a_1^0 a_3^1, a_1^1 a_3^0, a_2^0 a_3^1, a_2^1 a_3^0\}$$

$$S_{32}^* = \{(a_1^0 a_2^0, a_1^1 a_2^1), (a_1^0 a_3^0, a_1^1 a_3^1), (a_2^0 a_3^0, a_2^1 a_3^1), (a_1^0 a_2^1, a_1^1 a_2^0), (a_1^0 a_3^1, a_1^1 a_3^0), (a_2^0 a_3^1, a_2^1 a_3^0)\}.$$

It follows directly from the definition that $|S_{mk}| = \binom{m}{k} 2^k$ where for any set S, $|S|$ is the number of elements in S. Also $|S_{mk}^*| = |S_{mk}|/2$, if $k > 0$, and $|S_{m0}^*| = |S_{m0}| = 1$.

Suppose $\omega = a_{u_1}^{v_1} a_{u_2}^{v_2} \cdots a_{u_i}^{v_i}$ and $\rho = a_{r_1}^{t_1} a_{r_2}^{t_2} \cdots a_{r_j}^{t_j}$, are elements of S_{mi} and S_{mj}, respectively. The relation of association to be defined between ω and ρ will depend on the number of common factors (subscripts) and among the common factors the number of common levels (superscripts). For example $a_1{}^1 a_2{}^0 a_3{}^0 \in S_{m3}$ and $a_1{}^1 a_2{}^1 a_4{}^0 a_5{}^1 \in S_{m4}$ have two factors in common and one level in common, and will be called (2, 1) associates of each other.

Let $\Xi_m = \{S_{m0}, S_{m1}, \cdots, S_{mm}, S_{m0}^*, S_{m1}^*, \cdots, S_{mm}^*\}$. An association relation will now be defined between the elements of the sets in Ξ_m.

DEFINITION 2.3. The association scheme defined below is said to be the "factorial subassembly association (FSA) scheme" on Ξ_m.

(i) Let $\omega \in S_{mi}$ and $\rho \in S_{mj}$. Then ρ is said to be an (α_1, α_2) associate of ω if ρ has exactly α_1 factors in common with ω, such that among these α_1 factors, exactly α_2 factors have the same level in both ω and ρ. (It follows immediately that if ρ is an (α_1, α_2) associate of ω, then $\bar{\rho}$ is an $(\alpha_1, \alpha_1 - \alpha_2)$ associate of ω).

(ii) If $\omega \in S_{mi}$ and $(\rho, \bar{\rho}) \in S_{mj}^*$, then $(\rho, \bar{\rho})$ is said to be an $(\alpha_1, \alpha_2{}^*)$ associate of ω where $\alpha_2{}^* = \max [\alpha_2, \alpha_1 - \alpha_2]$.

(iii) Finally, under the above notation $(\rho, \bar{\rho}) \in S_{mj}^*$ is said to be an $(\alpha_1, \alpha_2{}^*)$ associate of $(\omega, \bar{\omega}) \in S_{mi}^*$.

Since the association relation depends only on the number of factors in common and the number of common factors at the same level, the association relation is obviously symmetric.

We shall now compute the parameters of the FSA scheme in a series of lemmas. The composite of these lemmas provides a proof that the FSA scheme is MDPB. For brevity let $\alpha = (\alpha_1, \alpha_2)$, and $\alpha^* = (\alpha_1, \alpha_2{}^*)$.

LEMMA 2.1. *If the* FSA *scheme is defined on* Ξ_m, *then*

(a) $$n_{ij} = \frac{(\theta_2+1)(\theta_2+2)}{2} - \frac{\theta_1(\theta_1+1)}{2},$$

(b) $$n_{ij^*} = n_{j^*i} = n_{i^*j^*} = \sum_{\theta=\theta_1}^{\theta_2} \left[\frac{\theta+2}{2}\right],$$

where $\theta_1 = \max [0, i+j-m]$, *and* $\theta_2 = \min [i, j]$, *and where* $[q]$ *denotes as usual, the greatest integer in* q.

PROOF. Let $\omega \in S_{mi}$. Consider the set S_{mj}, where j may or may not be different from i. With θ_1 and θ_2 defined as above, an element of S_{mj} must have at least θ_1, and at most θ_2 factors in common with ω. If an element of S_{mj} has θ factors in common with ω then it may have 0, 1, 2, $\cdots$, or θ levels in common with ω. That is, for a fixed number θ of factors in common there are $\theta+1$ distinct associate classes. Hence $n_{ij} = (\theta_1+1)+(\theta_1+1+1)+\cdots+(\theta_2+1)$, which reduces to the value given in part (a). Next consider the set S_{mj}^*. By the definition of the FSA scheme, for a fixed number θ of factors in common with ω, there can be either $(\theta+1)/2$ or $(\theta+2)/2$ distinct associate classes according as θ is odd or even. This can be expressed as the greatest integer in $(\theta+2)/2$, and n_{ij^*} is the sum of such expressions where θ takes values from θ_1 to θ_2. The remaining equalities in part (b) are easy to show. This completes the proof.

LEMMA 2.2. *If the* FSA *scheme is defined on* Ξ_m, *then*

(a) $n_{ij}^{\alpha} = \binom{i}{\alpha_2}\binom{i-\alpha_2}{\alpha_1-\alpha_2}\binom{m-i}{j-\alpha_1}2^{(j-\alpha_1)}$

(b) $n_{ij^*}^{\alpha^*} = n_{i^*j^*}^{\alpha^*} = n_{ij}^{\alpha}$, *if* $\alpha_2 \neq \alpha_1/2$

$= \frac{1}{2}n_{ij}^{\alpha}$, *if* $\alpha_2 = \alpha_1/2$.

(c) $n_{i^*j}^{\alpha^*} = 2n_{ij}^{\alpha}$, *if* $\alpha_2 \neq \alpha_1/2$

$= n_{ij}^{\alpha}$, *if* $\alpha_2 = \alpha_1/2$.

PROOF. Let ω be any element of S_{mi} and consider the number of $\alpha = (\alpha_1, \alpha_2)$ associates of ω in S_{mj}. Let $\rho \in S_{mj}$, and suppose ρ and ω are αth associates. Then we must find in how many ways we can select ρ. From the i factors in ω, α_2 factors may be chosen in $\binom{i}{\alpha_2}$ ways; these α_2 factors are to have levels in common with those of a set of α_2 factors in ρ. From the remaining $(i-\alpha_2)$ factors in ω, we can choose $(\alpha_1-\alpha_2)$ factors to have levels different from those of corresponding factors in ρ in $\binom{i-\alpha_2}{\alpha_1-\alpha_2}$ ways. Then from the $(m-i)$ factors not included in the symbol for ω we can choose $(j-\alpha_1)$ factors in $\binom{m-i}{j-\alpha_1}$ ways and the levels may be assigned to these factors in $2^{(j-\alpha_1)}$ ways; these $(j-\alpha_1)$ factors are to be in ρ, and not ω. This completes the proof of part (a).

