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A WEAK CONVERGENCE THEOREM FOR RANDOM SUMS IN A NORMED SPACE[1]

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0. Introduction. The present paper deals with the weak convergence of random sums of independent random variables which take values in a normed space. A similar problem but for independent random variables taking values in $D[0, 1]$, the set of all real-valued functions defined on $[0, 1]$ which are right continuous with left limits was studied by the same author in [3]. The main result of the paper is Theorem 2.1.

1. Basic notation and terminology. Throughout this work the pair $(\Omega, \mathscr{A})$ and the triple $(\Omega, \mathscr{A}, P)$ will denote a measurable space and a probability space respectively. Let R denote the set of all real numbers.

For $f:\Omega \to R$ we define

$$\int_* f dP = \sup \{\int g dP : g\ \mathscr{A}\text{-measurable}, g \leqq f, \int g dP \text{ is defined}\}.$$

Similarly we define $\int^* f dp$. If $A \subseteq \Omega$ we will write instead of $\int^* I_A dP$ and $\int_* I_A dP$, $\mu^*(A)$ and $\mu_*(A)$ respectively.

If $(\Omega, \mathscr{A}, \mathrm{P})$ and $(\Omega', \mathscr{A}', P')$ are two probability spaces $P \times P'$ will denote the product probability on $(\Omega \times \Omega', \mathscr{A} \times \mathscr{A}')$ ([4] page 145). Given $(\Omega, \mathscr{A}, P)$, $(\Omega', \mathscr{A}')$ and $X:\Omega \to \Omega'$ $\mathscr{A} - \mathscr{A}'$ measurable, we denote by PX^{-1} a probability on $(\Omega', \mathscr{A}')$ defined by $PX^{-1}(A') = P(X^{-1}(A'))$. We will also use the symbol $\mathscr{L}(X)$ for PX^{-1}. Let now (S, d) be a metric space, with distance function d. Let $C(S)$ denote the set of all bounded real valued continuous functions on S. We write $B_{x,r}$ (resp. $\overline{B_{x,r}}$) for the open (resp. closed) ball centered at $x \in S$ of radius $r > 0$:

$$B_{x,r} = \{y \in S : d(x, y) < r\} \overline{B_{x,r}} = \{y : d(x, y) \leqq r\}.$$

For $A \subseteq S$, we let A^c, $\bar{A}$, ∂A and A^δ denote respectively the complement of A, the closure of A, the boundary of A and the open δ-ball $\{y : y \in S, d(y, A) < \delta\}$ about A. We let $\mathscr{B}$ denote the Borel σ-algebra of S; this is the σ-algebra generated by the topology induced by d. It coincides with the minimal σ-algebra that makes measurable the bounded continuous real-valued functions on S. Let S_0 denote the σ-algebra generated by the balls of S. If the metric space is separable then clearly $\mathscr{B} = S_0$.

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It is easily seen that if K is compact $d(\cdot, K)$ is S_0-measurable, from which it follows that any compact set $K \in S_0$ and for any $\delta \geqq 0$ also $K^\delta \in S_0$. If $\mathscr{C}$ is a σ-algebra, $S_0 \subseteq \mathscr{C} \subseteq \mathscr{B}$ and μ is a probability measure on $\mathscr{C}$ we say that μ is tight iff $\forall\ \varepsilon > 0$, there exists K a compact such that $\mu(K) > 1-\varepsilon$. A subset A of S is said to be a P-continuity set (P defined on $\mathscr{B}$) iff $P(\partial(A)) = 0$. The class of all P-continuity sets is easily seen to be an algebra.

2. A weak convergence theorem for random sums in a normal space. Before proving the main result several propositions and lemmas will be stated. Some of these results are well known; in those cases no proofs will be given and only the appropriate references will be listed. (S, d) will be a fixed metric space, and unless explicit mention to the contrary is made, for any net of probability measures $\{\mu_\alpha\}_{\alpha\in\Gamma}$ we will always assume the existence of σ-algebras $\{\mathscr{A}_\alpha\}_{\alpha\in\Gamma}$, $S_0 \subseteq \mathscr{A}_\alpha \subseteq \mathscr{B}$, with μ_α defined on $\mathscr{A}_\alpha$.

