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A WEAK CONVERGENCE THEOREM FOR ORDER STATISTICS FROM STRONG-MIXING PROCESSES[1]

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This paper provides sufficient conditions for the weak convergence in the Skorohod space $D^d[a, b]$ of the processes $\{(Y_{1,[nt]}-b_n)/a_n, (Y_{2,[nt]}-b_n)/a_n, \cdots, (Y_{d,[nt]}-b_n)/a_n\}$, $0 < a \leqq t \leqq b$, where $Y_{i,n}$ is the ith largest among $\{X_1, X_2, \cdots, X_n\}$, a_n and b_n are normalizing constants, and $\langle X_n : n \geqq 1\rangle$ is a stationary strong-mixing sequence of random variables. Under the conditions given, the weak limits of these processes coincide with those obtained when $\langle X_n : n \geqq 1\rangle$ is a sequence of independent identically distributed random variables.

1. Introduction. Let $\langle X_n : n \geqq 1\rangle$ be a stationary strong-mixing sequence of random variables with common distribution function $F(x) = P\{X_n \leqq x\}$ and define the order statistics $Y_{i,n}$ by

$$\begin{aligned} Y_{i,n} &= i\text{th largest among } (X_1, X_2, \cdots, X_n) && i \leqq n; \\ &= Y_{n,n} && i > n. \end{aligned}$$

For constants $a_n > 0$ and b_n, set $\mathbf{z}_{n,d}(t) = \{y_{1,n}(t), y_{2,n}(t), \cdots, y_{d,n}(t)\}$ with

$$\begin{aligned} y_{i,n}(t) &= (Y_{i,1} - b_n)/a_n && 0 \leqq t \leqq 1/n; \\ &= (Y_{i,[nt]} - b_n)/a_n && t > 1/n. \end{aligned}$$

The processes $\mathbf{z}_{n,d}(t)$ with $0 \leqq a \leqq t \leqq b < \infty$ will be regarded as random elements of the product of d copies of $D[a, b]$, the space of all real-valued functions on $[a, b]$ that are right continuous and have left limits.

The possible limit laws of $\mathbf{z}_{n,d}(1)$ were described in Welsch (1970). In this paper we present sufficient conditions for the weak convergence of the processes $\mathbf{z}_{n,d}$. Loynes (1965) gave similar sufficient conditions for the convergence of $\mathbf{z}_{n,1}(1)$ but did not consider the joint distributions of $\mathbf{z}_{n,d}(1)$ for $d > 1$ or the weak convergence. Lamperti (1964) has given a complete solution to the weak convergence problem for $\mathbf{z}_{n,d}$ when $\langle X_n : n \geqq 1\rangle$ is composed of independent random variables.

2. Sufficient conditions for convergence. A stationary sequence is *strong-mixing* if

$$|P(AB) - P(A)P(B)| \leqq \alpha(k)$$

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whenever $A \in \mathscr{B}(X_1, X_2, \cdots, X_m)$ and $B \in \mathscr{B}(X_{m+k+1}, X_{m+k+2}, \cdots)$ for some m, where $\alpha(k) \downarrow 0$ as $k \to \infty$; here $\mathscr{B}(\cdots)$ denotes the σ-field generated by the random variables indicated.

To simplify the discussion we shall only consider the maximum and second maximum and let $M_n = Y_{1,n}$ and $S_n = Y_{2,n}$. The same techniques apply to higher dimensions but the results become more cumbersome to state. As was shown in Welsch (1970) the limit laws for M_n and S_n involve a distribution function $G(x)$ which Gnedenko (1943) proved has only three possible forms (except for scale and location parameters):

$$
\begin{aligned}
G_1(x) &= 0 && x \leqq 0 \\
&= \exp[-(x^{-\alpha})] && x > 0, \alpha > 0 \\
G_2(x) &= \exp[-(-x)^{\alpha}] && x < 0, \alpha > 0 \\
&= 1 && x \geqq 0 \\
G_3(x) &= \exp(-e^{-x}) && -\infty < x < \infty.
\end{aligned} \tag{2.1}
$$