For (b), note that each $\rho \in S_{mj}$ which is an (α_1, α_2) associate of ω, is paired with $\bar{\rho}$ which is a $(\alpha_1, \alpha_1-\alpha_2)$ associate of ω in S_{mj}. If $\alpha_2 \neq \alpha_1/2$ the two associate classes in S_{mj} are distinct, while if $\alpha_2 = \alpha_1/2$, the (α_1, α_2) and $(\alpha_1, \alpha_1-\alpha_2)$ associate classes of ω in S_{mj} are the same. Hence part (b) follows. Finally consider $(\omega, \bar{\omega}) \in S_{mi}^*$. An element of S_{mj} is an (α_1, α_2^*) associate of $(\omega, \bar{\omega})$ if it is either a (α_1, α_2) or $(\alpha_1, \alpha_1-\alpha_2)$ associate of ω. If $\alpha_2 \neq \alpha_1/2$ these two classes are distinct and if $\alpha_2 = \alpha_1/2$ the two classes are identical. This completes the proof.

LEMMA 2.3. *For the* FSA *scheme defined on* Ξ_m, *we have*

$$p(i, j, \alpha; k, \beta, \gamma) = \Sigma \binom{\alpha_2}{a_1, a_2}\binom{\alpha_1-\alpha_2}{b_1, b_2}\binom{i-\alpha_1}{c_1, c_2}\binom{j-\alpha_1}{d_1, d_2}\binom{m-i-j+\alpha_1}{f}2^f$$

where the summation is over all $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ *and* f *satisfying*

(i) $k = a_1+a_2+b_1+b_2+c_1+c_2+d_1+d_2+f$

(ii) $\beta_1 = a_1+a_2+b_1+b_2+c_1+c_2$; $\beta_2 = a_1+b_1+c_1$

(iii) $\gamma_1 = a_1+a_2+b_1+b_2+d_1+d_2$; $\gamma_2 = a_1+b_2+d_1$

and where, $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$; *and for any positive integers* $l_0, l_1, \cdots, l_g$,

$$\binom{l_0}{l_1, \cdots, l_g} = \frac{l_0!}{l_1!l_2!\cdots l_g!(l_0-l_1-\cdots-l_g)!}, \qquad \text{if } l_1+\cdots+l_g \leqq l_0,$$

$$= 0, \qquad \text{otherwise.}$$

PROOF. Let $\omega \in S_{mi}$ and $\rho \in S_{mj}$ be α associates. Consider $\pi \in S_{mk}$. From the α_2 factors with levels in common between ω and ρ, a_1 factors in π may be chosen to have levels in common with both ω and ρ, and a_2 factors to have levels different from each of ω and ρ in $\binom{\alpha_2}{a_1, a_2}$ ways. Then, from the $(\alpha_1 - \alpha_2)$ factors in common with levels different between ω and ρ, b_1 factors (in π) may be chosen to have levels in common with ω and b_2 factors (in π) to have levels in common with ρ in $\binom{\alpha_1-\alpha_2}{b_1, b_2}$ ways. Next from the $(i-\alpha_1)$ factors of ω not in common with any factor of ρ, c_1 factors (for π) may be chosen to have levels in common and c_2 factors to have levels different in $\binom{i-\alpha_1}{c_1, c_2}$ ways. Likewise, from the $(j-\alpha_1)$ factors of ρ not in common with any factor of ω we chose d_1 factors to have levels in common and d_2 factors to have levels different, in $\binom{j-\alpha_1}{d_1, d_2}$ ways. Finally from the remaining $(m-i-j+\alpha_1)$ factors not appearing in either ω or ρ or both, f factors may be chosen in $\binom{m-i-j+\alpha_1}{f}$ ways and the levels may be assigned to these f factors in any of 2^f ways. Condition (i) insures that the element so chosen is in S_{mk}, while conditions (ii) and (iii) insure that the element is a (β_1, β_2) associate of ω and a (γ_1, γ_2) associate of ρ. This completes the proof.

LEMMA 2.4. *For the* FSA *scheme defined on* Ξ_m, *we have*

(a) $$p(i,j,\alpha;k^*,\beta^*,\gamma^*) = p(i,j,(\alpha_1,\alpha_2);k,(\beta_1,\beta_2),(\gamma_1,\gamma_2)) + p(i,j,(\alpha_1,\alpha_2);k,(\beta_1,\beta_1-\beta_2),(\gamma_1,\gamma_1-\gamma_2)), \text{ if } \beta_2 > \beta_1/2 \text{ and } \gamma_2 > \gamma_1/2;$$
$$= p(i,j,(\alpha_1,\alpha_2);k,(\beta_1,\beta_2),(\gamma_1,\gamma_2)), \text{ if either } \beta_2 = \beta_1/2 \text{ and } \gamma_2 > \gamma_1/2, \text{ or } \beta_2 > \beta_1/2 \text{ and } \gamma_2 = \gamma_1/2;$$
$$= \tfrac{1}{2}p(i,j,(\alpha_1,\alpha_2);k,(\beta_1,\beta_2),(\gamma_1,\gamma_2)), \text{ if } \beta_2 = \beta_1/2 \text{ and } \gamma_2 = \gamma_1/2;$$

(b) $$p(i,j^*,\alpha^*;k,\beta,\gamma^*) = p(i,j,\alpha;k,\beta,\gamma) \text{ if } \gamma_2 = \gamma_1/2;$$
$$= p(i,j,\alpha;k,\beta,(\gamma_1,\gamma_2)) + p(i,j,\alpha;k,\beta,(\gamma_1,\gamma_1-\gamma_2)), \text{ if } \gamma_2 \neq \gamma_1/2;$$

(c) $$p(i,j^*,\alpha^*;k^*,\beta^*,\gamma^*) = p(i,j,\alpha;k^*,\beta^*,\gamma^*), \text{ where } \alpha_2 \geqq \alpha_1/2;$$

