The Annals of Mathematical Statistics
1971, Vol. 42, No. 5, 1686-1696

INVARIANT MEASURES ON SOME MARKOV PROCESSES[1]

BY Y. S. YANG

Nanyang University

0. Introduction, summary and notations. The principal result of this paper is the introduction of a readily verifiable sufficient condition for the existence of an invariant measure on transient Markov chains. Specifically, subject to mild additional regularity conditions, it is enough to check that for every compact set A in the state space X, there exists a compact $C \supset A$ such that

$$(0.1)\quad \lim_{y \in F, y \to \infty} \frac{P(y, A)}{P(y, C)} = 0 \qquad \text{if } \bar{F} \text{ is not compact and } F = \{y : P(y, A) > 0\},$$

$P(y, A)$ denote the 1-step transition probabilities of the Markov chain.

(0.1) is readily implied by simple conditions on the generating functional of the offspring distribution (Theorems 2.1 and 2.2) for discrete time multi-type Markov processes. These conditions together with some additional regularity-hypotheses are verified for

(i) A 1-dimensional neutron branching model of Harris,

(ii) Discrete-time age dependent branching processes,

(iii) Galton–Watson processes with immigration.

A generalization (to discrete-time temporally homogeneous Markov processes with σ-compact metric state space) of a condition of T. Harris (1957) for existence of invariant measures on transient Markov chains is also given. This condition is, unfortunately, difficult to check in specific examples. The sufficient condition (0.1) involves only 1-step transition probabilities, as opposed to the n-step transitions incorporated in the Harris condition, which is also necessary.

Throughout this paper, all subsets of a topological space considered are assumed Borel measurable. The closure, interior, boundary and complement of a set A are denoted by $\bar{A}$, A^0, ∂A and A^c respectively, and I_A denotes the indicator function of A. The sets of positive integers, nonnegative integers, are denoted by I and I_0 respectively. For any σ-compact metric space X, the following notations are used:

$M(X)$ = set of regular measures (Borel measures finite on compact sets) on X.

$B(X)$ = set of bounded measurable functions on X with sup norm.

$B'(X) = \{s \in B(X) : \|s\| < 1\}$.

$C(X)$ = set of bounded continuous functions on X.

$C_{00}(X) = \{s \in C(X) : s \text{ has compact support}\}$.

Received March 10, 1970; revised March 8, 1971.

[1] This paper is based on the first part of the author's thesis presented to the Graduate School of the University of Southern California under the direction of Professor T. E. Harris who suggested the problem and made many suggestions.

$C_0(X) = \{s \in C(X): \lim_{x\to\infty} s(x) = 0\}$.

$C'(X) = C(X) \cap B'(X)$, similarly for $C_0'(X)$ and $C'_{00}(X)$.

For any $\mathscr{C} \subset B(X)$, $\mathscr{C}^+ = \{s \in \mathscr{C}: s \geqq 0\}$. If $\mu \in M(X)$ and $\in B(X)$, then $\mu f = \int_X f(x)\mu(dx)$ when defined.

1. Invariant measures on certain Markov processes with σ-compact metric state space. Let $Z_0, Z_1, \cdots$ be a temporally homogeneous Markov process on X which is σ-compact metric. $P(x, A)$, the transition probability, is measurable in x for all $A \subset X$, and $P(x, X) \leqq 1$ for all $x \in X$. Let $Q(x, A) = \sum_{n=1}^{\infty} P^n(x, A)$. For $f \in B(X)$ let $P^n f(\cdot) = \int_X P^n(\cdot, dy)f(y)$, $n \in I$, and for $\pi \in M(X)$,

$$\pi P(\cdot) = \int_X P(x, \cdot)\pi(dx) \quad \text{and} \quad \pi P f = \int_X \pi(dx) \int_X P(x, dy) f(y).$$

ASSUMPTIONS.

(1.1) $Pf(\cdot) \in C_0(X)$ for all $f \in C_0(X)$.

(1.2) If A is open, then for any $x \in X$, there exists $n \in I$ such that $P^n(x, A) > 0$.

(1.3) $Q(x, A) < \infty$ for all $x \in X$ and compact A.

REMARK. (1.1) is a weaker assumption than that of the continuity of $P(x, A)$ in x and $\lim_{x\to\infty} P(x, A) = 0$ for all compact A.

The following lemma may be known.

LEMMA 1.1. *If conditions* (1.1) *to* (1.3) *hold, then for any compact A and open B, there exists a finite number c such that*

$$\frac{Q(x, A)}{Q(x, B)} < c \quad \textit{for all} \quad x \in X.$$

PROOF. It follows from (1.1) and (1.3) that for any $x \in A$, there exists $f \in C_{00}(X)$ and $n \in I$ such that $f \leqq I_B$ and $\inf_{u\in U} P^n f(u) > 0$ for some neighborhood U of x. Now for each $y \in X$,

$$Q(y, B) = \sum_{m=1}^{\infty} P^m(y, B) \geqq \sum_{m=1}^{\infty} P^{m+n}(y, B) = \int_X Q(y, dx) P^n(x, B)$$
$$\geqq \int_U Q(y, du) P^n(u, B) \geqq Q(y, U) \inf_{u\in U} P^n(u, B).$$

Hence

$$\frac{Q(y, U)}{Q(y, B)} \leqq \frac{1}{\inf_{u\in U} P^n(u, B)} < \infty$$

and the lemma follows from a simple compactness argument. □

THEOREM 1.1. *Suppose that* (1.1) *to* (1.3) *hold and that one of the following conditions* (1.4) *or* (1.5) *is satisfied*:

