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ON BRANCHING PROCESSES WITH RANDOM ENVIRONMENTS: I EXTINCTION PROBABILITIES[1]

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0. Introduction. Smith and Wilkinson (1969) (see also Smith (1968)) have formulated an intriguing model of a *branching process with random environment* (abbreviated as B.P.R.E.). The structure of the model can be delineated as follows. Let ζ_t be a discrete time ($t = 0, 1, 2, \cdots$) stochastic process of "environmental variables" taking values in some probability space Θ. We suppose associated with each $\zeta \in \Theta$ is a probability generating function (p.g.f.)

$$\varphi_\zeta(s) = \textstyle\sum_{j=0}^{\infty} p_j(\zeta) s^j, \qquad 0 \leqq s \leqq 1.$$

For each realization of the process $\zeta_t : \bar{\zeta} = (\zeta_0, \zeta_1, \zeta_2, \cdots)$ and the associated random sequence of p.g.f.'s, there evolves a population Z_n, $n = 0, 1, 2, \cdots$ governed by the laws of the standard temporally non-homogeneous branching process. Specifically, suppose Z_0 comprise the initial population number of the 0th generation. These individuals (alternatively called particles or objects) create progeny so that the population size at the first generation is

$$Z_1 = \textstyle\sum_{i=1}^{Z_0} X_{1i}$$

where $X_{1i} (i = 1, 2, \cdots, Z_0)$ are independent random variables with p.g.f. $\varphi_{\zeta_0}(s)$. The second generation population number Z_2 is composed from the progeny of the Z_1 individuals each producing independently according to the p.g.f. $\varphi_{\zeta_1}(s)$. Proceeding in this way the $n+1$th generation population number Z_{n+1} is determined as the cumulative progeny of the Z_n particles of the nth generation each creating independently according to the p.g.f. $\varphi_{\zeta_n}(s)$. We shall call the process generated in this way $\{Z_n(\bar{\zeta}),\ n = 0, 1, 2, \cdots\}$ the branching process *conditioned* on the environment $\bar{\zeta}$. The population process Z_n, $n \geqq 0$ without specification of $\bar{\zeta}$ in advance is referred to as the branching process with random environment (B.P.R.E.).

Smith and Wilkinson limited themselves to the case where $\varphi_{\zeta_n}(s)$ is a sequence of independent and identically distributed random variables (i.i.d.). In that special case the process Z_n, $n \geqq 0$ is Markovian. These authors devoted their efforts exclusively to ascertaining conditions for certain or noncertain extinction of the B.P.R.E. model.

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This paper and its sequel (1971) refine and extend their work in two directions. In addition to the extinction problem studied in a more general framework, we also set forth several limit laws on the variables Z_n paralleling the classic limit theorems of simple branching processes. We shall deal with more general situations of the environmental process, like e.g. when ζ_t is stationary and metrically transitive or ζ_t is a Markov chain. We will also treat some cases of a multi-type B.P.R.E.

The developments of this paper divide into two parts. Sections 2 and 3 delimit complete criteria for certainty or noncertainty of extinction. Our methods are simpler and the conclusions sharper even for the independence case investigated by Smith and Wilkinson. We discuss in Section 4 facets of the extinction problem in the multi-type B.P.R.E. model. A summary of the results of this paper appeared in July (1970).

1. Preliminaries and statement of main results. Let $(\Omega, \mathbb{F}, P)$ be a given probability space. Let $\mathscr{M}$ designate the collection of all probability distributions $\{\bar{p} = \{p_i\}_{i=0}^{\infty}, p_i \geqq 0, \sum p_i = 1\}$ on the nonnegative integers satisfying the further constraints

$$(1) \qquad \sum_{i=0}^{\infty} i p_i < \infty, \qquad 0 \leqq p_0 + p_1 < 1.$$

Clearly $\mathscr{M}$ is a Borel subset of the Banach space l_∞ of all bounded sequences of real numbers. Let $\zeta_i(\omega)$, $i = 0, 1, 2, \cdots$ be a sequence of random mappings from $(\Omega, \mathbb{F}, P)$ into $(l_\infty, \mathscr{B}_\infty)$ where $\mathscr{B}_\infty$ is the Borel σ-algebra in l_∞ generated by the product topology. We assume

$$(2) \qquad P\{\omega : \zeta_i(\omega) \in \mathscr{M} \quad \text{for all} \quad i\} = 1.$$

For any $\zeta \in \mathscr{M}$ associate the p.g.f.

$$(3) \qquad \varphi_\zeta(s) = \sum_{i=0}^{\infty} p_i(\zeta) s^i \qquad |s| \leqq 1.$$

We can regard $\bar{\zeta} = (\zeta_0(\omega), \zeta_1(\omega), \zeta_2(\omega), \cdots)$ as a realization of the "environmental process."

Let $Z_0(\omega)$ be a nonnegative integer valued random variable defined on $(\Omega, \mathbb{F}, P)$. We now generate the branching process $Z_n(\omega)$; $n = 0, 1, 2, \cdots$ as described earlier starting with Z_0 by means of $\{\varphi_{\zeta_1}(s)\}$.

It is convenient to introduce a series of σ-algebras. (For any collection D of random elements on $(\Omega, \mathbb{F}, P)$ let $\sigma(D)$ denote the sub σ-algebra of $\mathbb{F}$ generated by D.) We set

$$(4) \qquad \mathbb{F}(\bar{\zeta}) \equiv \sigma(\zeta_0, \zeta_1, \cdots, \zeta_n, \zeta_{n+1}, \cdots)$$
$$\mathbb{F}_n(\bar{\zeta}) = \sigma(Z_0, Z_1, \cdots, Z_n; \zeta_0, \zeta_1, \zeta_2, \cdots).$$

From the description of the model (or alternatively take as an axiom for the definition of B.P.R.E.) we postulate that Z_n satisfy the relations

$$(5) \qquad E(s^{Z_{n+1}} \mid \mathbb{F}_n(\bar{\zeta})) = [\varphi_{\zeta_n}(s)]^{Z_n} \qquad \text{a.s.}$$

and for any set of integers $1 \leqq n_1 < n_2 < \cdots < n_k$

$$(6) \quad E({s_1}^{Z_{n_1}} {s_2}^{Z_{n_2}} \cdots {s_k}^{Z_{n_k}} \mid \mathbb{F}(\bar{\zeta}); Z_0 = m) = [E({s_1}^{Z_{n_1}} {s_2}^{Z_{n_2}} \cdots s_{n_k}^{Z_{n_k}} \mid \mathbb{F}(\bar{\zeta}); Z_0 = 1)]^m$$

for $|s_i| \leqq 1$, $i = 1, 2, \cdots, k$.

The proof of existence of a process satisfying these axioms is routine. We refer to Harris (1963) for details of such constructions.