(d) $$p(i^*,j^*,\alpha^*;k,\beta^*,\gamma^*) = p(i,j,\alpha;k,(\beta_1,\beta_2),(\gamma_1,\gamma_2)) + p(i,j,\alpha;k,(\beta_1,\beta_2),(\gamma_1,\gamma_1-\gamma_2)) + p(i,j,\alpha;k,\beta_1,(\beta_1-\beta_2),(\gamma_1,\gamma_2)) + p(i,j,\alpha;k,(\beta_1,\beta_1-\beta_2),(\gamma_1,\gamma_1-\gamma_2)), \text{ if } \beta_2 > \beta_1/2 \text{ and } \gamma_2 > \gamma_1/2;$$
$$= p(i,j,\alpha;k,\beta,(\gamma_1,\gamma_2)) + p(i,j,\alpha;k,\beta,(\gamma_1,\gamma_1-\gamma_2)), \text{ if } \beta_2 = \beta_1/2 \text{ and } \gamma_2 > \gamma_1/2, \text{ or if } \beta_2 > \beta_1/2 \text{ and } \gamma_2 = \gamma_1/2;$$
$$= p(i,j,\alpha;k,\beta,\gamma), \text{ if } \beta_2 = \beta_1/2 \text{ and } \gamma_2 = \gamma_1/2;$$

(e) $$p(i^*,j^*,\alpha^*;k^*,\beta^*,\gamma^*) = p(i,j^*,\alpha^*,k^*,\beta^*,\gamma^*).$$

PROOF. (a) Let $\omega \in S_{mi}$, and $\rho \in S_{mj}$ be α associates. Then an element $(\sigma, \bar{\sigma}) \in S_{mk}^*$ is a (β_1, β_2^*) associate of ω and a (γ_1, γ_2^*) associate of ρ, if $\sigma \in S_{mk}$ is a

(i) (β_1, β_2) associate of ω and a (γ_1, γ_2) associate of ρ, or

(ii) (β_1, β_2) associate of ω and a $(\gamma_1, \gamma_1-\gamma_2)$ associate of ρ, or

(iii) $(\beta_1, \beta_1-\beta_2)$ associate of ω and a (γ_1, γ_2) associate of ρ, or

(iv) $(\beta_1, \beta_1-\beta_2)$ associate of ω and a $(\gamma_1, \gamma_1-\gamma_2)$ associate of ρ.

Now if $\beta_2 > \beta_1/2$ and $\gamma_2 > \gamma_1/2$ the four classes listed above are all disjoint. If σ is of type (i) then $\bar{\sigma}$ is of type (iv), and if σ is of type (ii) then $\bar{\sigma}$ is of type (iii); hence the number of pairs $(\sigma, \bar{\sigma}) \in S_{mk}^*$ which are β^* associates of ω and γ^* associates of ρ, is the number of elements of type (i) or (ii) in S_{mk}. This completes the proof of the first equality of part (a). If $\beta_2 = \beta_1/2$ and $\gamma_2 > \gamma_1/2$, then in S_{mk} types (i) and (iii) coincide, and types (ii) and (iv) coincide. Then if σ is of type (i), we have that $\bar{\sigma}$ is of type (ii); in this case the number of pairs $(\sigma, \bar{\sigma}) \in S_{mk}^*$, which are β^* associates of ω and γ^* associates of ρ is the number of elements of type (i) in S_{mk}. Similarly if $\beta_2 > \beta_1/2$ and $\gamma_2 = \gamma_1/2$. Finally, if $\beta_2 = \beta_1/2$ and $\gamma_2 = \gamma_1/2$, all four types are the same and clearly the number of pairs in S_{mk}^* is one-half the number of elements of type (i) in S_{mk}. This completes the proof of part (a).

(b) Let $\omega \in S_{mi}$ and $(\rho, \bar{\rho}) \in S_{mj}^*$ be $\alpha^* = (\alpha_1, \alpha_2^*)$ associates. An element $\sigma \in S_{mk}$ is a (γ_1, γ_2^*) associate of $(\rho, \bar{\rho})$, if it is either a (γ_1, γ_2) or $(\gamma_1, \gamma_1-\gamma_2)$ associate of $\rho \in S_{mj}$. If $\gamma_2 \neq \gamma_1/2$ these classes are disjoint, and if $\gamma_2 = \gamma_1/2$ they are identical. Part (b) now follows directly.

(c) This part is obvious since $(\sigma, \bar{\sigma}) \in S_{mk}^*$ is a (γ_1, γ_2^*) associate of $(\rho, \bar{\rho}) \in S_{mj}^*$, if and only if $(\sigma, \bar{\sigma})$ is a (γ_1, γ_2^*) associate of $\rho \in S_{mj}$.

(d) An element $\sigma \in S_{mk}$ is a (β_1, β_2^*) associate of $(\omega, \bar{\omega}) \in S_{mi}^*$, and a (γ_1, γ_2^*) associate of $(\rho, \bar{\rho}) \in S_{mj}^*$, if it is of types (i), (ii), (iii), or (iv) mentioned in the proof of part (a). If $\beta_2 \neq \beta_1/2$ and $\gamma_2 \neq \gamma_1/2$, all the four classes are disjoint. If $\beta_2 = \beta_1/2$ and $\gamma_2 \neq \gamma_1/2$, classes (i) and (iii) are identical, and so are classes (ii) and (iv). The same holds also if $\beta_2 \neq \beta_1/2$ and $\gamma_2 = \gamma_1/2$. If $\beta_2 = \beta_1/2$ and $\gamma_2 = \gamma_1/2$, all four classes are the same. From this, part (d) follows immediately.

(e) This follows immediately, by noting that $(\rho, \bar{\rho}) \in S_{mj}^*$ is a (α_1, α_2^*) associate of $(\omega, \bar{\omega}) \in S_{mi}^*$ if and only if $(\rho, \bar{\rho})$ is an (α_1, α_2^*) associate of $\omega \in S_{mi}$, and similarly for $(\sigma, \bar{\sigma}) \in S_{mk}^*$. This completes the proof of the theorem.

The following theorem follows from Lemmas 2.1 to 2.4, and the observation that the association relation is symmetric.

THEOREM 2.1. *The* FSA *scheme defined on* Ξ_m *is a* MDPB *scheme.*

3. Construction of MDPB designs. In this section we obtain a general method of constructing MDPB designs from an MDPB association scheme defined on the sets of factor levels. The method is illustrated by examples.

We consider the three dimensional case. Let S_1, S_2, and S_3 be three sets with s_1, s_2, and s_3 elements respectively. We shall suppose that some MDPB association scheme is defined on this class of sets.