(1.4) *For each compact A, there exists a compact $C \supset A$ such that*

$$\lim_{y\to\infty, y\in F} \frac{P(y, A)}{P(y, C)} = 0 \quad \textit{if } \bar{F} \textit{ is not compact, where} \quad F = \{y: P(y, A) > 0\}.$$

(1.5) *There exists a sequence* $\{z_k\}$ *in* X, $z_k \to \infty$ *such that for each sequence of compact sets* $Y_n \uparrow X$, $Y_n \subset Y_{n+1}^0$, *we have*

$$\lim_{n\to\infty} \limsup_{k\to\infty} \frac{L(z_k, A, Y_n)}{Q(z_k, A)} = 0$$

or each compact set A *with* $A^0 \neq \varnothing$, *where*

$$L(z, A, B) = P(z, A) + \sum_{n=1}^{\infty} \int_{x-B} {}_AP^n(z, dy)P(y, A)$$

and

$${}_AP^n(z, B) = P_z(Z_n \in B; Z_i \notin A, i = 1, \cdots, n-1),$$

and $Qf(\cdot)$ *is continuous for each* $f \in C_{00}(X)$.

Then there exists $\pi \in M(X)$ *such that* $\pi(X) = \infty$, $A^0 \neq \varnothing$ *implies that* $\pi(A) > 0$ *and* $\pi(A) = \pi P(A)$ *for all* $A \subset X$.

REMARKS. (1.5) is a direct generalization of the Harris condition (1957). (1.4), although special, involves just one-step transitions and hence is a useful sufficient condition, and readily applies to branching processes and other related processes. Also continuity of Qf for $f \in C_{00}(X)$ is not assumed in (1.4).

PROOF OF THEOREM 1.1. Let D be a fixed compact set with $D^0 \neq \varnothing$. Then for any compact set A with $A^0 \neq \varnothing$, by Lemma 1.1, there exist positive numbers α and β such that

$$\alpha \geqq \frac{Q(z, A)}{Q(z, D)} \geqq \beta \quad \text{for all} \quad z \in X. \tag{1.6}$$

Define $\pi_k \in M(X)$, $k \in I$ by $\pi_k(\cdot) = Q(z_k, \cdot)/Q(z_k, D)$, where $\{z_k\}$ is the sequence in (1.5) or any sequence such that $z_k \to \infty$ if (1.4) is assumed. From (1.6), $\{\pi_k\}$ is bounded uniformly in $k \in I$ when restricted to compact subset of X which is σ-compact metric. Hence it follows from Prohorov's theorem on weak compactness of measures on compact metric spaces (Prohorov (1956)), and a diagonalization process, that there exists a subsequence of $\{\pi_k\}$ again denoted by $\{\pi_k\}$ and $\pi \in M(X)$ such that $\pi_k f \to \pi f$ for all $f \in C_{00}(X)$. For any $Y \subset X$,

$$\begin{aligned} Q(z, \cdot) &= P(z, \cdot) + \textstyle\sum_{n=1}^{\infty} \int_x P^{n-1}(z, dy)P(y, \cdot) \\ &= P(z, \cdot) + \textstyle\int_x Q(z, dy)P(y, \cdot) \\ &= P(z, \cdot) + \textstyle\int_y Q(z, dy)P(y, \cdot) + \int_{x-y} Q(z, dy)P(y, \cdot). \end{aligned}$$

Dividing by $Q(z, D)$, we have

$$\frac{Q(z, \cdot)}{Q(z, D)} - \frac{P(z, \cdot)}{Q(z, D)} + \int_y \frac{Q(z, dy)}{Q(z, D)} P(y, \cdot) + \int_{x-y} \frac{Q(z, dy)}{Q(z, D)} P(y, \cdot). \tag{1.7}$$

Let $Y_n \uparrow X$, Y_n compact, $Y_{n+1} \subset Y_n^0$ and $\pi(\partial Y_n) = 0$. We will show that under condition (1.4) or condition (1.5), we have for any compact A,

$$(1.8) \qquad \lim_{n\to\infty} \limsup_{k\to\infty} \frac{P(z_k, A) + \int_{X-Y_n} Q(z_k, dy)P(y, A)}{Q(z_k, D)} = 0.$$

First assume (1.5) holds. Since A is compact, $A \subset Y_n$ for all sufficiently large n. For $B \cap A = \varnothing$, we have

$$\begin{aligned} Q(z, B) &= E_z \text{ (number of visits to } B \text{ without hitting } A) \\ &\quad + E_z \text{ (number of visits to } B \text{ preceded by at least one visit to } A) \\ &= \textstyle\sum_{m=1}^{\infty} {}_AP^m(z, B) + \sum_{m=1}^{\infty} \int_A P^m(z, dy)\, {}_AQ(y, B) \end{aligned}$$

where ${}_AQ(y, B) = \sum_{n=1}^{\infty} {}_AP^n(y, B) = E_y$ (number of visits to B without hitting A). Hence for all sufficiently large n,

$$\begin{aligned} P(z_k, A) + \textstyle\int_{X-Y_n} Q(z_k, dy)P(y, A) &= P(z_k, A) + \textstyle\int_{X-Y_n} \sum_{m=1}^{\infty} P^m(z_k, dy)P(y, A) \\ &\quad + \textstyle\int_A Q(z_k, dx) \int_{X-Y_n} Q(x, dy)P(y, A). \end{aligned}$$