THEOREM 1. *Let $\langle X_n : n \geqq 1 \rangle$ be a stationary strong-mixing sequence and assume that a sequence $\langle a_n > 0, b_n : n \geqq 1 \rangle$ exists so that*

$$F^n(a_n x + b_n) \to G(x). \tag{2.2}$$

Then $P\{M_n \leqq a_n x + b_n, S_n \leqq a_n y + b_n\}$ converges to the limiting distribution

$$
\begin{aligned}
H(x, y) &= G(y)\{1 + \log[G(x)/G(y)]\} && y < x \\
&= G(x) && y \geqq x
\end{aligned} \tag{2.3}
$$

provided that

$$\lim_{n\to\infty} k_n \sum_{j=1}^{p_n-1} (p_n - j) P\{X_1 > a_n x + b_n, X_{j+1} > a_n x + b_n\} = 0 \tag{2.4}$$

for x such that $0 < G(x) < 1$ and any system of integer-valued functions k_n and p_n that satisfy

$$k_n \to \infty, p_n \to \infty \tag{2.5a}$$

$$n/k_n p_n \to 1 \tag{2.5b}$$

$$k_n{}^2 \alpha([(n - k_n p_n)/k_n]) \to 0. \tag{2.5c}$$

The following two lemmas will be needed in the proof.

LEMMA 1 (Ibragimov). *Given a nonnegative monotone decreasing function of the positive integers, $\alpha(k)$, such that $\alpha(k) \to 0$ there exists a system of functions satisfying* (2.5).

PROOF. A minor modification of the proof of Theorem 1.3 of Ibragimov (1962). If we let $q_n = [(n - k_n p_n)/k_n]$ then $n - k_n(p_n + q_n) \geqq 0$.

LEMMA 2. *Condition* (2.4) *implies that*

$$\lim_{n\to\infty} k_n P\{M_{p_n} \geqq a_n x + b_n\} = -\log G(x). \tag{2.6}$$

PROOF. (cf. Loynes (1965)). The Bonferroni inequalities state that

$$S_1 - S_2 \leqq P\{M_{p_n} \geqq a_n X + b_n\} \leqq S_1$$

where $S_1 = \sum_{i=1}^{p_n} P\{X_i > a_n x + b_n\}$ and

$$S_2 = \sum_{1 \leqq i < j \leqq p_n} P\{X_i > a_n x + b_n, X_j \geqq a_n x + b_n\}.$$

Now $k_n S_2 \to 0$ from (2.4) and $k_n S_1 = k_n p_n P\{X_1 > a_n x + b_n\} \to -\log G(x)$ because of (2.2) and (2.5b).

PROOF OF THEOREM 1. We assume first that $x \geqq y$ and $0 < G(y), G(x) < 1$. Let

$$\tilde{M}_n = \max\{X_1, \cdots, X_{p_n}; X_{p_n+q_n+1}, \cdots, X_{2p_n+q_n}; \cdots; X_{(k_n-1)(p_n+q_n)+1}, \cdots, X_{k_n p_n+(k_n-1)q_n}\},$$

$$M_{n,i} = \max\{X_{(i-1)(p_n+q_n)+1}, \cdots, X_{ip_n+(i-1)q_n}\}, 1 \leqq i \leqq k_n$$

and define $\tilde{S}_n$ and $S_{n,i}$ similarly using the second maximum. Then

$$P\{\tilde{M}_n \leqq a_n x + b_n, \tilde{S}_n \leqq a_n y + b_n\} - P\{M_n \leqq a_n x + b_n, S_n \leqq a_n y + b_n \leqq (n - k_n p_n) P\{X_1 > a_n y + b_n\} \to 0 \tag{2.7}$$

because of (2.5b) and (2.2).

Furthermore

$$\begin{aligned} P\{\tilde{M}_n \leqq a_n x + b_n, \tilde{S}_n \leqq a_n y + b_n\} &= P\{M_n \leqq a_n y + b_n\} \\ &+ \textstyle\sum_{j=1}^{k_n} P\{a_n y + b_n < M_{n,j} \leqq a_n x + b_n, S_{n,j} \leqq a_n y + b_n; \\ &\qquad M_{n,i} \leqq a_n y + b_n, i = 1, \cdots, k_n, i \neq j\}. \end{aligned} \tag{2.8}$$

Applying the strong-mixing property repeatedly to (2.8) and using (2.5c) gives

$$\begin{aligned} |P\{\tilde{M}_n \leqq a_n x + b_n, \tilde{S}_n \leqq a_n y + b_n\} &- P^{k_n}\{M_{p_n} \leqq a_n y + b_n\} \\ - k_n P\{a_n y + b_n < M_{p_n} \leqq a_n x + b_n, S_{p_n} &\leqq a_n y + b_n\} P^{k_n-1}\{M_{p_n} \leqq a_n y + b_n\}| \\ &\leqq (k_n+1)(k_n-1)\alpha(q_n) \to 0. \end{aligned} \tag{2.9}$$