Suppose that α, β, and γ are three integers and consider the design $T(\alpha, \beta, \gamma) \subset \Omega$ where

$$(3.1)\quad T(\alpha,\beta,\gamma = \{(X_1,X_2,X_3) \mid X_i \in S_i (i=1,2,3), \text{ and } (X_1,X_2),(X_1,X_3) \text{ and } (X_2,X_3) \text{ are respectively } \alpha,\beta \text{ and } \gamma\text{th associates}\}.$$

It is apparent that

$$(3.2)\quad |T(\alpha,\beta,\gamma)| = s_3 n_{12}^{\alpha} p(1,2,\alpha;3,\beta,\gamma) = s_1 n_{23}^{\gamma} p(2,3,\gamma;1,\alpha,\beta) = s_2 n_{31}^{\beta} p(3,1,\beta;2,\gamma,\alpha).$$

Next we see that each level of a factor appears the same number of times; in fact

$$(3.3)\quad \mu_1 = n_{12}^{\alpha} p(1,2,\alpha;3,\beta,\gamma), \qquad \mu_2 = n_{23}^{\gamma} p(2,3,\gamma;1,\alpha,\beta), \qquad \mu_3 = n_{31}^{\beta} p(3,1,\beta;2,\gamma,\alpha).$$

Finally, we have

$$(3.4)\quad \begin{aligned} d_{12}^{\alpha} &= p(1,2,\alpha;3,\beta,\gamma), & d_{12}^{\delta} &= 0 \quad \text{if } \delta \neq \alpha \\ d_{13}^{\beta} &= p(1,3,\beta;2,\alpha,\gamma), & d_{13}^{\delta} &= 0 \quad \text{if } \delta \neq \beta \\ d_{23}^{\gamma} &= p(2,3,\gamma;1,\alpha,\beta), & d_{23}^{\delta} &= 0 \quad \text{if } \delta \neq \gamma. \end{aligned}$$

Thus we have shown the following:

THEOREM 3.1. *The design $T(\alpha, \beta, \gamma)$ defined by* (3.1) *is an* MDPB *design.*

COROLLARY 3.1. *If $T(\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2, \cdots, k$, are k designs of type* (3.1) *then*

$$(3.5)\quad T = T(\alpha_1,\beta_1,\gamma_1) \oplus T(\alpha_2,\beta_2,\gamma_2) \oplus \cdots \oplus T(\alpha_k,\beta_k,\gamma_k) \quad \textit{is an } \text{MDPB } \textit{design}.$$

Theorem 3.1 and Corollary 3.1 give a method of construction of three dimensional designs. There is no guarantee, of course, that such a design is completely connected and this must be checked using the methods given by Srivastava and Anderson (1970). Also the designs obtained in this manner may not be as economic as desired. If all of the nonzero d_{ij}^{ε} are greater than one, the size of the design may often be reduced by taking an appropriate subset of the design so that at least one of the d_{ij}^{ε} is one.

For convenience of presentation of designs a subassembly of order k, say, $a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{i_k}^{j_k}$, will be denoted by an m-vector with $j_1, j_2, \cdots, j_k$ respectively in positions $i_1, i_2, \cdots, i_k$, and dots elsewhere. For example, $a_1{}^1 a_2{}^0 \in S_{32}$ will be denoted by $(1, 0, \cdot)$. Likewise pairs of subassemblies in S_{mk}^* will be denoted by pairs of m-tuples.

EXAMPLE 3.1. $4 \times 6 \times 8$ DESIGN. Consider the sets S_{33}^*, S_{31}, and S_{33} of Ξ_3, and let the FSA scheme be defined on these sets. Then $T((1, 1), (2, 2), (1, 0))$ is an MDPB design with $N = 72$, $\mu_1 = 18$, $\mu_2 = 12$, $\mu_3 = 9$, $d_{12}^{(1,1)} = 3$, $d_{13}^{(3,2)} = 3$, and $d_{23}^{(1,0)} = 3$. From this design we may extract the 24 assemblies specified by

$[((j_1, j_2, j_3), (1+j_1, 1+j_2, 1+j_3)), (j_1', \cdot, \cdot), (1+j_1, j_2, j_3)]$, if $j_1' = j_1$;

$[((j_1, j_2, j_3), (1+j_1, 1+j_2, 1+j_3)), (j_1', \cdot, \cdot), (j_1, 1+j_2, 1+j_3)]$, if $j_1' = 1+j_1$;

and similarly for $(\cdot, j_2', \cdot)$ and $(\cdot, \cdot, j_3')$. This is a completely connected MDPB design with $N = 24$, and $d_{12}^{(1,1)} = d_{13}^{(3,2)} = d_{23}^{(1,0)} = 1$. The remaining 48 assemblies constitute an MDPB design with $d_{12}^{(1,1)} = d_{13}^{(3,2)} = d_{23}^{(1,0)} = 2$. The design with $N = 24$ is given in the following array.

TABLE 1

Level combination for 4×6×8 MDPB *design*

	(0, ·, ·)	(1, ·, ·)	(·, 0, ·)	(·, 1, ·)	(·, ·, 0)	(·, ·, 1)
((0, 0, 0), (1, 1, 1))	(1, 0, 0)	(0, 1, 1)	(0, 1, 0)	(1, 0, 1)	(0, 0, 1)	(1, 1, 0)
((0, 1, 1), (1, 0, 0))	(1, 1, 1)	(0, 0, 0)	(1, 0, 0)	(0, 0, 1)	(1, 0, 1)	(0, 1, 0)
((0, 1, 0), (1, 0, 1))	(1, 1, 0)	(0, 0, 1)	(1, 1, 1)	(0, 0, 0)	(0, 1, 1)	(1, 0, 0)
((0, 0, 1), (1, 1, 0))	(1, 0, 1)	(0, 1, 0)	(0, 1, 1)	(1, 0, 0)	(1, 1, 1)	(0, 0, 0)