The first two terms on the right are just $L(z_k, A, Y_n)$. By (1.5) and Lemma 1.1, we have

$$\lim_{n\to\infty} \limsup_{k\to\infty} \frac{L(z_k, A, Y_n)}{Q(z_k, D)} = \lim_{n\to\infty} \limsup_{k\to\infty} \frac{L(z_k, A, Y_n)}{Q(z_k, A)} \frac{Q(z_k, A)}{Q(z_k, D)} = 0.$$

Now pick $f \in C_{00}(X)^+$ such that $f \geqq I_A$ and $\|f\| = 1$, and $g_n \in C(X)$ such that $I_{X-Y_{n-1}} \geqq g_n \geqq I_{X-Y_n}$.
Then

$$\textstyle\int_{X-Y_n} {}_AQ(x\, dy)P(y, A) \leqq \int_X g_n(y)Q(x, dy)Pf(y) = h_n(x).$$

Since Qf is continuous, $\sum_{n=1}^{\infty} P^n f$ is uniformly convergent in any compact subset (Dini's theorem). Now $P^m Pf(y)g_n(y) \leqq P^{m-1}f(y)$ for all $m \in I$. Hence $\sum_{m=1}^{\infty} P^m Pf(y)$ $g_n(y)$ is uniformly convergent in any compact subset of X. Hence h_n is continuous. Also $h_n \downarrow 0$ as $n \to \infty$. Let $B \supset A$, B compact and $\pi(\partial B) = 0$, then

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{k\to\infty} \int_A \frac{Q(z_k, dx)}{Q(z_k, D)} \int_{X-Y_n} {}_AQ(x, dy)P(y, A) \\ &\qquad \leqq \lim_{n\to\infty} \limsup_{k\to\infty} \textstyle\int_B \pi_k(dx)h_n(x) = \lim_{n\to\infty} \int_B h_n(x)\pi(dx) = 0. \end{aligned}$$

Hence (1.8) holds.
Now assume (1.4) holds. Let C and F be as in (1.4). If $\bar{F}$ is compact, then (1.8) obviously holds. Hence assume $\bar{F}$ is not compact. By the same argument as in the proof of Lemma 1.1, there exist open sets $U_1, \cdots, U_m$ covering C and $n_1, \cdots, n_m \in I$ such that $\inf_{u \in U_i} P^{n_i}(u, A) = a_i > 0$, $i = 1, \cdots, m$.

Let $a = \min_{1 \leqq i \leqq m} a_i > 0$. Then for any $i = 1, \cdots, m$

$$Q(z, A) \geqq \sum_{n=1}^{\infty} P^{n+1+n_i}(z, A) = \int_X Q(z, dy) \int_X P(y, dx) P^{n_i}(x, A)$$
$$\geqq \int_{(X - Y_n) \cap F} Q(z, dy) P(y, U_i) a.$$

Hence for $i = 1, \cdots, m$, $Q(z, A) \geqq am^{-1} \int_{(X-Y_n) \cap F} Q(z, dy) P(y, C)$.
Given $\varepsilon > 0$, there exists $N \in I$ such that for all $n \geqq N$ and $y \in (X - Y_n) \cap F$,

$$\frac{P(y, A)}{P(y, C)} \leqq \frac{\varepsilon}{\dfrac{m}{a} + \sup_{z \in X} \dfrac{Q(z, C)}{Q(z, D)}}.$$

Hence for all $n \geqq N$,

$$\frac{P(z, A) + \sum_{k=1}^{\infty} \int_{X - Y_n} P^k(z, dy) P(y, A)}{Q(z, A)}$$
$$\leqq \frac{\int_{(X-Y_n) \cap F} Q(z, dy) P(y, A)}{\dfrac{a}{m} \displaystyle\int_{(X-Y_n) \cap F} Q(z, dy) P(y, C)} + \frac{P(z, A)}{Q(z, A)}$$
$$\leqq \frac{m}{a} \frac{\varepsilon}{\dfrac{m}{a} + \sup_{z \in X} \dfrac{Q(z, C)}{Q(z, A)}} + \frac{P(z, A) P(z, C)}{P(z, C) Q(z, A)} \leqq \varepsilon$$

for all z sufficiently "large," that is outside some compact set. In case $P(z, C) = 0$, $P(z, A)/Q(z, A) = 0$. Hence (1.8) holds.

Now for $f \in C_{00}(X)$, we have from (1.7)

$$\text{(1.9)} \qquad \pi_k f = \frac{Pf(z_k) + \int_{X - Y_n} Q(z_k, dy) Pf(y)}{Q(z_k, D)} + \int_{Y_n} \pi_k(dy) Pf(y).$$

Let $k \to \infty$ and then $n \to \infty$, then $\pi_k f \to \pi f$. On the right side of (1.9), the first term $\to 0$ by (1.8). Furthermore, since Y_n is compact and $\pi(\partial Y_n) = 0$,

$$\lim_{n \to \infty} \limsup_{k \to \infty} \int_{Y_n} \pi_k(dy) Pf(y) = \lim_{n \to \infty} \int_{Y_n} \pi(dy) Pf(y) = \pi Pf.$$

Hence $\pi f = \pi Pf$, whence $\pi(B) = \pi P(B)$ for all $B \subset X$.
If $B^0 \neq \varnothing$, then there exists compact $K \subset B$ such that $K^0 \neq \varnothing$.
Then from (1.6), $\pi(B) \geqq \pi(K) \geqq \limsup_{k \to \infty} \pi_k(K) > 0$. Finally we show $\pi_k(X) = \infty$. Let B be such that $\pi(B) > 0$ and $Q(x, B) < \infty$ for all $x \in X$. Then by the bounded convergence theorem,

$$\pi(B) = \pi P^n(B) = \int_X \pi(dx) P^n(x, B) \qquad n = 1, 2, 3, \cdots$$
$$= \lim_{n \to \infty} \int_X \pi(dx) P^n(x, B)$$
$$= \int_X \pi(dx) \lim_{n \to \infty} P^n(x, B) = 0.$$