An immediate consequence of (5) and (6) is

LEMMA 1.

$$E(s^{Z_{n+1}} \mid Z_0 = k, \mathbb{F}(\zeta)) = [\varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots(\varphi_{\zeta_n}(s))\cdots)]^k. \tag{7}$$

A direct implication of Lemma 1 is that when conditioning on $(\zeta_0, \zeta_1, \zeta_2, \cdots)$ the process behaves like a non-temporally homogeneous branching process and therefore the lines of descent are independent subject to this conditioning. Smith and Wilkinson noted that unconditionally the lines of descent are not independent. Nevertheless, contrary to the claims of Smith and Wilkinson much of the classical theory of Galton–Watson branching processes (especially the familiar limit theorems) carries over to the B.P.R.E.

The first objective of this paper is to ascertain complete criteria guaranteeing certain extinction of the process. It is useful to underscore the fact that in the Smith–Wilkinson case (where $\zeta_i(\omega)$ are *independent identically distributed* (i.i.d.)) Z_n is Markovian and then a standard result of Markov chain theory tells us that

$$P\{\omega: Z_n \to 0 \quad \text{or} \quad Z_n \to \infty\} = 1. \tag{8}$$

In contrast, when $\{\zeta_i(\omega)\}$ constitutes a stationary ergodic process then it is not at all automatic that these same exclusive alternatives prevail w.p. 1. Nevertheless, the conclusion of (8) is valid (see Theorem 7 of Section 3).

Henceforth, unless stated explicitly otherwise we assume that $\zeta_i(\omega), i = 0, 1, 2, \cdots$ is a *stationary ergodic process* (Loève (1962)). In this circumstance the conditions for certain extinction are the same as in the independence case. Let

$$B = \{\omega: Z_n(\omega) = 0 \quad \text{for some} \quad n\} \tag{9}$$

$$q_k = P(B \mid Z_0 = k) \quad q_k(\zeta) = P(B \mid Z_0 = k, \mathbb{F}(\zeta)).$$

We refer to B as the set of *extinction* and q_k, $q_k(\zeta)$ *extinction probabilities*. It is clear from (7) that

$$q_k(\zeta) = [q_1(\zeta)]^k \qquad \text{a.s.} \tag{10}$$

and

$$q_k = E([q_1(\zeta)]^k). \tag{11}$$

We see instantly on the basis of (11) that $\{q_k : k = 1, 2, \cdots\}$ is a *moment sequence*, a fact that seems to have excited Smith and Wilkinson who derived this point by more complicated means.

Since the sequence of events $B_n = \{\omega : Z_n(\omega) = 0\}$ increase to B we infer (see (7))

$$q(\zeta) = \lim_{n\to\infty} \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots(\varphi_{\zeta_n}(0)\cdots) = E(\chi_B \mid \mathbb{F}(\zeta); Z_0 = 1) \tag{12}$$

where χ_B denotes the indicator function of the set B. (Note that we write $q(\bar{\zeta})$ for $q_1(\bar{\zeta})$.) An immediate consequence of (12) is the important functional relationship

$$q(\bar{\zeta}) = \varphi_{\zeta_0}(q(T\bar{\zeta})) \tag{13}$$

where T denotes the *shift transformations*

$$T\bar{\zeta} = T(\zeta_0, \zeta_1, \zeta_2, \cdots) = (\zeta_1, \zeta_2, \zeta_3, \cdots). \tag{14}$$

Recalling the stipulations of (1) and (2) we may conclude from (13)

PROPOSITION 1. *The sets* $\{\omega : q(\bar{\zeta}) = 1\}$ *and* $\{\omega : q(T\bar{\zeta}) = 1\}$ *coincide modulo a set of probability zero.*

Thus $\{\bar{\zeta}; q(\bar{\zeta}) = 1\}$ is a T-invariant set of B and since T is ergodic by hypothesis, we infer that

$$P(q(\bar{\zeta}) = 1) = 0 \quad \text{or} \quad 1. \tag{15}$$

We now exhibit a necessary condition for noncertain extinction i.e., for $P(q(\bar{\zeta}) < 1) = 1$. For any real number a, we employ the symbols $a^+ = \max(0, a)$ and $a^- = -\min(a, 0)$.

THEOREM 1. *Suppose*

$$P(q(\bar{\zeta}) < 1) = 1 \quad \textit{and} \quad E(\log \varphi'_{\zeta_0}(1))^+ < \infty$$

(*prime designates as usual the derivative*). *Then*

$$E|\log \varphi'_{\zeta_0}(1)| < \infty, \qquad E \log \varphi'_{\zeta_0}(1) > 0 \tag{16}$$

and

$$E\left|\log \frac{1-q(\bar{\zeta})}{1-q(T\bar{\zeta})}\right| < \infty, \qquad E \log\left(\frac{1-q(\bar{\zeta})}{1-q(T\bar{\zeta})}\right) = 0. \tag{17}$$

We can extract quite easily (see Section 2) from the assertion (17) in the special case where ζ_i comprise a family of i.i.d. random variables (or when these variables generate an irreducible finite Markov chain) an integrability property.

THEOREM 2. *Suppose* $P(q(\bar{\zeta}) < 1) = 1$, $E(\log \varphi'_{\zeta_0}(1))^+ < \infty$ *and* $\varphi_{\zeta_i}(s)$, $i = 0, 1, 2, \cdots$ *are* i.i.d. *random variables* (*or* $\zeta_i(\omega)$, $i \geqq 0$ *form an irreducible finite Markov chain*). *Then*

$$E(-\log(1-\varphi_{\zeta_0}(0))) < \infty.$$

The result of Theorem 2 in the i.i.d. case was achieved first by W. L. Smith (1968).

The converse proposition to Theorem 1 is as follows.

THEOREM 3. *Suppose* $E(-\log(1-\varphi_{\zeta_0}(0))) < \infty$ *and* $E(\log \varphi'_{\zeta_0}(1))^- < E(\log \varphi'_{\zeta_0}(1))^+ \leqq \infty$. *Then*

$$P(q(\zeta) < 1) = 1.$$

Theorems 1 and 3 manifestly provide a full set of criteria for certain extinction except when $E(\log \varphi'_{\zeta_0}(1))^+ = E(\log \varphi'_{\zeta_0}(1))^- = \infty$. In summary, modulo mild integrability conditions extinction is certain iff $E(\log \varphi'_{\zeta_0}(1)) \leqq 0$.

The conclusions of Theorems 1 and 3 persist also in the circumstance where ζ_i unfolds as a positive recurrent irreducible Markov chain. The precise result is as follows.