EXAMPLE 3.2. 3×6×8 DESIGN. In this example we shall take the "sum" of two designs and then take a subset of the resulting design. Consider the sets S_{31}^*, S_{31}, and S_{33} of Ξ_3 and let the FSA scheme be defined on these three sets. Consider the design $T = T((0,0), (1,1), (1,0)) + T((1,1), (1,1), (1,0))$. For this design we have $|T| = N = 72$, $\mu_1 = 24$, $\mu_2 = 12$, $\mu_3 = 9$, $d_{12}^{(0,0)} = d_{12}^{(1,1)} = 4$, $d_{13}^{(1,1)} = 3$, and $d_{23}^{(1,0)} = 3$. Consider the design with assemblies $[((0,\cdot,\cdot), (1,\cdot,\cdot)), (j_1,\cdot,\cdot), (1+j_1, 0, 0)]$, $[((0,\cdot,\cdot), (1,\cdot,\cdot)), (j_1,\cdot,\cdot), (1+j_1, 1, 1)]$, $[((0,\cdot,\cdot), (1,\cdot,\cdot)), (\cdot, j_2, \cdot), (1+j_2, 1+j_2, j_2)]$ $[((0,\cdot,\cdot), (1,\cdot,\cdot)), (\cdot,\cdot,j_3), (1+j_3, j_3\ 1+j_3)]$. The remaining sixteen assemblies are obtained similarly from pairs $((\cdot, 0, \cdot), (\cdot, 1, \cdot))$ and $((\cdot,\cdot,0), (\cdot,\cdot,1))$. This design has $N = 24$, $\mu_1 = 8$, $\mu_2 = 4$, $\mu_3 = 3$, $d_{12}^{(1,1)} = 2$, and $d_{12}^{(0,0)} = d_{13}^{(1,1)} = d_{23}^{(1,0)} = 1$. The remaining 48 assemblies also constitute a MDPB design.

The structure of the FSA scheme, and the relations on the parameters derived in Srivastava and Anderson (1970), can be employed to construct four and higher dimensional PB designs. In order to illustrate this, the construction of a 4×6×8×12 design with $N = 48$ will be considered. This design is also used for an example in Srivastava and Anderson (1970). Again the FSA scheme with $m = 3$ is employed.

First we select four appropriate sets from Ξ_3; in this case S_{33}^*, S_{31}, S_{33}, and S_{32}. There are 27 parameters to be estimated, thus N must be at least 27. By Theorem 3.1 in Srivastava and Anderson (1970), N must be a multiple of $24 = \text{l.c.m.}[4, 6, 8, 12]$; hence N must be at least 48. The design to be constructed will have $N = 48$.

Consider the first three sets, with $(x, \bar{x}) \in S_{33}^*$, $y \in S_{31}$, and $Z \in S_{33}$. Each $(x, \bar{x}) \in S_{33}$ is an (1, 1) associate of each $y \in S_{31}$ so that each of the 24 pairs $((x, \bar{x}), y)$ will appear in exactly 2 assemblies. In S_{33} there are three (3, 2) associates of $(x, \bar{x})$ which are (1, 1) associates of y, say Z_1, Z_2, and Z_3. Now one of the pairs of triples of $(x, \bar{x})$, say x, matches the non-dot coordinate of y, and each of Z_1, Z_2, Z_3 match x and y in that coordinate. One of Z_1, Z_2, Z_3, say Z, differs from x in the remaining two coordinates. For example if $x = [(0, 0, 0), (1, 1, 1)]$ and $y = (0, \cdot, \cdot)$

then $Z_1 = (0, 1, 1)$, $Z_2 = (0, 0, 1)$ and $Z_3 = (0, 1, 0)$, and $Z = Z_1 = (0, 1, 1)$. This defines a set of 24 assemblies $\{((x, \bar{x}), y, Z)\}$, which are a subset of $T((1, 1), (3, 2), (1, 1))$. Similarly we define another set of 24 assemblies $\{((x, \bar{x}), y, \bar{Z})\}$, by taking a subset of $T((1, 1)(3, 2), (1, 0))$. To each assembly $((x, \bar{x}), y, Z)$ there are two assemblies ω_1 and ω_2 of S_{32}, which are (2, 1) associates of $(x, \bar{x})$ and (1, 1) associates of y and (2, 2) associates of Z. The final structure of the design is exemplified by the four typical assemblies $[(x, \bar{x}), y, Z, \omega_1]$, $[(x, \bar{x}), y, \bar{Z}, \omega_2]$, $[(x, \bar{x}), \bar{y}, Z, \omega_2]$, and $[(x, \bar{x}), \bar{y}, \bar{Z}, \omega_1]$. A particular example of four assemblies is given by $[((0, 0, 0), (1, 1, 1)), (0, \cdot, \cdot), (0, 1, 1), (0, 1, \cdot)]$, $[((0, 0, 0), (1, 1, 1)), (0, \cdot, \cdot), (1, 0, 0), (0, \cdot, 1)]$, $[((0, 0, 0), (1, 1, 1)), (1, \cdot, \cdot), (0, 1, 1), (0, \cdot, 1)]$ $[((0, 0, 0), (1, 1, 1)), (1, \cdot, \cdot), (1, 0, 0), (0, 1, \cdot)]$. It is easy to write out all the 48 assemblies and therefore the whole design does not need to be presented again here. The parameters of the design are

$$N = 48; \quad \mu_1 = 12, \mu_2 = 8, \mu_3 = 6, \mu_4 = 4; \quad d_{12}^{(1,1)} = 2; \quad d_{13}^{(3,3)} = 0,$$
$$d_{13}^{(3,2)} = 2; \quad d_{14}^{(2,2)} = 0, d_{14}^{(2,1)} = 2; \quad d_{23}^{(1,1)} = d_{23}^{(1,0)} = 1; \quad d_{24}^{(1,1)} = 2,$$
$$d_{24}^{(1,0)} = d_{24}^{(0,0)} = 0; \quad d_{34}^{(2,2)} = d_{34}^{(2,0)} = 1, d_{34}^{(2,1)} = 0.$$

The reader may have noticed that for each of the three designs discussed above, the factor levels within any particular assembly have an "overall relation" between themselves. Indeed, this "overall relation" consists of nothing else but certain kinds of ternary and higher associations, and will be more fully studied elsewhere in view of its effectiveness in producing designs with reduced number of observations.

4. An extension of the FSA scheme. Let s be a positive integer, and let $\Omega_m{}^s$ denote the set of all s^m assemblies for the s^m factorial experiment. Also let $a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{i_k}^{j_k}$ denote the subset of all assemblies of $\Omega_m{}^s$ in which the factors $i_1, i_2, \cdots, i_k$ are at levels $j_1, j_2, \cdots, j_k$, respectively, where the j's can take values $0, 1, \cdots, (s-1)$. $a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{i_k}^{j_k}$ is called a subassembly of order k. Let S_{mk}^s denote the set of all subassemblies of order k from the s^m factorial. It follows directly that $|S_{mk}^s| = \binom{m}{k} s^k$.

DEFINITION 4.1. $\omega \in S_{mi}^s$ is said to be an (α_1, α_2) associate of $\rho \in S_{mi}$, if ω and ρ have exactly α_1 factors in common and among these α_1 factors in common exactly α_2 have levels in common. (Note that when $s = 1$, the association relation would depend only upon α_1).