Now

$$P^{k_n}\{M_{p_n} \leqq a_n y + b_n\} = \left[1 - \frac{k_n P\{M_{p_n} > a_n y + b_n\}}{k_n}\right]^{k_n}$$

and therefore from Lemma 2

$$P^{k_n}\{M_{p_n} \leqq a_n y + b_n\} \to G(y). \tag{2.10}$$

We may rewrite $k_n P\{a_n y + b_n < M_{p_n} \leqq a_n x + b_n, S_{p_n} \leqq a_n y + b_n\}$ as

$$(2.11)\quad k_n P\{M_{p_n} > a_n y + b_n\} - k_n P\{M_{p_n} > a_n x + b_n\} - k_n P\{a_n y + b_n < M_{p_n} \leqq a_n x + b_n, S_{p_n} > a_n y + b_n\}.$$

Again using Lemma 2, the first two terms of (2.11) converge to $\log [G(x)/G(y)]$. Finally

$$(2.12)\quad \begin{aligned} & k_n P\{a_n y + b_n < M_{p_n} \leqq a_n x + b_n, S_{p_n} > a_n y + b_n\} \\ & \leqq k_n P\{S_{p_n} > a_n y + b_n\} \\ & \leqq k_n \textstyle\sum_{1 \leqq i < j \leqq p_n} P\{X_i > a_n y + b_n, X_j > a_n y + b_n\} \to 0 \end{aligned}$$

from condition (2.4). It now follows that $\lim_{n\to\infty} P\{\tilde{M}_n \leqq a_n x + b_n, \tilde{S}_n \leqq a_n y + b_n\} = H(x, y)$.

If $y > x$ the conclusion of the theorem follows immediately from (2.10). The remaining cases are treated by noticing that $G(\cdot)$ is continuous.

3. Weak convergence. When $G(x)$ is of type II or III we cannot allow $t = 0$ (i.e. $a = 0$) since this will lead to improper random variables. All of our weak convergence results will be stated for $a > 0$. Only minor modifications of the proofs are necessary to consider $a = 0$ when the limit law is of type I. Let $m_n(t) = y_{1,n}(t)$ and $s_n(t) = y_{2,n}(t)$.

THEOREM 2. *Under the same conditions as stated in Theorem* 1, $\langle m_n(t), s_n(t)\rangle$ *converges weakly in* $D^2[a, b]$ *to a random element* $\langle m(t), s(t)\rangle$ *characterized by*

$$(3.1)\quad \begin{aligned} & P\{m(t_1) \leqq x_1, s(t_1) \leqq y_1; m(t_2) \leqq x_2, s(t_2) \leqq y_2\} \\ & = G^{t_1}(y_1)\{1 + t_1 \log [G(x_1)/G(y_1)]\} \\ & \quad \cdot G^{t_2 - t_1}(y_2)\{1 + (t_2 - t_1) \log [G(x_2)/G(y_2)]\} \\ & \qquad \text{when } 0 < t_1 \leqq t_2, y_1 \leqq x_1 \leqq y_2 \leqq x_2 \text{ and} \\ & = G^{t_1}(y_1) G^{t_2 - t_1}(y_2)\{1 + t_1 \log [G(x_1)/G(y_1)]\} \\ & \quad + G^{t_1}(y_1)\{1 + t_1 \log [G(y_2)/G(y_1)]\} \\ & \quad \cdot G^{t_2 - t_1}(y_2)\{(t_2 - t_1) \log [G(x_2)/G(y_2)]\} \\ & \qquad \text{when } y_1 \leqq y_2 \leqq x_1 \leqq x_2. \end{aligned}$$

The higher dimensional laws have a similar form.