THEOREM 4. *Suppose the process* ζ_i, $i \geqq 0$ *is an irreducible positive recurrent Markov chain* $\mathscr{P}$ *with countable state space* $\{1, 2, \cdots\}$. *Associate with each state* j *a* p.g.f. $\varphi_j(s)$ *satisfying the conditions of* (2). *Let* $\{\pi_i\}_{i=0}^{\infty}$ *be the unique stationary measure of* $\mathscr{P}$. *Assume that*

$$\textstyle\sum_{j=1}^{\infty} \pi_j|\log(1-\varphi_j(0))| < \infty, \ \sum_{j=1}^{\infty} \pi_j(\log(\varphi_j'(1)))^+ < \infty.$$

Consider the corresponding B.P.R.E. *Then for any initial distribution of* ζ_0,

$$P(q(\zeta) < 1) = 0 \quad or \quad 1,$$

and extinction is certain iff $\sum_{j=1}^{\infty} \pi_j \log \varphi_j'(1) \leqq 0$.

Theorem 4 can be proved allowing a general state space.

The proof of Theorem 4 falls back on the results of Theorems 1–3 in the stationary case.

The above results constitute refinement of results given in Smith and Wilkinson (1969) and Smith (1968) who used analytic techniques of renewal processes in conjunction with methods of fluctuation theory for sums of independent random variables.

Fluctuation and renewal theory arguments are virtually impossible to apply in the stationary case. Our approach exploits decisively several variants of the ergodic theorem Loève (1962) and the detailed arguments for these more general cases seem to be simpler than in the special cases. The proofs of Theorems 1–4 occur in Section 2.

As mentioned earlier it is not evident in the stationary case that

$$P\{Z_n \to 0 \quad \text{or} \quad \infty\} = 1. \tag{18}$$

This assertion is correct and its proof depends on the following interesting theorem first discovered by Church (1967).

THEOREM 5. *Given a sequence of* p.g.f.'*s* $f_0(s), f_1(s), f_2(s), f_3(s), \cdots$ *we form the* p.g.f.'*s* $f_n(s) = f_0(f_1(\cdots f_n(s)) \cdots)$.

(i) *Then* $\lim_{n\to\infty} f_n(s) = g(s)$, $0 \leqq s < 1$ *always exists and* $g(s)$ *is strictly increasing* iff

$$\lim_{m\to\infty} \max_{|s|\leqq 1, k\geqq 0} |f_m(f_{m+1}(\cdots f_{m+k}(s)\cdots)) - s| = 0. \tag{19}$$

(ii) *The conclusion* (19) *holds* iff

$$\lim_{N\to\infty} \lim_{n\to\infty} \prod_{i=N}^{n} f_i'(0) = 1 \quad \textit{or equivalently} \quad \sum_{i=0}^{\infty} [1 - f_i'(0)] < \infty.$$

In Section 3 we elaborate the proof of Theorem 5. Our arguments inspired by those of Church substantially simplify his analysis.

With the aid of Theorem 5 we deduce

THEOREM 6.

$$\lim_{n\to\infty} \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots(\varphi_{\zeta_n}(s))\cdots)) = q(\bar{\zeta}) \qquad \textit{for } 0 \leqq s < 1 \text{ a.s.,} \tag{20}$$

and the only solution $\tilde{q}(\bar{\zeta})$ (*when one exists*) *of the functional equation*

$$\tilde{q}(\bar{\zeta}) = \varphi_{\zeta_0}(\tilde{q}(T\bar{\zeta})) \tag{21}$$

satisfying

$$P\{\tilde{q}(\bar{\zeta}) < 1\} = 1 \tag{22}$$

is $\tilde{q}(\bar{\zeta}) = q(\bar{\zeta})$ a.s.

That the assertion (20) implies (18) is a corollary to the following.

THEOREM 7. *Let* $\pi_n(s, \bar{\zeta}) = \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{n-1}}(s))\cdots)$. *Suppose* $E(\log \varphi'_{\zeta_0}(1))^+ < \infty$. *Then, for* $0 \leqq s < 1$,

$$\sum_{n=1}^{\infty} [\pi_n(s, \bar{\zeta}) - \pi_n(0, \bar{\zeta})] < \infty, \qquad \text{w.p. } 1 \tag{23}$$

whenever

(i) $E(\log \varphi'_{\zeta_0}(1)) < 0$, *or*

(ii) $E(\log \varphi'_{\zeta_0}(1)) > 0$, *and* $E(-\log(1 - \varphi_{\zeta_0}(0))) < \infty$.

The final section investigates the extinction problem for the multi-type (say p-types) B.P.R.E. model. We outline the criteria in the case where ζ_i are i.i.d. To review quickly the formulation we have associated with each ζ a p-vector p.g.f. $\boldsymbol{\varphi}_\zeta(\mathbf{s}) = (\varphi_\zeta^{(1)}(\mathbf{s}), \varphi_\zeta^{(2)}(\mathbf{s}), \cdots, \varphi_\zeta^{(p)}(\mathbf{s}))$. Let $A_\zeta = \|\partial \varphi_\zeta^{(i)}(1)/\partial s_j\|$ be the mean matrix of the p.g.f. $\boldsymbol{\varphi}_\zeta(\mathbf{s})$. We assume $E|\log\|A_\zeta\|| < \infty$ where the norm $\|\ \|$ is defined as $\|A\| = \max_i \sum_{j=1}^{p} |a_{ij}|$ and A_ζ is a strictly positive matrix w.p.1. It was proved by Furstenburg and Kesten (1960) that

$$\lim_{n\to\infty} n^{-1} \log ||A_{\zeta_n} A_{\zeta_{n-1}} \cdots A_{\zeta_0}|| = \pi \qquad \text{exists w.p. } 1, \pi < \infty \tag{24}$$

and also $\lim_{n\to\infty} n^{-1} E \log ||A_{\zeta_n} A_{\zeta_{n-1}} \cdots A_{\zeta_0}|| = \pi$. It is easy to see that for the one type process $\pi = E \log \varphi_\zeta'(1)$.

Criteria for extinction or nonextinction is the substance of the following Theorem. (The notation $\mathbf{x} \ll \mathbf{y}$ signifies that every component of $\mathbf{y} - \mathbf{x}$ is positive.

The inner product is denoted by $<, >$ and $\mathbf{1}$ denotes the vector with each component equal to one.)