Since the association relation depends only on the number of factors in common and the number of common factors at the same level, the association scheme is obviously symmetric. The following lemma establishes the values of the parameters for this scheme on the class of sets $\{S_{m0}^s, S_{m1}^s, \cdots, S_{mm}^s\}$.

LEMMA 4.1. *The parameters of the association scheme on the class of sets* $\{S_{m0}^s, S_{m1}^s, \cdots, S_{mm}^s\}$ *are*

(a) $$n_{ij} = \frac{(\theta_2+1)(\theta_2+2)}{2} - \frac{\theta_1(\theta_1+1)}{2} \quad \textit{where} \quad \theta_1 = \max[0, i+j-m]$$
$$\textit{and} \quad \theta_2 = \min[i, j]$$

(b) $n_{ij}^{(\alpha_1,\alpha_2)} = \binom{i}{\alpha_2}\binom{i-\alpha_2}{\alpha_1-\alpha_2}\binom{m-i}{j-\alpha_1}(s-1)^{\alpha_1-\alpha_2}s^{j-\alpha_1}$

(c) $p(i,j,(\alpha_1,\alpha_2);k(\beta_1,\beta_2)(\gamma_1,\gamma_2)$

$$= \sum \binom{\alpha_2}{a_1,a_2}\binom{\alpha_1-\alpha_2}{b_1,b_2,b_3}\binom{i-\alpha_1}{c_1,c_2}\binom{j-\alpha_1}{d_1,d_2}\binom{m-i-j+\alpha_1}{f}(s-2)^{b_3}(s-1)^{a_2+c_2+d_2}s^f$$

where the summation is over all $a_1, a_2, b_1, b_2, b_3, c_1, c_2, d_1, d_2, f$ *satisfying*

(i) $k = a_1+a_2+b_1+b_2+b_3+c_1+c_2+d_1+d_2+f$

(ii) $\beta_1 = a_1+a_2+b_1+b_2+b_3+c_1+c_2$; $\beta_2 = a_1+b_1+c_1$

(iii) $\gamma_1 = a_1+a_2+b_1+b_2+b_3+d_1+d_2$; $\gamma_2 = a_1+b_2+d_1$.

The proof is similar to Lemmas 2.1a, 2.2a, and 2.3 and will be omitted to avoid repetition. Lemma 4.1 and the symmetry of the association relation gives

THEOREM 4.1. *The association scheme on the class of sets* $\{S_{m0}^s, S_{m1}^s, \cdots, S_{mm}^s\}$ *an* MDPB *scheme.*

It may be remarked here that the above association scheme is nontrivial even for $s = 1$. In fact, the well-known triangular association scheme is a special case of the above scheme when $s = 1$ and $k = 2$.

For $s \geqq 2$, a number of other schemes can be defined apart from the general FSA scheme defined above. In particular, we may define schemes on sets "smaller" than the S_{mj}^s. The MDPB designs obtained by using schemes defined on "small" sets generally involve factors with "small" number of levels. Since in practice, factors with a relatively small number of levels are more common than factors with a large number of levels, these other schemes are potentially quite useful. When $s = 2$, we did define one such scheme previously using sets whose elements are pairs of complementary assemblies. As will be seen from the designs presented in this paper, these latter sets have been quite useful in the construction of MDPB designs.

In the next section, we shall use several other small sets, including sets that come from s^m factorials, with either $m > 3$, and/or $s > 2$. We would not try to define MDPB association schemes directly over such sets, since such discussion would make this paper too long. However, we shall present useful and non-trivial MDPB designs obtained by using such sets. The fact that the designs presented are MDPB and are also "connected" can be checked by the reader, and will not be proved here for lack of space.

5. Some new MDPB designs. The designs to be presented are summarized in the table below. The above designs will be described in the above order, except for the $(4\times6\times8)$ and $(3\times6\times8)$ designs which will be discussed first in relation to Examples 3.1 and 3.2. This would serve to illustrate our notation, which we now introduce.

If $\gamma(\geqq 2)$ is any positive integer, then $R(\gamma)$ will denote the set of elements in the ring of integers mod γ. If $(\theta_0, \theta_1, \cdot, \theta_{t-1})$ is any t-element vector ($t \geqq 1$), then $\theta_0, \theta_1, \cdot, \theta_{t-1}$ will be called respectively the 0th, 1st, $\cdots$, and $(t-1)$th element of

TABLE 2
Some connected MDPB *designs*

Type	N	n_e	Type	N	n_e
$2\times 5\times 10$	20	5	$5\times 10\times 10$	40	17
$2\times 7\times 21$	42	14	$6\times 6\times 6$	30	14
$3\times 5\times 10$	30	14	$6\times 6\times 9$	36	17
$3\times 6\times 8$	24	9	$6\times 6\times 12$	36	14
$3\times 12\times 24$	48	11	$6\times 9\times 18$	54	23
$4\times 6\times 6$	24	10	$8\times 8\times 8$	40	18
$4\times 6\times 8$	24	8	$9\times 9\times 9$	45	20
$5\times 5\times 10$	30	12	$10\times 15\times 30$	90	37

the vector. Thus in the vector (2, 1, 4, 7), 4 is the 2nd element. (This labeling of elements as 0th, 1st, $\cdots$, etc. instead of 1st, 2nd etc. helps in a neater presentation of the designs.) If $u_1, \cdots, u_n$ is any set of (not necessarily distinct) objects, then $P(u_1, \cdots, u_n)$ shall denote the set of all distinct permutations of $(u_1, u_2, \cdots, u_n)$. Thus $P(0, 1, 2)$ has the six elements 012, 021, 102, 120, 201, and 210. Also, $L(x_1, \cdots, x_k; j_1, \cdots, j_k; k, t)$ shall denote a t-plet, which has the object x_u at position j_u $(u = 1, 2, \cdots, k)$, and which has a dot at the remaining $(t-k)$ places. Thus $L(2, l; 2, 4; 2, 5)$ stands for $(\cdot, \cdot, 2, \cdot, l)$. If $v_1, \cdots, v_n$ are n distinct objects, then $(v_1, \cdots, v_n)^*$ would stand for the unordered set of these n objects. Also $K_1{}^*(s_1, s_2; (x_1, x_2, \cdots, x_{s_1}))$ will denote the set of all unordered s_2-tuples which can be made out of the set of s_1 *distinct* symbols $x_1, x_2, \cdots, x_{s_1}$. Notice that $K_1{}^*$ has $\binom{s_1}{s_2}$ elements. Thus $K_1{}^*(5, 2; R(5))$ would consist of the elements $(0, 1)^*$, $(0, 2)^*$, $(0, 3)^*$, $(0, 4)^*$, $(1, 2)^*$, $(1, 3)^*$, $(1, 4)^*$, $(2, 3)^*$, $(2, 4)^*$, $(3, 4)^*$. (Notice that $(0, 1)^*$ and $(1, 0)^*$ represent the same objects.)