This contradicts $\pi(B) = 0$. Hence $\pi(X) = \infty$. □

COROLLARY 1.1. *Let P_{ij} be the transition matrix of a transient Markov chain with the denumerable set I as states. If for each $i \in I$, there exists a finite $B \subset I$ such that $i \in B$ and*

$$\lim_{k\to\infty,k\in F}\frac{p_{ki}}{\sum_{j\in Y}P_{kj}} = 0 \quad \textit{if } F \textit{ is infinite where} \quad F = \{k: P_{ki} > 0\},$$

then an invariant measure exists on any infinite subset T of I such that $\sum_{j\in 1-T}P_{ij} = 0$ for all $i \in T$ and the states of T communicate.

REMARK. It is possible to have $\sum_{j\in T}P_{ij} < 1$ for some $i \in T$, since we allow substochastic processes. And thus T may not be closed.

PROOF OF COROLLARY 1.1. For each $i \in T$, let B be as in the hypothesis. For all $k \in T$, $\sum_{j\in B}P_{kj} = \sum_{j\in B\cap T}P_{kj}$.. Hence $\lim_{k\to\infty,k\in T\cap F}(P_{ki}/\sum_{j\in B\cap T}P_{kj}) = 0$ if F is infinite. Apply Theorem 1.1 with $X = T$. □

2. Invariant measures on branching processes. Let X, σ-compact metric, be the set of types of a branching process as considered in Chapter 3 of Harris (1963). For $n \in 1$, let X_n = symmetrized n-fold product with itself, that is the quotient space X^n/N, where X^n is the n-fold Cartesian product of X with itself and N is the equivalence relation of permutation. Let $X_0 = \{\varnothing\}$. Denote the topological sum $X_0 + X_1 + \cdots$ of X_n, $n \in I_0$ by $\tilde{X}$. $\tilde{X}$ is a σ-compact metric space. If $\tilde{x} \in \tilde{X}$, then $\tilde{x}$ can be identified with a finite set of points in X and hence a finite counting measure. And we have $\tilde{x}s = \int_X s(x)\tilde{x}(dx) = \sum_{x\in\tilde{x}}s(x)$, $s \in B(X)$. Also if $s \in \overline{B'(X)^+}$, let $s_{\tilde{x}} = \pi_{x\in\tilde{x}}s(x) \in \overline{B'(X)^+}$. A general branching process can be considered as a temporally homogeneous Markov process $\tilde{Z}_0, \tilde{Z}_1, \cdots$ in a state space $\tilde{X}$ of points $\tilde{x}$. Each $\tilde{x}$ is a finite set $\{x_1, x_2, \cdots\}$, $x_i \in X$ and represents a set of objects with types $x_1, x_2, \cdots, \tilde{Z}_n \in X_0$ will mean no object present. We shall identify X with $X_1 C\tilde{X}$. $P(\tilde{x}, \cdot)$, the transition probability, will be a probability measure on $\tilde{X}$ for each $\tilde{x} \in \tilde{X}$. We shall write $P(x,\cdot)$ for $P(\tilde{x}, \cdot)$ when $\tilde{x}$ consists of the single point $x \in X$. Let $P_{\tilde{x}}$ be the probability measure on the sample space with $\tilde{Z}_0 = \tilde{x}$, $\tilde{x} \in \tilde{X}$. For $s \in B'(X)^+, x \in X$, let $F(s) = F(s, x) = \int_{\tilde{x}} s^{\tilde{y}}p(x, d\tilde{y})$ be the generating functional of $\tilde{Z}_1$ with $\tilde{Z}_0 = x \in X = X_1$. The branching property, which can be expressed as $\int_{\tilde{X}} s^{\tilde{y}}P(\tilde{x}, d\tilde{y}) = \Pi_{x\in\tilde{x}} F(s, x)$ for all $\tilde{x} \in \tilde{X}$, has to be satisfied. An invariant measure is a σ-finite measure on the Borel sets of $\tilde{X} - X_0$ satisfying $\pi(\cdot) = \int_{\tilde{X}-\tilde{X}_0} \pi(d\tilde{y})P(\tilde{y},\cdot)$.

THEOREM 2.1. *Suppose X is compact metric and the following conditions hold*:

(2.1) $\inf_{x\in X}P(x, X_0) > 0$;

(2.2) *For any $x \in X$ and open $\tilde{A} \subset \tilde{X}$, there exists $m \in I$ such that $P^m(x, \tilde{A}) > 0$;*

(2.3) *$F(s,\cdot) \in C'(X)^+$ for all $s \in C'(X)^+$.*

Then a regular invariant measure π exists on $\tilde{X} - X_0$ such that $\pi(\tilde{A}) > 0$ for all open $\tilde{A} \subset \tilde{X} - X_0$.