THEOREM 8. *Assume there exist constants C and D such that* w.p.1

$$0 < C \leqq \frac{\partial \varphi_{\zeta_0}^{(i)}(\mathbf{1})}{\partial s_j} \leqq D < \infty \tag{25}$$

$$0 < \frac{\partial^2 \varphi_{\zeta_0}^{(i)}(\mathbf{1})}{\partial s_j \partial s_k} \leqq D < \infty$$

and

$$E|\log \langle \mathbf{1} - \boldsymbol{\varphi}_{\zeta_0}(\mathbf{0}), \mathbf{1}\rangle| < \infty. \tag{26}$$

Then we have

(i) $\pi < 0$ *implies* $P(\mathbf{q}(\bar{\zeta}) = \mathbf{1}) = 1$

(ii) $\pi > 0$ *implies* $P(\mathbf{q}(\bar{\zeta}) \ll \mathbf{1}) = 1$

where

$$\mathbf{q}(\bar{\zeta}) = \lim_{n > \infty} \boldsymbol{\varphi}_{\zeta_0}(\boldsymbol{\varphi}_{\zeta_1}(\cdots \boldsymbol{\varphi}_{\zeta_n}(0) \cdots)).$$

2. Extinction probabilities. In this section we shall prove Theorems 1–4 on extinction probability. For ease of reference, we restate the theorems.

THEOREM 1. *Assume $E(\log \varphi_{\zeta_0}'(1))^+ < \infty$ and $P\{q(\bar{\zeta}) < 1\} = 1$. Then the following hold*:

(i) $E\left|\log\left(\dfrac{1-q(\bar{\zeta})}{1-q(T\bar{\zeta})}\right)\right| < \infty$ *and* $E\log\left(\dfrac{1-q(\bar{\zeta})}{1-q(T\bar{\zeta})}\right) = 0.$

(ii) $E|\log \varphi_{\zeta_0}'(1)| < \infty$ *and* $E \log \varphi_{\zeta_0}'(1) > 0.$

PROOF. From the basic functional equation (13) satisfied by $q(\bar{\zeta})$ and the hypothesis $P\{q(\bar{\zeta}) < 1\} = 1$ we get

$$h(\bar{\zeta}) = f(\bar{\zeta}) + h(T\bar{\zeta}) \tag{27}$$

where $0 \leqq h(\bar{\zeta}) = -\log(1-q(\bar{\zeta}))$ (finite w.p.1) and

$$f(\bar{\zeta}) = -\log\left(\frac{1-\varphi_{\zeta_0}(q(T\bar{\zeta}))}{1-q(T\bar{\zeta})}\right) = -\log\left(\frac{1-q(\bar{\zeta})}{1-q(T\bar{\zeta})}\right).$$

Iterating the above yields

$$h(\bar{\zeta}) = f(\bar{\zeta}) + f(T\bar{\zeta}) + \cdots + f(T^n\bar{\zeta}) + h(T^{n+1}\bar{\zeta}) \tag{28}$$

and since $h(\cdot)$ is nonnegative it follows that

$$\textstyle\sum_{i=0}^n f(T^i\bar{\zeta}) \leqq h(\bar{\zeta}).$$

Breaking $f(T_i\bar{\zeta})$ into positive and negative parts we rewrite this inequality as

$$n^{-1}\textstyle\sum_{i=0}^n f^+(T^i\bar{\zeta}) - n^{-1}\sum_{i=0}^n f^-(T^i\bar{\zeta}) \leqq n^{-1}h(\bar{\zeta}).$$

However,

$$0 \leqq E(f^-(\bar\zeta)) = E(-f(\bar\zeta); f(\bar\zeta) \leqq 0)$$
$$= E\left(\log\left(\frac{1-\varphi_{\zeta_0}(q(T\bar\zeta))}{1-q(T\bar\zeta)}\right); \frac{1-\varphi_{\zeta_0}(q(T\bar\zeta))}{1-q(T\bar\zeta)} \geqq 1\right)^2$$
$$\leqq E(\log \varphi'_{\zeta_0}(1); \varphi'_{\zeta_0}(1) \geqq 1)$$

(using the fact that $(1-\varphi'_{\zeta_0}(x))/(1-x)$ is increasing in [0, 1])

$$= E(\log \varphi'_{\zeta_0}(1))^+ < \infty.$$

Therefore, applying the ergodic theorem to $f^-(\bar\zeta)$ implies

(29) $$0 \leqq \limsup n^{-1} \textstyle\sum_{i=0}^n f^+(T^i\bar\zeta) \leqq E(\log \varphi'_{\zeta_0}(1))^+ < \infty.$$

Because of nonnegativity of f^+, again by the ergodic theorem we deduce $Ef^+(\bar\zeta) < \infty$ and hence that $E|f(\bar\zeta)| < \infty$. To finish (i) we need to show that $Ef(\bar\zeta) = 0$. This also is an easy consequence of the ergodic theorem. In fact, note first that

(30) $$Ef(\bar\zeta) = \lim_{n\to\infty} n^{-1} \textstyle\sum_{i=0}^n f(T^i\bar\zeta) = \lim_{n\to\infty} [n^{-1}h(\bar\zeta) - n^{-1}h(T^{n+1}\bar\zeta)].$$

Since $\lim_{n\to\infty} n^{-1}h(\bar\zeta) = 0$ it follows that $\lim_{n\to\infty} n^{-1}h(T^{n+1}\bar\zeta)$ exists and by stationarity has the same distribution as $\lim n^{-1}h(\bar\zeta)$, thus proving $Ef(\bar\zeta) = 0$.

We next turn to (ii). By hypothesis, $E(\log \varphi'_{\zeta_0}(1))^+ < \infty$. We shall establish $E(\log \varphi'_{\zeta_0}(1))^- < E(\log \varphi'_{\zeta_0}(1))^+$. To this end, we have

$$E(\log \varphi'_{\zeta_0}(1))^- = E(-\log \varphi'_{\zeta_0}(1); \varphi'_{\zeta_0}(1) \leqq 1)$$
$$\leqq E\left(-\log\frac{1-\varphi_{\zeta_0}(q(T\bar\zeta))}{1-q(T\bar\zeta)}; \varphi'_{\zeta_0}(1) \leqq 1\right)$$

(again since $(1-\varphi_{\zeta_0}(x))/(1-x) \leqq \varphi'_{\zeta_0}(1)$ for x in [0,1])

$$\leqq E(f(\bar\zeta); f(\bar\zeta) \geqq 0)$$
$$= Ef^+(\bar\zeta) \leqq E(\log \varphi'_{\zeta_0}(1))^+.$$

If $E(\log \varphi'_{\zeta_0}(1))^- = E(\log \varphi'_{\zeta_0}(1))^+$ then $E \log \varphi'_{\zeta_0}(1) = 0$ and $Ef(\bar\zeta)$ being zero it follows that

$$E[\log \varphi'_{\zeta_0}(1) + f(\bar\zeta)] = 0$$

while $P\{\log \varphi'_{\zeta_0}(1) + f(\bar\zeta) \geqq 0\} = 1$ with $P\{\log \varphi'_{\zeta_0}(1) + f(\bar\zeta) > 0\} > 0$ except if $P\{\zeta_0; p_1(\zeta_0) = 1\} = 1$ which is precluded by the stipulations of (1) and (2). This completes the proof. □

Stating the conclusion of the above theorem in contrapositive form yields specifically

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[2] The notation $E(f; A)$ indicates as customary that the expectation is evaluated restricted to the subset A.