Instead of being presented in tabular form (like the design $4\times 6\times 8$ in Table 1), we shall present various designs in a compact form. Besides saving space, this will also help the reader to understand the structure of the designs. Once the general technique is clear, the reader could perhaps construct many other designs using the same approach.

5.1. $4\times 6\times 8$. We take the set of all assemblies of the form $[\{((0, x_1, x_2), (1, x_1+1, x_2+1))^*\}; \{L(y; j; 1, 3)\}; \{(z_0, z_1, z_2)\}]$, where

(i) $x_1, x_2, y \in R(2)$, and $j \in R(3)$,

(ii) $z_j = y+1$, and $z_k = x_j+x_k+y$ $(k \neq j)$, where $x_0 = 0$.

Notice that an assembly is presented in the form $[\{a\}; \{b\}; \{c\}]$, with semicolons separating the levels of various factors, the levels themselves being denoted by the objects in the curly brackets. Now as x_1, x_2 vary over $R(2)$ (i.e., x_1, x_2 take values 0, 1 of $R(2)$), the object $((0, x_1, x_2), (1, x_1+1, x_2+1))^*$ clearly takes the four values for the first factor indicated in Table 1. Similarly, as y takes values 0 and 1, and j equals 0, 1 and 2, the symbol $L\{y; j; 1, 3\}$ clearly assumes the various values shown in Table 1 as the levels of the second factor. Now consider the "levels," say, $((0, 1, 1), (1, 0, 0))^*$ of the first factor, and $(\cdot, 1, \cdot)$ of the second factor. Here

$j = 1$, $y = 1$. Hence $z_1 = y+1 = 0$, $z_0 = x_1+x_0+y = 1+0+1 = 0$, $z_2 = x_1+x_2+y = 1+1+1 = 1$, so that $(z_0, z_1, z_2) = (0, 0, 1)$, as given in Table 1. Similarly, the "level" of the third factor can be calculated for the other pairs of the "levels" of the first two factors.

5.2. $3\times 6\times 8$. The assemblies are $[\{i\}; \{(y,j)\}; \{(z_0, z_1, z_2)\}]$, where

(i) $i, j \in R(3)$, $y \in R(2)$,

(ii) if $i = j$, then $z_j = 1+y$, and (the ordered pair) (z_{j+1}, z_{j+2}) takes two values (0, 0) and (1, 1),

(iii) if $j = i+1$, then $z_j = z_{j+2} = 1+y$, $z_{j+1} = y$, and

(iv) if $j = i+2$, then $z_j = z_{j+1} = 1+y$, and $z_{j+2} = y$.

It will be instructive to the reader to compare the above representation of the levels of the various factors with that given in Example 3.2. For example, the level $((0,\cdot,\cdot), (1,\cdot,\cdot))^*$ of the first factor corresponds to $i = 0$ above. Similarly the level $(\cdot, 1, \cdot)$ of the second factor, which could have been denoted by $\{L(1; 1; 1, 3)\}$, corresponds to (y, j) above with $y = 1, j = 1$. The authors have kept this apparently dual notation, since it seems to help intuitively in the construction of various designs.

5.3. $2\times 5\times 10$. Assemblies are $[\{z\}; \{x\}; \{(x, y)^*\}]$. Here,

(i) $x \in R(5)$.

(ii) Given x, y may take any value such that $(x, y)^* \in K_1^*(5, 2; R(5))$,

(iii) if $y-x = 1$ or 3, then $z = 1$; if $y-x = 2$ or 4, then $z = 2$.

5.4. $2\times 7\times 21$. Assemblies are $[\{z\}; \{x\}; \{(x, y)^*\}]$, such that

(i) $x \in R(7)$.

(ii) Given x, y may take any value such that $(x, y)^* \in K_1^*(7, 2; R(7))$,

(iii) if $y-x = 1, 3, 5$, take $z = 1$; if $y-x = 2, 4, 6$, take $z = 2$.

5.5. $3\times 5\times 10$. Use $[\{z\}; (x); (y_1, y_2)^*\}$, with

(i) $x \in R(5)$,

(ii) $(y_1, y_2)^* \in K_1^*(5, 2; R(5))$, such that $y_1 \neq x$, $y_2 \neq x$,

(iii) $(y_1-x, y_2-x)^* = (1, 2)^*$, $(1, 3)^*$, $(2, 3)^*$, $(4, 1)^*$, $(3, 4)^*$, $(2, 4)^*$ respectively imply $z = 1, 1, 2, 2, 3, 3$,

5.6. $3\times 12\times 24$. Assemblies are $[\{x\}; \{(j, y)\}; \{k, (z_0, z_1)\}]$. Here

(i) $x, y \in R(3)$; $j \in R(4)$,

(ii) if $y-x = 0$ or 1, then $k = j$,

(iii) if y-$x = 0$, then (z_0, z_1) takes two values $(y+1, y+2)$, and $(y+2, y+1)$,

(iv) if $y-x = 1$, then $(z_0, z_1) = (y, y-1)$,

(v) if $y-x = 2$, then $(z_0, z_1) = (y, y+1)$,

(vi) if $y-x = 2$, then $j = (0, 1, 2, 3)$ respectively implies $k = (1, 0, 3, 2)$ when $x = 0$, $(2, 3, 0, 1)$ when $x = 1$, and $(3, 2, 1, 0)$ when $x = 2$.

5.7. $4\times 6\times 6$. Use $[\{k\}; \{(x_0, x_1, x_2)\}; \{(y_0, y_1, y_2)\}]$. Here

(i) (x_0, x_1, x_2), and $(y_0, y_1, y_2) \in P(0, 1, 2)$, such that $x_j = y_j$ for at least one j,

(ii) if $(x_0, x_1, x_2) = (y_0, y_1, y_2)$, then take $k = 3$,

(iii) if (x_0, x_1, x_2) and (y_0, y_1, y_2) have only the jth element common (i.e. $x_l = y_l$, only when $l = j$), then take $k = j$.