PROOF. Iglehart (1968), Theorem 5, has shown that it is only necessary to verify that the finite dimensional laws of $\langle m_n, s_n\rangle$ converge and that each of the marginal processes $m_n(t)$ and $s_n(t)$ is tight in $D[a, b]$. We begin by using Theorem 1 to show that the one-dimensional distribution functions converge. For convenience we will assume that the limit law $G(x)$ is $G_1(x)$. The proof for $G_2(x)$ and $G_3(x)$ is not essentially different. A theorem of Khintchine [Gnedenko and Kolmogorov (1968) Theorem 2, page 42] and (2.2) imply that for $0 \leqq s_1 < s_2$,

$$a_n/a_{[ns_2]-[ns_1]} \to (s_2 - s_1)^{-1/\alpha} \quad \text{and} \tag{3.2}$$
$$(b_n - b_{[ns_2]-[ns_1]})/a_{[ns_2]-[ns_1]} \to 0.$$

Now

$$\begin{aligned} P\{m_n(t) \leqq x, s_n(t) \leqq y\} \\ = P\{M_{[nt]} \leqq a_{[nt]}[(a_n x + b_n - b_{[nt]})/a_{[nt]}] + b_{[nt]}, \\ S_{[nt]} \leqq a_{[nt]}[(a_n y + b_n - b_{[nt]})/a_{[nt]}] + b_{[nt]}\} \end{aligned}$$

and it follows from (3.2) and a standard argument [Gnedenko and Kolmogorov (1968) page 41] that if $(t^{-1/\alpha}x, t^{-1/\alpha}y)$ is a point of continuity for the limit law, H, then by Theorem 1

$$\begin{aligned} \lim_{n\to\infty} P\{m_n(t) \leqq x, s_n(t) \leqq y\} &= H(t^{-1/\alpha}x, t^{-1/\alpha}y) \\ &= G^t(y)\{1 + t\log[G(x)/G(y)]\}. \end{aligned} \tag{3.3}$$

The last equation follows from the fact that $G_1(t^{-1/\alpha}x) = G_1{}^t(x)$.

For the two-dimensional case we let

$$\begin{aligned} r_n &= [([nt_2] - [nt_1])^{\frac{1}{2}}] \\ M_{n,1} &= \max\{X_1, \cdots, X_{[nt_1]}\} \\ M_{n,2} &= \max\{X_{[nt_1]+1}, \cdots, X_{[nt_2]}\} \\ \tilde{M}_{n,2} &= \max\{X_{[nt_1]+r_n+1}, \cdots, X_{[nt_2]}\} \end{aligned}$$

with $S_{n,1}$, $S_{n,2}$, and $\tilde{S}_{n,2}$ defined similarly. It is not hard to show that $P\{m_n(t_1) \leqq x_1, s_n(t_1) \leqq y_1, m_n(t_2) \leqq x_2, s_n(t_2) \leqq y_2\}$

$$= P\{M_{n,1} \leqq a_n x_1 + b_n, S_{n,1} \leqq a_n y_1 + b_n, M_{n,2} \leqq a_n x_2 + b_n, S_{n,2} \leqq a_n y_2 + b_n\} \tag{3.4a}$$

when $y_1 \leqq x_1 \leqq y_2 \leqq x_2$ and

$$\begin{aligned} &= P\{M_{n,1} \leqq a_n x_1 + b_n, S_{n,1} \leqq a_n y_1 + b_n, M_{n,2} \leqq a_n y_2 + b_n\} \\ &\quad + P\{M_{n,1} \leqq a_n y_2 + b_n, S_{n,1} \leqq a_n y_1 + b_n, a_n y_2 + b_n \leqq M_{n,2} \leqq a_n x_2 + b_n, \\ &\quad S_{n,2} \leqq a_n y_2 + b_n\} \end{aligned} \tag{3.4b}$$

when $y_1 \leqq y_2 \leqq x_1 \leqq x_2$. Considering (3.4a) first, we have

$$\begin{aligned} &P\{M_{n,1} \leqq a_n x_1 + b_n, S_{n,1} \leqq a_n y_1 + b_n, \tilde{M}_{n,2} \leqq a_n x_2 + b_n, \tilde{S}_{n,2} \leqq a_n y_2 + b_n\} \\ &\quad - P\{M_{n,1} \leqq a_n x_1 + b_n, S_{n,1} \leqq a_n y_1 + b_n, M_{n,2} \leqq a_n x_2 + b_n, S_{n,2} \leqq a_n y_2 + b_n\} \\ &\leqq r_n P\{X_1 > a_n y_2 + b_n\} \to 0 \end{aligned} \tag{3.5}$$

since $r_n/n \to 0$, and the strong-mixing property implies that

$$|P\{M_{n,1} \leqq a_n x_1 + b_n, S_{n,1} \leqq a_n y_1 + b_n; \tilde{M}_{n,2} \leqq a_n x_2 + b_n, \tilde{S}_{n,2} \leqq a_n y_2 + b_n$$
$$- P\{M_{n,1} \leqq a_n x_1 + b_n, S_{n,1} \leqq a_n y_1 + b_n\}$$
$$P\{\tilde{M}_{n,2} \leqq a_n x_2 + b_n, \tilde{S}_{n,2} \leqq a_n y_2 + b_n\}| \leqq \alpha(r_n) \to 0.$$