COROLLARY 1. *Assume* $E(\log \varphi'_{\zeta_0}(1))^+ < \infty$. *Then* $E(\log \varphi'_{\zeta_0}(1))^+ \leqq E(\log \varphi'_{\zeta_0}(1))^- \leqq \infty$ *implies* $P\{q(\bar{\zeta}) = 1\} = 1$.

Next we determine sufficient conditions for $P(q(\bar{\zeta}) < 1) = 1$ by proving

THEOREM 3. *Assume* $E[-\log(1-\varphi'_{\zeta_0}(0))] < \infty$ *and* $E(\log \varphi'_{\zeta_0}(1))^- < E(\log \varphi'_{\zeta_0}(1))^+ \leqq \infty$. *Then* $P\{q(\bar{\zeta}) < 1\} = 1$.

PROOF. First note that

$$\begin{aligned} E(\log \phi'_{\zeta_0}(1))^- &= E[-\log \varphi'_{\zeta_0}(\bar{\zeta});\ \varphi'_{\zeta_0}(1) \leqq 1] \\ &\leqq E[-\log(1-\varphi_{\zeta_0}(0));\ \varphi'_{\zeta_0}(1) \leqq 1] \\ &\leqq E[-\log(1-\varphi_{\zeta_0}(0))] < \infty. \end{aligned}$$

Now set

$$Y_n(\bar{\zeta}) = \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_n}(0)\cdots)). \tag{31}$$

Then, from the definition of the shift transformation T

$$Y_n(\bar{\zeta}) = \varphi_{\zeta_0}(Y_{n-1}(T\bar{\zeta})) \tag{32}$$

and since the $\varphi_{\zeta_i}(s)$ are nontrivial w.p.1 (because of conditions (2) and (3)) $P\{Y_n(\bar{\zeta}) < 1 \text{ for all } n\} = 1$. Thus, w.p.1, $0 \leqq -\log(1-Y_n(\bar{\zeta})) < \infty$ for all n. Defining $\mu_n = E(-\log(1-Y_n(\bar{\zeta}))$ we employ induction to infer that $\mu_n < \infty$ for all n. In fact, $\mu_0 < \infty$ by hypothesis. From (32)

$$-\log(1-Y_n(\bar{\zeta})) = -\log\left(\frac{1-\varphi_{\zeta_0}(Y_{n-1}(T\bar{\zeta}))}{1-Y_{n-1}(T\bar{\zeta})}\right) - \log(1-Y_{n-1}(T\bar{\zeta})). \tag{33}$$

Since $(1-\varphi_{\zeta_0}(x))/(1-x)$ is increasing for x in [0, 1] it follows that

$$\left[-\log\frac{(1-\varphi_{\zeta_0}(Y_{n-1}(T\bar{\zeta}))}{1-Y_{n-1}(T\bar{\zeta})}\right]^+ \leqq -\log(1-\varphi_{\zeta_0}(0)) \tag{34}$$

which is integrable by hypothesis. Thus

$$0 \leqq \mu_n \leqq E(-\log(1-\varphi_{\zeta_0}(0)) + \mu_{n-1}.$$

This completes the induction and in this analysis we noted the fact of the integrability of

$$-\log\left(\frac{1-Y_n(\bar{\zeta})}{1-Y_{n-1}(T\bar{\zeta})}\right).$$

Set

$$\theta_n = E\left(-\log\left(\frac{1-Y_n(\bar{\zeta})}{1-Y_{n-1}(T\bar{\zeta})}\right)\right); \tag{35}$$

we obtain the recurrence relation

$$\mu_n = \theta_n + \mu_{n-1} \tag{36}$$

and therefore

$$\mu_n = \textstyle\sum_{j=1}^n \theta_j + \mu_0. \tag{37}$$

Assume contrary to the assertion of the theorem that $P(q(\bar{\zeta}) < 1) < 1$ holds. Then by the zero-one law, (15), $P(q(\bar{\zeta}) < 1) = 0$ and so $Y_{n-1}(T\bar{\zeta})$ increases to 1 w.p.1. This fact implies by monotone convergence that $\mu_n \uparrow \infty$. On the other hand, since $Y_{n-1}(T\bar{\zeta}) \uparrow 1$ w.p.1, we deduce that $\theta_n \downarrow \theta = E(-\log \varphi'_{\zeta_0}(1))$ which is < 0 by hypothesis. (Note that in the last step the assumption $E(\log \varphi'_{\zeta_0}(1))^+ < \infty$ is unnecessary.) Thus $\sum_{j=1}^{n} \theta_j \to -\infty$ and a contradiction results from (37). □

We shall next extend these results to the case when the $\{\zeta_i\}$ form an irreducible positive recurrent Markov chain. The restriction to a countable state space is not crucial but makes the arguments more transparent and less encumbered by unimportant technical details. Thus the ζ_i take values in a countable set, say, the integers $\{1, 2, \cdots\}$. Let $P = \|p_{ij}\|$ be the associated transition probability matrix of the process and let $\{\pi_j\}$ denote the stationary probability distribution. Consider the induced Markov chain stationary process $\{\eta_i\}$, $i = 0, 1, 2, \cdots$ such that η_0 has the distribution $\{\pi_i\}$ and the transition probabilities are given by $\|p_{ij}\|$. It is clear that $\{\eta_i\}$ form a stationary ergodic sequence.

Our procedure will be to express the probability of extinction for the ζ-process in terms of that for the related η-process.

LEMMA 2. *Let* $e_i = P\{\lim_{n\to\infty} \varphi_{\zeta_0}(\cdots(\varphi_{\zeta_n}(0)) = 1 \mid \zeta_0 = i\}$. *Then* $e_i = 1$ *for all* i *or* $e_i = 0$ *for all* i *according to* $P(q(\bar{\eta}) = 1) = 1$ *or* 0.

PROOF. Since $\{\eta_i\}$ is stationary and forms a Markov chain with transition probabilities the same as $\{\zeta_i\}$ we get

$$P\{q(\bar{\eta}) = 1\} = \textstyle\sum_i P\{q(\bar{\eta}) = 1 \mid \eta_0 = i\} P\{\eta_0 = i\} = \sum_i e_i \pi_i. \tag{38}$$

But $0 < \pi_i < 1$ for all i. We know that $P(q(\bar{\eta}) = 1)$ can assume only the two values 0 or 1. The conclusion of the lemma is now immediate. □

A simple corollary of Lemma 2 in conjunction with the results of Theorems 1 and 3 is Theorem 4 of Section 1. We take up next Theorem 2 cited in Section 1.