PROOF. (1.1), which now refers to the space $\tilde{X} - X_0$, follows from (2.3) by a result (Lemma 0.2) in Ikeda, Nagasawa, and Watanabe (1968). For $n \in I$, let

$t = \inf\{m > 0: \tilde{Z}_m \in X_n\}$ $t \leqq \infty$ and let $p \leqq \infty$ be the number of $m > 0$ such that $\tilde{Z}_m \in X_n$ and $\Omega_0 = \{t < \infty\}$. For $\tilde{x} \in X_k$, $P_{\tilde{x}}(\Omega_0) \leqq 1-(\inf_{\tilde{x} \in X} P(x, X_0))^k$, hence $\sup_{x \in X_k} P_{\tilde{x}}(\Omega_0) = a_k < 1$ for all $k \in I$. It follows from the strong Markov property that $P_{\tilde{x}}(p \geqq m+1) \leqq a_k \sup_{\tilde{x} \in X_n} P_{\tilde{x}}(p \geqq m)$, $k \in I$, $\tilde{x} \in X_k$. Hence inductively $\sup_{\tilde{x} \in X_k} P_{\tilde{x}}$ (hitting X_n at least m times) $\leqq a_k a_n^{m-1}$. We then have $\sup_{x \in X_k} Q(\tilde{x}, X_n) < \infty$ and $\sup_{\tilde{x} \in \tilde{A}} Q(\tilde{x}, \tilde{B}) < \infty$ for all compact $\tilde{A}, \tilde{B} \subset \tilde{X} - X_0$.

Now for any $\tilde{x} \in \tilde{X} - X_0$, $\tilde{x} = (x_1, \cdots, x_n)$ and open $\tilde{A} \subset \tilde{X}$, by (2.2), there exists $m \in I$ such that $P^m(x_1, \tilde{A}) > 0$, hence $P^m(\tilde{x}, \tilde{A}) \geqq (\inf_{x \in X} P(x, X_0))^{n-1} P^m(x_1, \tilde{A}) > 0$, and (1.2) holds. Now to show that (1.4) holds. Let $g(x) = P(x, X_0)$, $x \in X$. From (2.2), for any $x \in X$, there exists $h \in I$ such that $P(x, X_h) > 0$. From (1.1), there exists an open neighborhood U of x such that $\inf_{u \in U} P(u, X_h) > 0$. Hence from compactness of X, there exists $c > 0$ and $J = \{h_1, \cdots, h_m\} \subset I$ such that for all $x \in X$, there exists $h \in J$ such that $P(x, X_h) > c$. Let $l = \max\{h_1, \cdots, h_m\}$. Then $\inf_{\tilde{x} \in X_n} P(\tilde{x}, X_n + X_{n+1} + \cdots + X_{nl}) > c^n > 0$ for all $n \in I$. Let $\tilde{A}_n = X_n + \cdots + X_{nl}$. Consider any $\tilde{x} = (x_1, \cdots, x_k) \in X_k$, $k > (n+1)l$. Let C_j be the class of all sets of jx's from $\{x_1, \cdots, x_k\}$, $j = 1, \cdots, k$. Note that every element of C_j is an element of X_j. If a generation with k objects is transformed into one with $n < k$ objects, at least $k-n$ objects in the initial generation must have no children. Hence

$$P(\tilde{x}, X_n) \leqq \textstyle\sum_{j=k-n}^{k} \sum_{C_j} g(x_{i_1}) \cdots g(x_{ij}).$$

For each subset $\tilde{y}$ of $\tilde{x}$ with $k-(n+1)$ objects, that is $\tilde{y} \in C_{k-(n+1)}$, a possibility for $\tilde{Z}_1 \in \tilde{A}_{n+1}$ given $\tilde{Z}_0 = \tilde{x}$ is that all the objects in $\tilde{y}$ should have no children and the remaining $n+1$ objects to transform into $n+1$ to $(n+1)l$ objects. Since different objects reproduce independently of one another, we have

$$\begin{aligned} P(\tilde{x}, \tilde{A}_{n+1}) &\geqq \textstyle\sum_{C_{k-n-1}} g(x_{i_1}) \cdots g(x_{i_{k-n-1}}) \inf_{z \in X_{n+1}} P(\tilde{z}, \tilde{A}_{n+1}) \\ &\geqq \textstyle\sum_{C_{k-n-1}} g(x_{i_1}) \cdots g(x_{i_{k-n-1}}) c^{n+1} \end{aligned}$$

and

$$\frac{P(\tilde{x}, X_n)}{P(\tilde{x}, X_n + \tilde{A}_{n+1})} \leqq \frac{P(\tilde{x}, X_n)}{P(\tilde{x}, \tilde{A}_{n+1})} \leqq \frac{\sum_{j=k-n}^{k} \sum_{C_j} g(x_{i_1}) \cdots g(x_{i_j})}{c^{n+1} \sum_{C_{k-n-1}} g(x_{i_1}) \cdots g(x_{i_{k-n-1}})}.$$

For each $\{x_{i_1}, \cdots, x_{i_{k-m}}\} \in C_{k-m}$, $0 \leqq m \leqq n$, there correspond $k-m$ elements of C_{k-m-1}, namely the $k-m$ subsets of $\{x_{i_1}, \cdots, x_{i_{k-m}}\}$ with $k-m-1$ elements. But each such subset could have come from $k-(k-m-1) = m+1$ elements of C_{k-m}. Hence

$$\begin{aligned} \frac{B_m}{B_{m+1}} &= \frac{\sum_{C_{k-m}} g(x_{i_1}) \cdots g(x_{i_{k-m}})}{\sum_{C_{k-m-1}} g(x_{j_1}) \cdots g(x_{i_{k-m-1}})} = \frac{\sum_{C_{k-m}} g(x_{i_1}) \cdots g(x_{i_{k-m}})}{(k-m)/(m+1) \sum_{C_{k-m}} g(x_{i_1}) \cdots g(x_{i_{k-m}})} \\ &= \frac{m+1}{k-m} \end{aligned}$$

for $0 \leqq m < k-1$.