An argument like that used in (3.5) allows us to remove the tildes in $P\{\tilde{M}_{n,2} \leqq a_n x_2 + b_n, \tilde{S}_{n,2} \leqq a_n y_2 + b_n\}$ and it is only necessary to prove that

$$\lim_{n\to\infty} P\{M_{n,2} \leqq a_n x_2 + b_n, S_{n,2} \leqq a_n y_2 + b_n\}$$
$$= G^{t_2 - t_1}(y_2)\{1 + (t_2 - t_1)\log[G(x_2)/G(y_2)]\},$$

which follows from (3.3). This method can be used to prove (3.4b) and for the convergence of any finite-dimensional distribution.

It remains to demonstrate that the marginal measures are tight. Let $y_n(t) = m_n(t)$ or $s_n(t)$ and use $y(t)$ to denote the limit process. Following Billingsley (1968), we define functionals on $D[a, b]$ for each $\delta > 0$ by letting

$$w_x[a, a+\delta) = \sup\{|x(t_1) - x(t_2)|; a \leqq t_1, t_2 < a+\delta\}$$
$$w_x[b-\delta, b) = \sup\{|x(t_1) - x(t_2)|; b-\delta \leqq t_1, t_2 < b\}, \quad \text{and}$$
$$w_x''(\delta, a, b) = \sup\{\min(|x(t) - x(t_1)|, |x(t_2) - x(t)|);$$
$$a \leqq t_1 \leqq t \leqq t_2 \leqq b, t_2 - t_1 \leqq \delta\}.$$

According to Theorem 15.3 of Billingsley (1968) the family $\{y_n\}$ is tight if and only if for each positive ε and η, there exist a $\beta > 0$, a δ with $0 < \delta < b-a$, and an integer n_0 such that

(3.6a) $$P\{\sup_{a\leqq t\leqq b} |y_n(t)| > \beta\} \leqq \eta \qquad n \geqq 1$$

(3.6b) $$P\{w''_{y_n}(\delta, a, b) > \varepsilon\} \leqq \eta \qquad n \geqq n_0$$

(3.6c) $$P\{w_{y_n}[a, a+\delta) > \varepsilon\} \leqq \eta \qquad n \geqq n_0$$

(3.6d) $$P\{w_{y_n}[b-\delta, b) > \varepsilon\} \leqq \eta \qquad n \geqq n_0.$$

Since $y_n(t)$ is monotone increasing in t

$$P\{\sup_{a\leqq t\leqq b} |y_n(t)| > \beta\} = P\{\max[|y_n(a)|, |y_n(b)|] > \beta\}$$
$$\leqq P\{|y_n(a)| > \beta\} + P\{|y_n(b)| > \beta\}$$
$$\to P\{|y(a)| > \beta\} + P\{|y(b)| > \beta\}$$

and therefore β can be chosen to satisfy (3.6a).

Now assume ε and η have been specified. Choose γ so that $G(\gamma) > 0$ and $P\{y(a) \leqq \gamma\} < \eta/2$. Then

$$P\{w_{y_n}[a, a+\delta) > \varepsilon\} \leqq P\{y_n(a+\delta) - y_n(a) > \varepsilon\}$$
$$\leqq P\{y_n(a) \leqq \gamma\} + P\{\max(X_{[na]+1}, \cdots, X_{[n(a+\delta)]}) > a_n\gamma + b_n\}$$

and it follows that

$$\limsup_{n\to\infty} P\{w_{y_n}[a, a+\delta) > \varepsilon\} \leqq \eta/2+1-G^{\delta}(\gamma)$$

which can be made less than η for sufficiently small δ. Condition (3.6c) is verified in a similar way.