THEOREM 2. *Let* $E(\log \varphi'_{\zeta_0}(1))^+ < \infty$, $P(q(\bar{\zeta}) < 1) = 1$ *and suppose* ζ_i *is a process of either of the following kinds.* $\{\zeta_i\}$ *consists of* i.i.d. *random variables or* $\{\zeta_i\}$ *fluctuate as a finite state irreducible Markov chain, then* $E(-\log(1-\varphi_{\zeta_0}(0)) < \infty$.

PROOF. Since the $\{\zeta_i\}$ form a Markov chain in either case we obtain

$$\begin{aligned} E\left|\log\left(\frac{1-q(\bar{\zeta})}{1-q(T\bar{\zeta})}\right)\right| &= E\left|\log\left(\frac{1-\varphi_{\zeta_0}(q(T\bar{\zeta}))}{1-q(T\bar{\zeta})}\right)\right| \\ &= E\left\{E\left[\left|\log\left(\frac{1-\varphi_{\zeta_0}(q(T\bar{\zeta}))}{1-q(T\bar{\zeta})}\right)\right|\zeta_0\right]\right\} \\ &= E\left\{E\left[\left|\log\left(\frac{1-\varphi_{\zeta_0}(X_{\zeta_0})}{1-X_{\zeta_0}}\right)\right|\right]\right\} \end{aligned}$$

where X_{ζ_0} is a random variable having the same distribution as that of $q(T\bar{\zeta})$ when conditioned on ζ_0,

$$\geqq E\left\{E\left[\left|\log\left(\frac{1-\varphi_{\zeta_0}(X_{\zeta_0})}{1-X_{\zeta_0}}\right)\right|;\ X_{\zeta_0}\leqq 1-\varepsilon\right]\right\} \quad \text{where } \varepsilon>0 \text{ is arbitrary}$$

$$\geqq E\{E[|\log(1-\varphi_{\zeta_0}(X_{\zeta_0}))|;\ X_{\zeta_0}\leqq 1-\varepsilon]\}-|\log\varepsilon|$$

$$\geqq E\{|\log(1-\varphi_{\zeta_0}(0)|P[X_{\zeta_0}\leqq 1-\varepsilon\,|\,\zeta_0]\}-|\log\varepsilon|$$

$$\geqq [\inf_{\zeta_0}P\{X_{\zeta_0}\leqq 1-\varepsilon\,|\,\zeta_0\}]E|\log(1-\varphi_{\zeta_0}(0)|-|\log\varepsilon|.$$

Both in the cases where ζ_i are i.i.d. and where $\{\zeta_i\}$ is the state variable of a finite state irreducible Markov chain, it follows owing to the fact $P\{q(\bar{\zeta})<1\mid\zeta_0\}=1$ for almost all ζ_0, that $\delta=\inf_{\zeta_0}P\{X_{\zeta_0}\leqq 1-\varepsilon\,|\,\zeta_0\}>0$ for ε sufficiently small. (When ζ_i consists of i.i.d. random variables then $P(X_{\zeta_0}\leqq 1-\varepsilon\mid\zeta_0)$ is independent of ζ_0. When $\{\zeta_i, i>0\}$ goes according to a finite Markov chain, then ζ_0 can only assume a finite number of values.) In these cases, we obtain

$$E|\log(1-\varphi_{\zeta_0}(0)|\leqq\delta^{-1}\left[E\left|\log\left(\frac{1-q(\bar{\zeta})}{1-q(T\bar{\zeta})}\right)\right|+\log|\varepsilon|\right]<\infty. \quad \square$$

3. Some results on composition of probability generating functions and applications. In this section we are concerned with the following problem. Let $\{f_i(s)\}$, $i=1,2,\cdots$ be a sequence of p.g.f.'s and set

$$f_{(k)}(s)=f_1(f_2(\cdots f_k(s))). \tag{39}$$

We seek answers to questions like: When does $\lim_{k\to\infty}f_{(k)}(s)=g(s)$ exist and if so when is $g(s)$ strictly increasing in the right open interval $[0,1)$. These questions were first raised by J. D. Church (1967), who settled them completely. (Unfortunately, Church's paper is virtually inaccessible.) We present here simpler proofs of these results and some applications to the B.P.R.E. model. The following theorem summarizes Church's pertinent discoveries.

THEOREM 5. *Let* $f_{(k)}$ *be as defined in* (39). *Assume that for every* $i, f_i(s)\not\equiv 1$. *Then*

(i) $\lim_{k\to\infty}f_{(k)}(s)=g(s)$ *exist for all* s *in* $[0,1]$.

(ii) *Either* $g(s)=g(0)$ *for all* s *in* $[0,1)$ *or* $g(s)$ *is strictly increasing in* $[0,1)$ *with the latter holding if and only if*

$$\textstyle\sum_{i=1}^{\infty}(1-f_i'(0))<\infty. \tag{40}$$

The proof of this theorem is carried out in a series of lemmas.

LEMMA 1′. *If* $\lim_{k\to\infty}f_{(k)}(s)=g(s)$ *exist in* $[0,1]$ *and* $g(s)$ *is strictly increasing in* $[0,1)$ *then* $\lim_{k\to\infty}f_k(s)=s$ *for all* s *in* $[0,1]$, *and the convergence is uniform in* s.

PROOF. Fix s in [0, 1). Let $\varepsilon > 0$ be arbitrary such that $s+\varepsilon < 1$. Suppose there exists a sequence $\{n_k\}$ of integers going to ∞ such that $\lim_{k\to\infty} f_{n_k}(s) > s+\varepsilon$. Then for large k, $f_{n_k}(s) > s+\varepsilon$ and thus $f_{(n_k)}(s) = f_{(n_k-1)}(f_{n_k}(s)) \geqq f_{(n_k-1)}(s+\varepsilon)$. Letting $k \to \infty$ we see that $g(s) \geqq g(s+\varepsilon)$ which is impossible since g is strictly increasing. Thus $\limsup_{k\to\infty} f_{n_k}(s) \leqq s$. A similar argument yields $\liminf_{k\to\infty} f_{n_k}(s) \geqq s$, yielding the conclusion

$$\lim_{k\to\infty} f_k(s) = s \qquad \text{for } s \text{ in } [0,1).$$

Since $f_k(1) = 1$ for all k, Lemma 1 is proved except for the fact of uniform convergence which is assured since the function $f(s) \equiv s$ is a p.g.f. □

LEMMA 2′. *Let* $\{f_r\}$ *and* $\{g_r\}$ *be two sequences of* p.g.f.'*s such that*

$$\lim_{r\to\infty} f_r(g_r(s)) = s \qquad \textit{for } s \textit{ in } [0,1].$$