Hence we have

$$(2.4)\quad \frac{P(\tilde{x}, X_n)}{P(\tilde{x}, X_n - \tilde{A}_{n-1})} \leqq \frac{1}{C^{n+1}} \frac{\sum_{j=0}^{n} B}{B_{n+1}} = \frac{1}{C^{n+1}} \sum_{j=0}^{n} \frac{B_j}{B_{j+1}} \frac{B_{j+1}}{B_{j+2}} \cdots \frac{B_n}{B_{n+1}}$$

$$\leqq \frac{1}{C^{n+1}} \sum_{j=0}^{n} \frac{j+1}{k-j} \frac{j+2}{k-j-1} \cdots \frac{n+1}{k-n} \to 0 \quad \text{as} \quad k \to \infty.$$

Now a set $\tilde{A} \subset \tilde{X}$ is compact if and only if $\tilde{A} \subset X_0 + \cdots + X_h$ for some $h \in I$. Hence (1.4) follows from (2.4). □

THEOREM 2.2. *Suppose X is σ-compact metric,* (2.1), (2.2) *hold and $F(s,\cdot) \in C'(X)^+$ for all $s \in C'_{00}(X)^+$, and suppose X can be imbedded in a compact metric space Y such that $\bar{X} = Y$ and for any $y \in Y - X$, $\lim_{x \to y, x \in X} F(s,x)$ exists for all $s \in C'_{00}(X)^+$ and $\sup_{s \in C'_{00}(X)^+} \lim_{x \to y, x \in X} F(s,x) = 1$. Then a regular invariant measure π exists on $X - X_0$ such that $\pi(X_n) < \infty$ for all $n \in I$ and $\pi(\tilde{A}) > 0$ for all open $\tilde{A} \subset \tilde{X} - X_0$.*

PROOF. It can be shown (Yang (1969)) that for all $y \in Y - X$, there exists $P(y,\cdot) \in M(\tilde{X})$ such that $P(y, \tilde{X}) = 1$ such that $\lim_{x \to y, x \in X} F(s,x) = F(s,y)$ for all $s \in \overline{C'(X)^+}$ where $F(\cdot, y)$ is the generating functional of $P(y,\cdot)$. By a theorem in topology (page 216 in Dugundje (1966)), $F(s,\cdot) \in C'(Y)^+$ for all $s \in C'(X)^+$. $P(y,\cdot)$ is extended to $\tilde{Y}$ by defining $P(y, \tilde{Y} - \tilde{X}) = 0$ for all $y \in Y$. Then for $s \in C'(Y)^+$, $F(s, y) = F(s\,|_x, y)$ for all $y \in Y$. $P(y,\cdot)$, $y \in Y$ defines a branching process with Y as set of types. The hypotheses of Theorem 2.1 are satisfied. Hence there exists a regular invariant measure π on $\tilde{Y} - Y_0$. Now $P(\tilde{y}, \tilde{Y} - \tilde{X}) = 0$ for all $\tilde{y} \in \tilde{Y}$ implies $\pi(\tilde{Y} - \tilde{X}) = \int_{\tilde{Y} + Y_0} P(\tilde{x}, \tilde{Y} - \tilde{X})\pi(d\tilde{x}) = 0$. Hence for all $\tilde{A} \subset \tilde{X} - X_0$, $\pi(\tilde{A}) = \int_{\tilde{X} + X_0} P(\tilde{x}, \tilde{A})\pi(d\tilde{x})$. □

EXAMPLE 2.1. *A one-dimensional neutron model* (page 63 of Harris (1963)) $X = [o, L]$. We take $x \in X$ to represent the position of a birth of two neutrons instead of the position of a neutron at birth. The process is the same as in Harris (1963) except that initially there are two neutrons instead of one. There is an $a > 0$ such that e^{-ax} is the probability of a neutron travelling a distance $\geqq x$ in the rod without collision. Then $P_x(\tilde{Z}_1 \in X_0) = (\frac{1}{2} e^{ax} + \frac{1}{2} e^{-a(L-x)})^2$. Hence $\inf_{x \in X} P(x, X_0) > 0$ and (2.2) holds. For any subsets A, B of X, $P_x(\tilde{Z}_1 \in A) = \frac{1}{2}(e^{-ax} + e^{-a(L-x)}) \int_A a\, e^{-a|y-x|}\, dy$, $P_x(\tilde{Z}_1 \in A \times B) \geqq \frac{1}{4} \int_A a\, e^{-a|y-x|}\, dy \int_B a\, e^{-a|y-x|}\, dy$ where $A \times B$ is assumed symmetrized that is, $A \times B \subset X_2$. From these, it can be shown that (2.2) holds. For $s \in B'(X)^+$,

$$F(s,x) = P(x, X_0) + \tfrac{1}{2}(e^{-ax} + e^{-a(L-x)}) \int_0^L a\, e^{-a|y-x|} s(y)\, dy$$
$$+ \tfrac{1}{2} \int_0^L a\, e^{-a|z-x|} s(z) \left(\int_0^z a\, e^{-a|y-x|} s(y)\, dy \right) dz.$$

Hence (2.3) holds and Theorem 2.2 applies.