With γ chosen as above

$$(3.7)\qquad P\{w''_{y_n}(\delta, a, b) > \varepsilon\} \leqq P\{y_n(a) < \gamma\} + P\{w''_{y_n}(\delta, a, b) > \varepsilon, y_n(a) > \gamma\}.$$

It is clear that in evaluating the functional $w''_{y_n}(\delta, a, b)$ the points t_1, t, and t_2 each lie in intervals of the form $[a+i\delta, a+(i+1)\delta]$. If $t_2 - t_1 < \delta$ then these intervals either coincide or abut. Therefore

$$(3.8)\quad P\{w''_{y_n}(\delta, a, b) > \varepsilon, y_n(a) > \gamma\}$$
$$\leqq \sum_{i=0}^{[(b-a)/\delta]-1} P\{w''_{y_n}(\delta, a+i\delta, a+(i+2)\delta) > \varepsilon, y_n(a) > \gamma\}.$$

If $y_n(a) > \gamma$ there must be at least two random variables from among $\{X_{[nu]+1}, \cdots, X_{[n(u+2\delta)]}\}$ which exceed $a_n\gamma + b_n$ in order to have $w''_{y_n}(\delta, u, u+2\delta) > \varepsilon$. Formally this implies that

$$(3.9)\quad P\{w''_{y_n}(\delta, u, u+2\delta) > \varepsilon, y_n(a) > \gamma\}$$
$$\leqq P\{\text{second max } \{(X_{[nu]+1}, \cdots, X_{[n(u+2\delta)]})\} > a_n\gamma + b_n\}$$

and combining (3.7), (3.8), and (3.9) gives

$$\limsup_{n\to\infty} P\{w''_{y_n}(\delta, a, b) > \varepsilon\} \leqq \eta/2 + \{1 - G^{2\delta}(\gamma)[1-2\delta\log G(\gamma)]\}(b-a)/\delta.$$

It is easy to show that

$$\lim_{\delta\to 0} \frac{1-G^{2\delta}(\gamma)[1-2\delta\log G(\gamma)]}{\delta} = 0$$

which completes the proof of Theorem 2.

Recent results due to Whitt (1970), Corollary 4.2, page 20, allow the use of this same proof for the space $D[a, \infty)$.

4. Gaussian processes. If $\langle X_n : n \geqq 1\rangle$ is also a Gaussian process, then (2.4) can be translated into a condition on the covariance sequence.

THEOREM 3. *Let $\langle X_n : n \geqq 1\rangle$ be a Gaussian stationary strong-mixing sequence with $E(X_n) = 0$, $E(X_n{}^2) = 1$ and covariance sequence $\langle r_n : n \geqq 1\rangle$* where *$r_n = E(X_1 X_{n+1})$. If (2.2) holds and*

$$(4.1)\qquad r_n \log n = O(1)$$

then the results of Theorems 1 and 2 are valid.

PROOF. We remark that if

$$a_n = (2\log n)^{-\frac{1}{2}}$$
$$b_n = (2\log n)^{\frac{1}{2}} - \tfrac{1}{2}(2\log n)^{-\frac{1}{2}}(\log\log n + \log 4\pi)$$

then $F^n(a_n x + b_n) \to G_3(x)$ where $F(\cdot)$ is the normal law with mean zero and unit variance.

Since (4.1) implies that $r_n \to 0$, there exists a δ such that $\sup_n |r_n| = \delta < 1$. If $\delta(n) = \sup_{k \geqq n} |r_k|$ then (4.1) becomes

$$\delta(n)\log n = O(1). \tag{4.2}$$

We shall now verify condition (2.4). Define: $c_n = a_n x + b_n$, $T_n(r_j) = P\{X_1 > a_n x + b_n,\ X_{j+1} > a_n x + b_n\}$; then

$$T_n'(r_j) = (2\pi)^{-1}(1-r_j^2)^{-\frac{1}{2}} \exp[-c_n^2/(1+r_j)].$$

The mean-value theorem states that

$$T_n(r_j) - T_n(0) = r_j T_n'(\tilde{r}_j)$$

where $\tilde{r}_j$ is between zero and r_j.