Then, for each s in [0, 1]

$$\lim_{r\to\infty} f_r(s) = s = \lim_{r\to\infty} g_r(s). \tag{41}$$

PROOF. Let for each r, X_r and N_r be nonnegative integer valued random variables with p.g.f.'s $f_r(s)$ and $g_r(s)$ respectively. Then $f_r(g_r(s))$ is the p.g.f. of the random variable

$$\begin{aligned} Y_r &\equiv \textstyle\sum_{i=1}^{N_r} X_{r,i} && \text{if } N_r > 0 \\ &\equiv 0 && \text{if } N_r = 0 \end{aligned}$$

where $\{X_{r,i};\, i = 1, 2, \cdots\}$ is a sequence of independent random variables distributed as X_r and N_r is independent of $\{X_{r,i};\, i = 1, 2, \cdots\}$. Since $\lim_{r\to\infty} f_r(g_r(s)) = s$ for s in [0, 1] it follows that $Y_r \to 1$ in distribution. Now N_r and X_r are nonnegative integer valued and so $Y_r \to 1$ in distribution requires that both N_r and X_r tend to 1 in distribution. These facts clearly imply (41). □

The following lemma establishes the first part of Theorem 5.

LEMMA 3′. *Suppose there exists an* s_0 *in* (0, 1) *such that*

$$\beta \equiv \limsup_{k\to\infty} f_{(k)}(s_0) > \lim_{k\to\infty} f_{(k)}(0) \equiv \alpha.$$

$$\lim_{k\to\infty} f_{(k)}(s) = g(s) \textit{ exists for all } s \textit{ in } [0,1]$$

and $g(s)$ *is strictly increasing in* [0, 1].

PROOF. First note that the sequence $\{f_{(k)}(0)\}$ is nondecreasing and hence $\alpha = \lim_{k\to\infty} f_{(k)}(0)$ exists. Next by definition of β there exists a sequence $\{m_k\}$ of integers such that

$$\lim_{k\to\infty} f_{(m_k)}(s_0) = \beta.$$

This family $\{f_{(m_k)}(s)\}$ is *normal* (see Hille (1962) Chapter 15) in $|s| \leqq 1$ and so there exists a further subsequence $\{n_k\}$ of $\{m_k\}$ and a function $g(s)$ on $|s| < 1$ such that

$\lim_{k\to\infty} f_{(n_k)}(s) = g(s)$ exists uniformly on compact subsets of $|s| < 1$. Clearly, $g(s_0) = \beta$ and $g(0) = \alpha$. Since all the $f_{(n_k)}$ are p.g.f.'s, the coefficients of $g(s)$ are nonnegative. Thus $\beta > \alpha$ implies $g(s)$ is strictly increasing in [0, 1]. According to Lemma 1′ and noting the identity $f_{(n_{k+1})}(s) = f_{(n_k)}(f_{n_k+1}(f_{n_k+2}\cdots(f_{n_{k+1}}(s))\cdots)$ we deduce that

$$\lim_{k\to\infty} f_{n_k+1}(f_{n_k+2}\cdots f_{n_{k+1}-1}(f_{n_{k+1}}(s))\cdots) = s$$

for s in [0, 1]. For any q let $k(q)$ be such that $n_{k(q)} \leqq q < n_{k(q)+1}$. Clearly,

$$f_{(q)}(s) = f_{n_{k(q)}}(f_{n_{k(q)}+1}(f_{n_{k(q)}+2}\cdots(f_q(s)\cdots)).$$

By Lemma 2′

$$\lim_{q\to\infty} f_{n_{k(q)}+1}(f_{n_{k(q)}+2}\cdots(f_q(s)) = s \qquad \text{for all } s \text{ in } [0, 1].$$

Also $f_{(n_k)}(s) \to g(s)$ uniformly on compact subsets of $|s| < 1$. Thus

$$\lim_{q\to\infty} f_{(q)}(s) = g(s) \qquad \text{for all } s \text{ in } [0, 1).$$

On setting $g(1) = 1$ the conclusion of Lemma 3 follows. □

We now turn to the analysis for (ii) of Theorem 5, i.e., the task of ascertaining necessary and sufficient conditions guaranteeing $g(s)$ to be strictly increasing. This is the content of the next two lemmas.

LEMMA 4′. *If* $g'(s) = 0$ *for some* s *in* [0, 1) *then* $\sum_i (1 - f_i'(0)) = \infty$.

PROOF. For any s in [0, 1)

$$\begin{aligned} g'(s) &= \lim_{k\to\infty} f_{(k)}'(s) \\ &= \lim_{k\to\infty} \textstyle\prod_{j=1}^{k} f_j'(f_{j+1}(\cdots f_k'(s))\cdots) \\ &\geqq \lim_{k\to\infty} \textstyle\prod_{j=1}^{k} f_j'(0) = \prod_{j=1}^{\infty} f_j(0). \end{aligned}$$

Thus if $g'(s) = 0$ for some s in [0, 1) then $\prod_{j=1}^{\infty} f_j'(0) = 0$. The lemma is now immediate since $0 \leqq f_j'(0) \leqq 1$ for all j. □

LEMMA 5′. *If* $g(s)$ *is strictly increasing in* [0, 1) *then* $\sum_i (1 - f_i'(0)) < \infty$.

PROOF. Let $\varepsilon_j(s) = \lim_{k\to\infty} f_j(f_{j+1}(\cdots f_k(s))\cdots)$ for s in [0, 1) which exists by virtue of part (i) of the theorem. Then $g(s) = f_{(j)}(\varepsilon_j(s))$. Since $g(s)$ is strictly increasing arguing as in Lemma 1′ we conclude that $\varepsilon_j(s) \to s$ uniformly in [0, 1]. Thus,

$$1 = \lim_{j\to\infty} \varepsilon_j'(s) = \lim_{j\to\infty} \textstyle\prod_{k=j}^{\infty} f_k(\varepsilon_{k+1}(s))$$

and

$$0 = \lim_{j\to\infty} \varepsilon_j''(s) = \lim_{j\to\infty} (\varepsilon_j'(s)) \sum_{k=j}^{\infty} \frac{f_k''(\varepsilon_{k+1}(s))}{f_k'(\varepsilon_{k+1}(s))} \varepsilon_{k+1}'(s).$$

Evaluating these at $s = 0$ and noting that $\lim_{j\to\infty} f_j'(s) = 1$ uniformly in s in $[0, 1]$ we get

$$\lim_{j\to\infty} \prod_{k=j}^{\infty} f_k'(\varepsilon_{k+1}(0)) = 1$$

and

$$\lim_{j\to\infty} \sum_{k=j}^{\infty} f_k''(\varepsilon_{k+1}(0)) = 0.$$

In view of this last result

$$\limsup_{j\to\infty} \prod_{k=j}^{\infty} \frac{f_k'(\varepsilon_{k+1}(0))}{f_k'(0)} \leqq \limsup_{j\to\infty} \exp\left(\sum_{k=j}^{\infty}\left(\frac{f_k'(\varepsilon_{k+1}(0))}{f_k'(0)} - 1\right)\right)$$
$$\leqq \exp\left(\limsup_{j\to\infty} \sum_{k=j}^{\infty} \frac{f_k''(\varepsilon_{k+1}(0))}{f_k'(0)}\right) = 1.$$

