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ASYMPTOTIC BEHAVIOR OF HIGH ORDER MEANS[1]

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The following simple but interesting question was suggested by S. W. Joshi. Given a nonnegative Borel measure μ on [0, 1] with $\mu([0, 1]) = 1$, the mean m_1 can be defined as the unique root in [0, 1] of the equation $\int_0^1 (x-m)\,d\mu(x) = 0$; the root m_r in [0, 1] of $f_r(m) = \int_0^1 |x-m|^r \operatorname{sign}(x-m)\,d\mu(x) = 0$, for $r \geqq 1$, can be considered a generalized mean, in the spirit of the more general ϕ-means of [1] for which results similar to these being presented here can be obtained. The question arises as to how m_r depends on r and μ; in this note we show that m_r converges to the midpoint of the interval of essential support of the measure μ as r tends to infinity.

Let $a = \sup\{l; \mu([l, 1])\} = \mu([0, 1])$ and let $b = \inf\{r; \mu([0, r]) = \mu([0, 1])\}$. Clearly then $\mu([a, b]) = \mu([0, 1])$, $\mu([a, a+\varepsilon]) > 0$ for every ε in $(0, b-a]$, and $\mu([b-\varepsilon, b]) > 0$ for every ε in $(0, b-a]$; we call the interval $[a, b]$ the interval of essential support of the measure μ, although one might discard an endpoint if it itself has μ-measure zero. Our result can now be stated as follows. *For each* $r \geqq 1$, *a unique* m_r *exists, and* $\lim_{r\to\infty} m_r = m_\infty \equiv (a+b)/2$.

The proof is simple. Since $f_r(m)$ is a continuous function of m and satisfies $f_r(0) = f_r(a) \geqq 0$ and $f_r(1) = f_r(b) \leqq 0$, at least one root exists in [0, 1]. Furthermore, by using the Lebesgue dominated convergence theorem we can show easily that $f_r(m)$ is differentiable and $f_r'(m) = -r\int_0^1 |x-m|^{r-1}\,d\mu(x)$ which is less than zero for all m unless $a = b = m$, in which case $m_r = m_\infty$ for all r and there is nothing to prove. Thus a unique root m_r exists and lies in $[a, b]$. Now it remains to prove that m_r tends to m_∞ when $a \neq b$. Let m be a fixed number satisfying $m_\infty < m \leqq b$. We shall show that for all large r we have $f_r(m) < 0$ which implies $m_r < m$ which in turn implies $\limsup_{r\to\infty} m_r \leqq m_\infty$ since $m > m_\infty$ was arbitrary; for this purpose let ε satisfy $0 < \varepsilon < 2(m-m_\infty)$, $\varepsilon < m-a$. Then

$$\begin{aligned} f_r(m) &= -\int_a^{a+\varepsilon}(m-x)^r\,d\mu(x) - \int_{a+\varepsilon}^{m}(m-x)^r\,d\mu(x) + \int_m^b (x-m)^r\,d\mu(x) \\ &\leqq -\int_a^{a+\varepsilon}(m-x)^r\,d\mu(x) + \int_m^b (x-m)^r\,d\mu\ x) \\ &\leqq -(m-a-\varepsilon)^r\mu([a, a+\varepsilon]) + (b-m)^r\mu([m, b]) \\ &= (m-a-\varepsilon)^r\mu([a, a+\varepsilon])\left\{-1 + \left(\frac{b-m}{m-a-\varepsilon}\right)^r \frac{\mu([m, b])}{\mu([a, a+\varepsilon])}\right\}. \end{aligned}$$

Since $m-a-\varepsilon > 0$, since $\mu([a, a+\varepsilon]) > 0$, and since $(b-m)/(m-a-\varepsilon) < 1$ because $\varepsilon < 2(m-m_\infty) = 2(m-(b+a)/2)$, we conclude that $f_r(m) < 0$ for sufficiently

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large r. By precisely similar arguments (or by replacing x by $1-y$) we find that for any fixed $m < m_\infty$, we have $f_r(m) > 0$ for large r and hence $m_r > m$ and hence $\liminf_{r\to\infty} m_r \geqq m_\infty$. Therefore the two inequalities together yield $\lim_{r\to\infty} m_r = m_\infty$.□

REFERENCE

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