For n sufficiently large, $T_n'(\cdot)$ is an increasing function of its argument and therefore

$$|T_n(r_j) - T_n(0)| \leqq |r_j| T_n'(|r_j|).$$

Now

$$k_n \textstyle\sum_{j=1}^{p_n-1} (p_n - j) T_n(r_j) \leqq k_n p_n^2 T_n(0) + k_n p_n \sum_{j=1}^{p_n} |r_j| T_n'(|r_j|) \tag{4.3}$$

and

$$k_n p_n^2 T_n(0) = \frac{k_n p_n}{n}\left(\frac{p_n}{n}\right) n^2 P^2\{X_1 > a_n x + b_n\} \to 0$$

because $p_n/n \to 0$ and $nP\{X_1 > a_n x + b_n\}$ is bounded. Since $k_n p_n/n \to 1$ the last term in (4.3) will converge to zero if

$$\lim_{n\to\infty} n \textstyle\sum_{j=1}^{p_n} |r_j| n^{-2/(1+|r_j|)} (\log n)^{1/(1+|r_j|)} = 0. \tag{4.4}$$

If α is a real number satisfying $0 < \alpha < (1-\delta)/(1+\delta)$ then for n large

$$n \textstyle\sum_{j=1}^{[p_n^\alpha]} |r_j| n^{-2/(1+|r_j|)} (\log n)^{1/(1+|r_j|)} \leqq (p_n/n)^\alpha \delta\, n^{1+\alpha-2/(1+\delta)} \log n$$

which tends to zero because of the choice of α and the fact that $p_n/n \to 0$.

The remaining part of the sum in (4.4) is dominated by

$$(p_n/n)\{\delta([p_n^\alpha])\log n\} \exp(2\delta([p_n^\alpha])\log n). \tag{4.5}$$

Let $t_n = n/(k_n p_n)$. Then

$$\delta([p_n^\alpha])\log n \leqq \delta \log t_n + \delta \log k_n + \delta([p_n^\alpha])\log p_n$$

and (4.5) is smaller than

$$(p_n k_n^{\delta}/n)t_n^{\delta}[\delta \log t_n + \delta \log k_n + \delta([p_n^{\alpha}]) \log p_n] \cdot \exp\{\delta([p_n^{\alpha}]) \log p_n\}$$

which tends to zero because of condition (4.1) and the fact that $t_n \to 1$ implies that $(p_n k_n^{\delta} \log k_n)/n \to 0$. This completes the verification of condition (2.4).

This proof is based on one given by Berman (1964) for the convergence of $z_{n,1}(1)$ when $\langle X_n : n \geqq 1 \rangle$ is a Gaussian sequence and

$$r_n \log n \to 0. \tag{4.6}$$

We are able to weaken (4.6) because of the strong-mixing assumption.

5. Concluding remarks. A sequence is M-dependent if, in the definition of strong-mixing, $\alpha(k) = 0$ for $k \geqq$ M. When $\langle X_n : n \geqq 1 \rangle$ is M-dependent condition (2.4) follows immediately if (cf. Watson (1954))

$$\lim_{n\to\infty} nP\{X_1 > a_n x + b_n, X_j > a_n x + b_n\} = 0 \tag{5.1}$$

for $j > 1$ and x such that $G(x) > 0$.

Newell (1964) has constructed a 1-dependent process that fails to satisfy (2.4). Let $\langle Z_n : n \geqq 1 \rangle$ be a sequence of independent identically distributed random variables and set $X_n = \max(Z_n, Z_{n+1})$. Then (2.4) becomes

$$\lim_n [k_n(p_n - 1)P\{X_1 > a_n x + b_n, X_2 > a_n x + b_n\} + k_n P^2\{X_1 > a_n x + b_n\} \sum_{j=2}^{p_n - 1} (p_n - j)] = 0.$$

The last term is dominated by $(k_n p_n^2/n^2)n^2 P^2\{X_1 > a_n x + b_n\}$ and tends to zero. When a_n, b_n satisfy (2.2) it is easy to show that

$$\lim_n k_n p_n P\{X_1 > a_n x + b_n, X_2 > a_n x + b_n\} = \tfrac{1}{2}.$$

Some results on the weak convergence of such processes are contained in Welsch (1969).

The limit laws that occur in the statement of Theorem 2 were called extremal processes by Dwass (1964), (1966) and they coincide with the limit laws obtained if $\langle X_n : n \geqq 1 \rangle$ is an independent sequence. This leads to a kind of "invariance theorem" with respect to dependence. Using the results of Theorem 2 and Theorem 5.5 of Billingsley (1968) it is possible to compute the limiting distributions of functionals of $\langle m_n(t), s_n(t) \rangle$ by considering a sequence $\langle \hat{X}_n : n \geqq 1 \rangle$ of independent, identically distributed random variables. The independence generally makes the distributions of the functionals easier to compute, and the limiting values apply to the original strong-mixing process if the conditions of Theorem 1 are satisfied.

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