5.8. $5\times 5\times 10$. Use $[\{x\}; \{y\}; \{(z_1, z_2)^*\}]$, with

(i) $x, y \in R(5)$,

(ii) if $x \neq y$, then $(z_1, z_2)^* = (x, y)^*$,

(iii) if $x = y$, then $(z_1, z_2)^*$ takes the two values $(x-1, x+1)^*$, and $(x-2, x+2)^*$.

5.9. $5\times 10\times 10$. Use $[\{z\}; \{(x_1, x_2)^*\}; \{(y_1, y_2)^*\}]$, where

(i) $(x_1, x_2)^*$ and $(y_1, y_2)^* \in K_1^*(5, 2; R(5))$, such that the number of common elements in the unordered sets $(x_1, x_2)^*$ and $(y_1, y_2)^*$ does not equal 1,

(ii) if $(x_1, x_2)^*$ and $(y_1, y_2)^*$ do not have any element in common, then z is the element of $R(5)$ other than the four elements x_1, x_2, y_1, and y_2,

(iii) if $(x_1, x_2)^* = (y_1, y_2)^*$, then $z = x_1+2, x_2+2, x_1-1$, or x_2-1 according as (x_1-x_2) equals $1, -1, 2$, or (-2).

5.10. $6\times 6\times 6$. Use $[\{(x, i)\}; \{(y, j)\}; \{(z, k)\}]$, where

(i) $x, y \in R(3)$, and $i, j \in R(2)$, such that if $x = y$, then $i \neq j$,

(ii) if $x = y$, $i = j$, then $z = x$, $k = i$,

(iii) if $x \neq y$, then $k = i+j$, and z is the element of $R(3)$ distinct from x and y.

5.11. $6\times 6\times 9$. Use $[\{(x_0, x_1, x_2)\}; \{(z_0, z_1)\}; \{L(y, j; 1, 3)\}]$. Here

(i) $(x_0, x_1, x_2) \in P(0, 1, 2)$, and $j, y \in R(3)$,

(ii) if j is such that $y = x_j$, then (z_0, z_1) equals (x_{j+1}, x_{j+2}).

5.12. $6\times 6\times 12$. Use $[\{(x, i)\}; \{(y, j)\}; \{(z, k_1, k_2)\}]$, where

(i) $i, j \in R(2)$, and $x, y \in R(3)$,

(ii) $z = x+y$,

(iii) if $x = y$, then $k_1 = 1+i$, $k_2 = 1+j$, and

(iv) if $x \neq y$, then $k_1 = i$, $k_2 = j$.

5.13. $6\times 9\times 18$. Use $[\{(x_0, x_1, x_2)\}; \{L(y, j; 1, 3)\}; \{(z_0, z_1, z_2)\}]$, where

(i) $(x_0, x_1, x_2) \in P(0, 1, 2)$,

(ii) $j, y \in R(3)$,

(iii) $z_j = x_j$, $z_{j+1} = x_{j+1}$, and

(iv) if $x_j = y$, then $z_{j+2} = x_j$, and if $x_j \neq y$, then $z_{j+2} = x_{j+1}$.

5.14. $8\times 8\times 8$. Take $[\{(x_0, x_1, x_2)\}; \{(y_0, y_1, y_2)\}; \{(z_0, z_1, z_2)\}]$. Here

(i) x's, and y's $\in R(2)$, such that the number of values of j such that $(x_j = y_j)$, does not equal 1,

(ii) if $(y_0, y_1, y_2) = (x_0, x_1, x_2)$ or $(1+x_0, 1+x_1, 1+x_2)$ take $(z_0, z_1, z_2) = (y_0, y_1, y_2)$,

(iii) if $x_k = y_k$ for all k except $k = j$, then take $z_{j+1} = 1+x_{j+1}$, and $z_l = x$ (if $l \neq j+1$).

5.15. $9\times 9\times 9$. Assemblies are $[\{L(y_1, j_1; 1, 3)\}; \{L(y_2, j_2; 1, 3)\}; \{L(y_3, j_3; 1, 3)\}]$, where

(i) $y_1, y_2, j_1, j_2 \in R(3)$, with the condition that we have either $(j_1 = j_2, y_1 = y_2)$, or $(j_1 \neq j_2, y_1 \neq y_2)$,

(ii) if $j_1 = j_2$, and $y_1 = y_2$, then $j_3 = j_1$, and $y_3 = y_1$,

(iii) if $j_1 \neq j_2$, and $y_1 \neq y_2$, then j_3 and y_3 are such that $(j_1, j_2, j_3)^*$ and $(y_1, y_2, y_3)^*$ both equal $(0, 1, 2)^*$.

5.16. $10\times 15\times 30$. Assemblies are $[\{(l_1, l_2)^*\}; \{L(y, j; 1, 5)\}; \{L(z, z, z; i_1, i_2, i_3; 3, 5)\}]$. Here

(i) $j \in R(5)$, $y \in R(3)$,

(ii) Given j, $(l_1, l_2)^*$ can take any value in the set $K_1^*(5, 2; R(5))$, except $(j-1, j-2)^*$, $(j+1, j+2)^*$, $(j+1, j-2)^*$, and $(j-1, j+2)^*$,

(iii) if $j = l_1$ or l_2, then $z = y$,

(iv) if $(l_1, l_2)^* = (j-\theta, j+\theta)^*$, where $\theta = 1$ or 2, then $z = y+1$, and $(i_1, i_2, i_3)^*$ consist of the three elements of $R(5)$ excluding $(j-\theta)$ and $(j+\theta)$, or in brief $(i_1, i_2, i_3)^* = R(5)-(j-\theta, j+\theta)^*$,

(v) if $(l_1, l_2)^* = (j, j-1)^*$ or $(j, j+1)^*$, then $(i_1, i_2, i_3)^* = R(5)-(l_1\ l_2)^*$, and

(vi) if $(l_1, l_2)^* = (j, j+\theta)^*$, where $\theta = 2$ or (-2), then $(i_1, i_2, i_3)^* = R(5)-(j, j-\theta)^*$.

The above designs are presented to illustrate the general approach to the construction of designs involving various numbers of levels of a factor. For some more designs, see for example Anderson (1968). It may be stressed here that although a few of the above designs can be (and, in fact, were at first) constructed by using a geometrical approach, the factorial approach was found far more satisfactory and productive. The former is usually much more confusing and difficult, and was found many a time even to lead to error. On the contrary, the factorials have a certain richness and symmetry which makes them very flexible and versatile for being used for purposes of design construction.

Finally, we may remark that the above designs can also be used as fractionally replicated main effect plans for asymmetrical factorials. If confounding is desired, one or more factors may be considered as block factors.

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