DEFINITION 2.1. *Let $\{\mu_\alpha\}_{\alpha\in\Gamma}$ be a net of probabilities defined on σ-algebras $\mathscr{A}_\alpha \supseteq S_0$. We say that μ_α converges weakly to μ and write $\mu_\alpha \to_\omega \mu$, or $\mu = \lim \mu_\alpha$ iff for all bounded continuous functions f*

$$\lim_\alpha \int^* f\, d\mu_\alpha = \lim_\alpha \int_* f\, d\mu_\alpha = \int f\, d\mu$$

where μ is a probability defined on $\mathscr{B}$.

DEFINITION 2.2. *A net $\{\mu_\alpha\}_{\alpha\in\Gamma}$ of probabilities defined on σ-algebras $\mathscr{A}_\alpha \supseteq S_0$ is said to be δ-tight iff*

$$\sup_K \{\inf_{\delta \leqq 0} \liminf_\alpha \mu_\alpha(K^\delta) : K \textit{ compact}\} = 1.$$

For properties regarding the mode of convergence given by Definition 2.1, the concept of δ-tightness, and related notions the reader is referred to [1], [2], [5], [8] and [9].

PROPOSITION 2.1 *If μ is a tight probability measure defined on a σ-algebra $\mathscr{C}$, $S_0 \subseteq \mathscr{C} \subseteq \mathscr{B}$, then μ can be extended to a probability on $\mathscr{B}$.*

PROOF. Notice that if μ is tight $\int^* f du = \int_* f d\mu$ for all $f \in C(S)$. Define $M(f) = \int^* f d\mu$ for $f \in C(S)$. M is positive, linear, $M(1) = 1$ and σ-smooth. The result now follows from Daniell's representation theorem ([7] Section II.7).

Let (S, d) and (S', d') be two metric spaces. If we endow the product $S \times S'$ with the max. metric:

$$d''((x, x'), (y, y')) = \max\{d(x, y), d'(x', y')\}$$

then $(S \times S')_0 = S_0 \times S_0'$. If μ and ν are two tight probabilities on σ-algebras $\mathscr{A} \supseteq S_0$ and $\mathscr{A}' \supseteq S_0'$ then by Proposition 2.1 $\mu \times \nu$ which is defined on $\mathscr{A} \times \mathscr{A}'$ can be extended to the Borel σ-algebra of $S \times S'$, we denote that extension by $\mu \otimes \nu$.

In the following propositions we will always consider the following situation. $(E, \|\cdot\|)$ a normed space, $\mathscr{D}$ a σ-algebra containing S_0, $\{\mu_\alpha\}_{\alpha\in\Gamma}$ and $\{\nu_\alpha\}_{\alpha\in\Gamma}$ are nets of probabilities on $\mathscr{D}$, and if $\phi: E \times E \to E$, $\phi(x, y) = x+y$, is the addition operation in E, we will assume that ϕ is $\mathscr{D} \times \mathscr{D} - \mathscr{D}$ measurable. If μ and ν are tight probabilities on the Borel σ-algebra of E then we define $\mu * \nu = (\mu \otimes \nu)\phi^{-1}$. An $\mathscr{A} - \mathscr{D}$ measurable random variable will be called a random element of E.

PROPOSITION 2.2. *If $\mu_\alpha \to_\omega \mu$, $\nu_\alpha \to_\omega \nu$, and μ and ν are tight then*

$$(\mu_\alpha \times \nu_\alpha)\phi^{-1} \to_\omega \mu * \nu.$$

PROOF. If $\mu_\alpha \to_\omega \mu$ and $\nu_\alpha \to_\omega \nu$ with μ and ν tight then $\mu_\alpha \times \nu_\alpha \to_\omega \mu \otimes \nu$ by Theorem 1.6. of [9]. Then $(\mu_\alpha \times \nu_\alpha)\phi^{-1} \to_\omega \mu * \nu$ by a corollary to Theorem 1.2 (b) of [9]. See also Dudley [2] or Lemma 5 of LeCam [5].

