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ON THE CONVERGENCE OF BINOMIAL TO POISSON DISTRIBUTIONS

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1. Summary. Following the work of Poisson (1837), there has been considerable practical and theoretical interest in how well the Poisson distribution approximates the binomial distribution. The approximation, which was initially suggested by a limit theorem (see (1) below), has been shown in numerical examples to be very good for certain binomial parameters within a useful range. (See Feller (1950), page 143.) Subsequently, (nonasymptotic) theoretical results have confirmed the approximation's accuracy. (See (2.1) and (2.2) below.) The purpose of this note is to demonstrate, with an elementary argument, that the binomial distributions converge very strongly to the Poisson distributions.

2. Introduction and results. The probability that a binomial variable with parameters n, p takes the value ra nonnegative integer) is

$$b_r(n, p) = \binom{n}{r} p^r (1-p)^{n-r}$$

(with $\binom{n}{r} = 0$ if $r > n$).

The corresponding probability for a Poisson variable with parameter λ is

$$p_r(\lambda) = e^{-\lambda} \lambda^r / r!.$$

It is well known that

$$\lim_{n\to\infty} b_r(n, \lambda/n) = p_r(\lambda). \tag{1}$$

A number of results are known which show how fast the limit in (1) is approached. These include

$$\lim_{n\to\infty} \sum_{r=0}^{\infty} |b_r(n, \lambda/n) - p_r(\lambda)| = 0 \tag{2}$$

and several stronger forms of (2). For example,

$$\sum_{r=0}^{\infty} |b_r(n, \lambda/n) - p_r(\lambda)| \leqq \frac{\lambda}{n} \sqrt{\frac{2}{1-\lambda/n}} \tag{2.1}$$

(Vervaat (1970)) and

$$\sum_{r=0}^{\infty} |b_r(n, \lambda/n) - p_r(\lambda)| \leqq 1 \cdot 2\lambda/n \qquad \text{if } n \geqq 4 \tag{2.2}$$

(Kerstan (1964)). Additional references are given in Vervaat (1970).

We now give a considerable strengthening of (2) in another direction:

THEOREM. *If* $h(r) \geqq 0$ *for* $r \geqq 0$, *then*

$$\lim_{n\to\infty} \sum_{r=0}^{\infty} h(r) |b_r(n, \lambda/n) - p_r(\lambda)| = 0 \tag{3}$$

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if and only if

$$\sum_{r=0}^{\infty} h(r)p_r(\lambda) < \infty. \tag{4}$$

PROOF. The key to the proof is the fact that

$$S = \sup\,[b_r(n, \lambda/n)/p_r(\lambda) : r \geqq 0, n \geqq 1+\lambda] < \infty. \tag{5}$$

Let $a_{r,n}$ denote the ratio in (5). Then $a_{r,n}/a_{r-1,n} = (n-r+1)/n-\lambda)$ for $1 \leqq r \leqq n$, $n \geqq 1+\lambda$ and hence,

$$a_{r,n} \leqq a_{s,n} \qquad \text{for } r \geqq 0, n \geqq 1+\lambda, \tag{6}$$

where $s = [1+\lambda]$ the largest integer $\leqq 1+\lambda$). But (1) implies $\lim_{n\to\infty} a_{s,n} = 1$ which, together with (6), yields (5). Finally, since the rth summand in (3) is bounded below by 0 and above by $(S+1)h(r)p_r(\lambda)$, (3) follows from (1), (4) and the dominated convergence theorem. Conversely, if (3) holds, then

$$0 \leqq \sum_{r=n+1}^{\infty} h(r)p_r(\lambda) \leqq \sum_{r=0}^{\infty} h(r)|b_r(n, \lambda/n) - p_r(\lambda)| \to 0 \qquad \text{as } n \to \infty$$

and (4) follows.

We note that $\sum_{r=0}^{\infty} h(r)p_r(\lambda)$ is the expected value of (hX) when X is a Poisson variable with expected value λ. In particular, since $E(e^{tX})$ is finite, it follows that

$$\lim_{n\to\infty} \sum_{r=0}^{\infty} e^{tr}\,|b_r(n, \lambda/n) - p_r(\lambda)| = 0$$

for all values of t.

It is clear that one could generalize the argument which shows (4) implies (3) to include other densities than the binomial and the Poisson. Unfortunately, a condition such as (or like) (5) is very strong and seldom holds in interesting examples.

We wish to thank our referee who has shown us how the dominated convergence theorem could be used to make our original argument easier.

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