EXAMPLE 2.2. *The discrete-time age dependent process.* Let $X = I_0$ with discrete topology and Y its one-point compactification. An object of age x has probability $p(x)$ of changing into one object of age $x+1$ and probability $p_n(x)$, $n \in I_0$ of changing into n objects of age 0, with $p(x)+\sum_{n=0}^{\infty} p_n(x) = 1$. We assume (a) $\sum_{n=2}^{\infty} p_n(x) > 0$ for each $x \in I_0$; (b) $\lim_{x\to\infty} p_n(x)$ exists for $n \in I_0$; and (c) $p_0(x) > 0$ for all $x \in I_0$, $\lim_{x\to\infty} p_0(x) > 0$ and $\lim_{x\to\infty} p(x) = 0$. If (a) is not satisfied, we then have a well-known example of a Markov chain which does not admit an invariant measure if $p_0(x) > 0$ for some $x \in I_0$. If $p(x) = 0$ for some $x \in I$, let $y = \min\{x : p(x) = 0\}$ and suppose $y \geqq 3$, then the process can be considered to be of finite type, that is $X = \{0, 1, 2, \cdots, y\}$ and an invariant measure exists on $\tilde{X} - X_0$ by Theorem 2.1. Otherwise $p(x) > 0$ for all $x \in X$. In this case, for $s \in B'(X)^+$, $F(s, x) = p(x)s(x+1)+\sum_{n=0}^{\infty} p_n(x)s(0)^n$. If $s \in C'_{00}(X)^+$, that is $s(x) = 0$ for all sufficiently large x, then $\lim_{x\to\infty} F(s,x) = \lim_{x\to\infty} \sum_{n=0}^{\infty} p_n(x)s(0)^n$.

Hence

$$\sup_{s \in C_{00}'(X)^+} \lim_{x\to\infty} F(s,x) = \lim_{\lambda\uparrow 1} \lim_{x\to\infty} \sum_{n=0}^{\infty} p_n(x)\lambda^n = 1.$$

Condition (2.2) can be shown to hold, and condition (2.1) follows from (c). Hence Theorem 2.2 applies.

We now consider the case when X is finite. We then have a denumerable Markov chain, and slightly stronger results can be obtained.

LEMMA 2.1. *If $\sum_{i=0}^{\infty} f_i = 1$, $f_i \geqq 0$, $I > f_0 > 0$, $f(s) = \sum_{i=0}^{\infty} f_i s^i$, $0 \leqq s \leqq 1$, then for any $i \in I_0$, there exist infinitely many $m > i$ such that the coefficient of s^m $f(s)^k$ is positive for all $k > m$, and*

$$\lim_{k\to\infty} \frac{\textit{coefficient of } s_i \textit{ in } f(s)^k}{\textit{coefficient of } s^m \textit{ in } f(s)^k} = 0.$$

PROOF. Consider the Galton–Watson process with f as generating function. Let P_{ij} be the transition matrix. Then $P_{ki} \leqq P_k$ (at most i objects do not die in the next generation) $= \sum_{l=0}^{i} \binom{k}{k-l} f_0^{k-l}$. Let j be the smallest integer such that $P_{ij} = f_j > 0$. If m is any multiple of j such that $m = nj$ and $n > i$, then $P_{km} \geqq \binom{k}{k-n} f_0^{k-n} f_j^n > 0$ for all $k > m$. Hence the coefficient of s^m in $f(s)^k > 0$ for all $k > m$. Also

$$\frac{P_{ki}}{P_{km}} \leqq \frac{\sum_{l=0}^{i} \binom{k}{k-l} f_0^{k-l}}{\binom{k}{k-n} f_0^{k-n} f_j^n} = \frac{\sum_{l=0}^{i} \binom{k}{k-l} f_0^{h-l}}{\binom{k}{k-n} f_j^n}.$$

Now for $0 \leqq l \leqq i < n$

$$\frac{\binom{k}{k-l}}{\binom{k}{k-n}} = \frac{n!}{l!} \frac{1}{(k-n-1)\cdots\cdots(k-l)} \to 0 \quad \text{as} \quad k\to\infty.$$

Hence

$$\frac{\text{coefficient of } s^i \text{ in } f(s)^k}{\text{coefficient of } s^m \text{ in } f(s)^k} \leqq \frac{1}{f_j^n} \sum_{l=0}^{i} \frac{f_0^{n-l}\binom{k}{k-l}}{\binom{k}{k-n}} \to 0 \quad \text{as} \quad k\to\infty.$$

THEOREM 2.3. *For a Galton–Watson process with immigration, that is $P_{ki} =$ coefficient of s^i in $g(s)f(s)^k$ where g and f are probability generating functions, if $0 < f(0) < 1$, an invariant measure exists on any infinite subset T of I_0 such that*

$\sum_{j\in I_0-T} P_{tj} = 0$ *or* $\sum_{j\in I-T} P_{tj} = 0$ *if* $g(0) = 1$ *for all* $t \in T$ *and the states of* T *communicate.*

REMARKS. Conditions under which invariant measures or distributions exist in the case $f'(1) \leqq 1$ were discussed in Heathcote (1965) and Seneta (1968). When $g(0) = 1$, we have the Galton–Watson process; the result was stated without proof in Harris (1963), while proofs with moment assumption were given in Harris (1963), Seneta (1969) and others. The problem of existence and uniqueness of invariant measures on a branching process allowing immigration has been treated by Z. Seneta (1969), (1970), using functional equation methods assuming $f'(1) < \infty$.