PROPOSITION 2.3. *If $\{(\mu_\alpha \times \nu_\alpha)\phi^{-1}\}_{\alpha \in \Gamma}$ is δ-tight and $\{\mu_\alpha\}_{\alpha \in \Gamma}$ is δ-tight then $\{\nu_\alpha\}_{\alpha \in \Gamma}$ is also δ-tight.*

PROOF. The proof follows from the relation $K^\delta - K^\delta \subseteq (K-K)^{2\delta}$ which is valid for any $\delta > 0$ and any set K, and the inequality

$$(\mu_\alpha \times \nu_\alpha)\phi^{-1}(K^\delta) \leqq \nu_\alpha((K-K)^{2\delta}) + \mu_\alpha((K^\delta))$$

where K is now a compact set and $\delta > 0$.

PROPOSITION 2.4. *If μ and ν are two tight probability measures on the Borel σ-algebra of E and $\mu = \mu * \nu$ then $\nu = \delta_0$ where $\delta_0(\{0\}) = 1$.*

PROOF. If $\hat{\mu}(y) = \int_E e^{i\langle x,y\rangle}\mu(dx)$ where $y \in E'$, the topological dual of E, is the characteristic function of μ, then it is not difficult to show that $\hat{\nu} = 1 = \hat{\delta}_0$. From this equality it follows that $\nu = \delta_0$ on the minimum σ-algebra which makes measurable all the elements of E'. Since ν and δ_0 are tight this implies that $\nu = \delta_0$ on the Borel σ-algebra.

PROPOSITION 2.5. *$(\mu_\alpha \times \nu_\alpha)\phi^{-1} \to_\omega \mu$ and $\mu_\alpha \to_\omega \mu$ where μ is tight. Then $\nu_\alpha \to_\omega \delta_0$.*

PROOF. Since $\{\mu_\alpha\}_{\alpha \in \Gamma}$ and $\{(\mu_\alpha \times \nu_\alpha)\phi^{-1}\}_{\alpha \in \Gamma}$ converge are δ-tight and by Proposition 2.3 $\{\nu_\alpha\}_{\alpha \in \Gamma}$ is δ-tight. It is enough to show now that if $\{\nu_{\alpha'}\}_{\alpha' \in \Gamma'}$ is a subnet of $\{\nu_\alpha\}_{\alpha \in \Gamma}$ which converges then $\nu_{\alpha'} \to_\omega \delta_0$. If $\nu_{\alpha'} \to_\omega \nu$ then ν is tight and we have $(\mu_{\alpha'} \times \nu_{\alpha'})\phi^{-1} \to_\omega \mu$ by assumption and $(\mu_{\alpha'} \times \nu_{\alpha'})\phi^{-1} \to_\omega \mu * \nu$ by assumption and Proposition 2.2. Therefore $\mu = \mu * \nu$. The result now follows from Proposition 2.4.

We come now to the main result of this paper. We also assume now that for any real number λ the mapping $x \to \lambda x$ is $\mathscr{D} - \mathscr{D}$ measurable. From this assumption it follows that the inequality given by Proposition 2.2 of [3] holds. Let $\{X_n\}_{n=1,2,\dots}$ be a sequence of independent $\mathscr{A} - \mathscr{D}$ measurable random variables, and $\{\tau_n\}_{n=1,2,\dots}$ a sequence of positive integer-valued random variables satisfying Condition I of [3]. The main result of this paper is the following. Let $\{X_n\}_{n=1,2,\dots}$ be a sequence of independent random variables and

$$S_n = \sum_{i=1}^n X_i, \qquad S_{\tau_n} = \sum_{i=1}^{\tau_n} X_i.$$

THEOREM 2.1. (a) *If the random variables in the sequence are identically distributed and* $\mathscr{L}(S_n/n^{\frac{1}{2}}) \to_\omega \mu$ *then* $\mathscr{L}(S_{\tau_n}/\tau_n^{\frac{1}{2}}) \to_\omega \mu$.