PROOF OF THEOREM 2.3. If the process is recurrent on T, then an invariant measure always exists. Hence we assume that the process is transient. Also in the case $g(0) = 1$, $0 \in I_0$ is an absorbing state, we then consider $T \subset I$ rather than $T \subset I_0$. Let

$$g(s) = \textstyle\sum_{i=0}^{\infty} g_i s^i, \quad g_i \geqq 0, \qquad \sum_{i=0}^{\infty} g_i = 1$$

and

$$f(s) = \textstyle\sum_{i=0}^{\infty} f_i s^i, \quad f_i \geqq 0, \qquad \sum_{i=0}^{\infty} f_i = 1 \quad \text{and} \quad 0 < f_0 < 1.$$

For any $i \in I_0$, $P_{ki} = \sum_{u+v=i} g_u f_v^{(k)}$, where $f_v^{(k)}$ is the coefficient of s^v in the expansion of $f(s)^k$. Given $\varepsilon > 0$, by Lemma 2.1, for each $v = 0, 1, \cdots, i$, there exists $m_v > i$ such that $i < m_0 < m_1 < \cdots < m_i$, $m_v \in I$, and $N \in I$ such that $f_{mv}^{(k)} > 0$ for all $k \geqq N$ and

$$f_v^{(k)} < \frac{\varepsilon}{i+1} f_{mv}^{(k)} \qquad v = 0, \cdots, i. \tag{2.5}$$

Let $j = m_i + i$, then for any $u, v \in I_0$ such that $u+v = i$, we have $m_0 \leqq u + m_v \leqq j$. Hence

$$g_u f_{mv}^{(k)} \leqq \textstyle\sum_{t=m_0}^{j} \sum_{u+l=t} g_u f_l^{(k)} = \sum_{t=m_0}^{j} P_{kt}, \tag{2.6}$$

$$\textstyle\sum_{u+v=i} g_u f_{mv}^{(k)} \leqq (i+1) \sum_{t=m_0}^{j} P_{kt}. \tag{2.7}$$

But $P_{ki} > 0$ implies $\sum_{u+v=i} g_u f_v^{(k)} > 0$, where $g_u f_v^{(k)} > 0$ for some $u+v = i$ implying $\sum_{t=m_0}^{j} P_{kt} > 0$ from (2.6). Also

$$\begin{aligned} \frac{P_{ki}}{\sum_{t=m_0}^{j} P_{kt}} = \frac{\sum_{u+v=i} g_u f_v^{(k)}}{\sum_{t=m_0}^{j} P_{kt}} &< \frac{\varepsilon}{i+1} \frac{\sum_{u+v=i} g_u f_{mv}^{(k)}}{\sum_{t=m_0}^{j} P_{kt}} \quad \text{from (2.5)} \\ &\leqq \frac{\varepsilon}{i+1} \frac{(i+1)\sum_{t=m_0}^{j} P_{kt}}{\sum_{t=m_0}^{j} P_{kt}} \quad \text{from (2.7)} \\ &= \varepsilon. \end{aligned}$$

Hence by Corollary 1.1, an invariant measure exists on T. □

The proof of the following theorem is similar and is given in Yang (1969).

THEOREM 2.4. *For a branching process with finite number of types, if an object of each type has a positive probability of having no children in the next generation, then an invariant measure exists on any infinite subset* $\tilde{T}$ *of* $\tilde{X} - X_0$, *where* X *is the set of types, such that* $P(\tilde{x}, (\tilde{X} - X_0) - \tilde{T}) = 0$ *for any* $\tilde{x} \in \tilde{T}$ *and the states of* $\tilde{T}$ *communicate.*

REMARK. In the subcritical case, similar results can be obtained from results in Joffe and Spitzer (1967).

REFERENCES

DUGUNDJI, J. (1966). *Topology*. Allyn and Bacon, New York.

HARRIS, T. E. (1963). *The Theory of Branching Processes*. Springer-Verlag, Berlin.

HARRIS, T. E. (1957). Transient Markov chains with stationary measures. *Proc. Amer. Math. Soc.* **8** 937–942.

HEATHCOTE, C. R. (1965). A branching process allowing immigration. *J. Royal Stat. Soc. B* **27** 138–143.

IKEDA, N., NAGASAWA, M., and WATANABE, S. (1968). Branching Markov process I. *J. Math. Kyoto Univ.* **8**

JOFFE, A. and SPITZER, F. (1967). On multitype branching processes with $\rho < 1$. *J. Math. Anal. Appl.* 409–430.

KARLIN, S. and MCGREGOR, J. (1967). Properties of the stationary measure on the critical case simple branching process. *Ann. Math. Statist.* **38** 977–991.

KESTEN, H., NEY, P., and SPITZER, F. (1966). The Galton–Watson process with mean one and finite variance. *Theor. Probability Appl.* **11** 579–611.

PROHOROV, Y. (1956). Convergence of random processes and limit theorems in probability theory. *Theor. Probability Appl.* **1** 157–214.

SENETA, E. (1969). Functional equations and the Galton-Watson process. *Adv. Appl. Prob.* **1** 1–42.

SENETA, E. (1968). The stationary distribution of a branching process allowing immigration: A note in the critical case. *J. Roy. Statist. Soc. B* **30** 176–179.

SENETA, E. (1970). On invariant measures for simple branching processes summary. *Bull. Austral. Math. Soc.* **2** 359–362.

SPITZER, F. (1967). Two explicit Martin boundary constructions. *Symposium on Probability Methods in Analysis*, Lecture Notes in Mathematics No. 31. Springer–Verlag, Berlin.

YANG, Y. S. (1969). Stationary measures on branching processes. Ph.D. thesis, Univ. Southern Calif.