(b) *If* $\mathscr{L}(S_n/n^{\frac{1}{2}}) \to_\omega \mu$ *and* μ *is tight then* $\mathscr{L}(S_{\tau_n}/\tau_n^{\frac{1}{2}}) \to_\omega \mu$.

PROOF. First notice that S_{τ_n} and $S_{\tau_n}/\tau_n^{\frac{1}{2}}$ are $\mathscr{A}-\mathscr{D}$ measurable. As in Theorem 1 of [3] the problem is reduced to show that

$$P[||S_{c_n}-S_{a_n}|| \geqq b_n^{\frac{1}{2}}\varepsilon/2] \to 0 \qquad \text{as } n \to \infty$$

where $b_n \leqq a_n < c_n$, $b_n \to \infty$ and $c_n/b_n \to 1$.
The proof of (a) is similar to the one given in Theorem 1 of [3].

To prove (b) we write

$$(c_n/b_n)^{\frac{1}{2}}S_{c_n}/c_n^{\frac{1}{2}} + (S_{c_n}-S_{a_n})/b_n^{\frac{1}{2}} + (a_n/b_n)^{\frac{1}{2}}S_{a_n}/a_n^{\frac{1}{2}}.$$

If we let

$$\mathscr{L}((c_n/b_n)^{\frac{1}{2}}S_{c_n}/c_n^{\frac{1}{2}}) = \lambda_n$$

$$\mathscr{L}((S_{c_n}-S_{a_n})/b_n^{\frac{1}{2}}) = \nu_n$$

$$\mathscr{L}((a_n/b_n)^{\frac{1}{2}}S_{a_n}/a_n^{\frac{1}{2}}) = \mu_n$$

we have $\lambda_n = (\mu_n \times \nu_n)\phi^{-1}$ and $\lambda_n \to_\omega \mu$, $\mu_n \to_\omega \mu$ with μ tight. Then by Proposition 2.5: $\nu_n \to_\omega \delta_0$ which is what we wanted to prove.

REMARK. The main result of this paper, though seemingly more general than the one in [3], does not contain it. If we consider in $D[0, 1]$ the norm $||x|| = \sup_{0 \leqq t \leqq 1} |x(t)|$, $D[0, 1]$ becomes a Banach space (not separable). It is not difficult to show that the σ-algebra generated by the Skorokhod topology $\mathscr{D}$, coincides with the σ-algebra S_0 generated by the balls (with respect to $||\cdot||$). Therefore in Theorem 1 of [3] the limit measure μ is defined only on the σ-algebra generated by the balls while in Theorem 2.1 the limit measure is a Borel measure. If the limit measure in the first case were concentrated on $C[0, 1]$, the set of all continuous functions on [0, 1], then Theorem 1 of [3] would be a particular case of Theorem 2.1 because, since the uniform topology coincides with the Skorokhod topology on $C[0, 1]$, μ is tight and by Proposition 2.1, can be extended to $\mathscr{B}$, the Borel σ-algebra in $D[0, 1]$ for $||\cdot||$. By an argument similar to the one used in [1], page 151 (where we have to replace μ_n by μ_n^* all along, μ_n being $\mathscr{L}(S_n/n^{\frac{1}{2}})$), we have that $\mathscr{L}(S_n/n^{\frac{1}{2}}) \to_\omega \mu$ in the sense of Definition 2.1. Therefore Theorem 2.1 can be applied. In particular Theorem 1 of [3] remains true if the sequence of independent random elements $\{X_n\}_{n=1,2,\ldots}$ does not consist of identically distributed ones provided the limit measure is concentrated on $C[0, 1]$.

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