It is now clear that

$$\lim_{j\to\infty} \prod_{k=j}^{\infty} f_k'(0) \geqq \left[\lim_{j\to\infty} \prod_{k=j}^{\infty} f_k'(\varepsilon_{k+1}(0)\right]\left[\limsup_{j\to\infty} \prod_{k=j}^{\infty} \frac{f_k'(\varepsilon_{k+1}(0))}{f_k'(0)}\right]^{-1}$$
$$\geqq [\lim_{j\to\infty} \varepsilon_j'(0)] = 1.$$

But $0 \leqq f_j'(0) \leqq 1$ for all j and so $\sum_{j=1}^{\infty} [1 - f_j'(0)] < \infty$ thus yielding the assertion of the lemma. □

As a corollary of Theorem 5 we get the following results concerning $\lim_{n\to\infty} \varphi_{\zeta_0}(\cdots(\varphi_{\zeta_n}(s))$.

THEOREM 6. *Let* $\{\zeta_n : n = 0, 1, 2, \cdots\}$ *be a stationary ergodic sequence in the sense of Section* 2. *Then we have* (i)

$$P\{\lim_{n\to\infty} \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_n}(s))\cdots) = q(\zeta) \ \textit{for all} \ 0 \leqq s < 1\} = 1 \tag{42}$$

where we recall that $q(\zeta) = \lim_{n\to\infty} \varphi_{\zeta_0}(\cdots(\varphi_{\zeta_n}(0))\cdots)$.

(ii) *The only solution* $\tilde{q}(\zeta)$ (*when one exists*) *of the functional equation*

$$\tilde{q}(\zeta) = \varphi_{\zeta_0}(\tilde{q}(T\zeta)), \tag{43}$$

satisfying $P(\tilde{q}(\zeta) < 1) = 1$ *is* $\tilde{q}(\zeta) = q(\zeta)$ a.s.

PROOF. (i) From stationarity, ergodicity and by virtue of the stipulations (2) and (3) we see that $E(1 - \varphi_{\zeta_i}'(0)) > 0$ and hence $P\{\zeta; \sum_{i=0}^{\infty}(1 - \varphi_{\zeta_i}'(0)) = \infty\} = 1$. Now appeal to Theorem 5 to get (42).

(ii) If $\tilde{q}(\zeta)$ is any solution of (43) then iterating this equation we get

$$\tilde{q}(\zeta) = \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_n}(\tilde{q}(T^n\zeta)\cdots)))$$
$$\geqq \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_n}(0)\cdots)) \qquad \text{for any } n$$

and letting $n \to \infty$ leads to relation

$$P\{\bar{\zeta};\ \tilde{q}(\bar{\zeta}) \geqq q(\bar{\zeta})\} = 1. \tag{44}$$

Now suppose $P(\bar{\zeta};\tilde{q}(\bar{\zeta}) < 1) = 1$. Then for any fixed s $(0 \leqq s < 1)$,

$$\begin{aligned} P\{\bar{\zeta};\ \tilde{q}(\bar{\zeta}) &\leqq \varphi_{\zeta_0}(\varphi_{\zeta_1} \cdots \varphi_{\zeta_n}(s)\cdots)\} \\ &\geqq P\{\bar{\zeta};\ \tilde{q}(T^n\bar{\zeta}) \leqq s\} = P(\bar{\zeta};\ \tilde{q}(\bar{\zeta}) \leqq s) \end{aligned}$$

the last equality due to stationarity. Letting $n \to \infty$ and using the previous theorem we get

$$P\{\bar{\zeta};\ \tilde{q}(\bar{\zeta}) \leqq q(\bar{\zeta})\} \geqq P\{\bar{\zeta};\ \tilde{q}(\bar{\zeta}) \leqq s\}$$

for any s satisfying $0 \leqq s < 1$. Now, let $s \uparrow 1$ to conclude $P\{\bar{\zeta};\tilde{q}(\bar{\zeta}) \leqq q(\bar{\zeta})\} = 1)$ which in conjunction with (44) implies the inference

$$P\{\bar{\zeta};\ \tilde{q}(\bar{\zeta}) = q(\bar{\zeta})\} = 1. \quad \square$$

We now turn to estimate the rate at which $\pi_n(s, \bar{\zeta}) - \pi_n(0, \bar{\zeta}) \to 0$ as $n \to \infty$ where

$$\pi_n(s, \bar{\zeta}) = \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{n-1}}(s)\cdots)). \tag{45}$$

THEOREM 7. *Let* $E(\log \varphi'_{\zeta_0}(1))^+ < \infty$. *Then* $\sum_{n=1}^{\infty}(\pi_n(s,\bar{\zeta}) - \pi_n(0,\bar{\zeta})) < \infty$ *for each* s $(0 \leqq s < 1)$ a.s. *whenever*

(i) $E(\log \varphi'_{\zeta_0}(1)) < 0$, *or*

(ii) $E(\log \varphi'_{\zeta_0}(1)) > 0$ *and* $E(-\log(1-\varphi_{\zeta_0}(0)) < \infty$.

We need the following known fact whose proof we present as we do not have an available reference. (We are grateful to N. Kaplan who suggested the relevance of this lemma.)

LEMMA 6′. *Let* $h_n(\bar{\zeta}) \to h(\bar{\zeta})$ a.s. *and suppose there exists* $g(\bar{\zeta})$ *integrable satisfying* $|h_n(\bar{\zeta})| \leqq g(\bar{\zeta})$ a.s. *for all* n. *Then*

$$\lim_{n\to\infty} n^{-1} \textstyle\sum_{j=0}^{n-1} h_j(T^{n-j}\bar{\zeta}) = Eh(\bar{\zeta}). \tag{46}$$

PROOF. Let $g_L(\bar{\zeta}) = \sup_{j>L} |h_j(\bar{\zeta}) - h(\bar{\zeta})|$. Then

$$0 \leqq g_L(\bar{\zeta}) \leqq 2g(\bar{\zeta}) \quad \text{and} \quad \lim_{L\to\infty} g_L(\bar{\zeta}) = 0, \qquad \text{a.s.} \tag{47}$$

By dominated convergence theorem it follows that $Eg_L(\bar{\zeta}) \to 0$ as $L \to \infty$. Now

$$\begin{aligned} &|n^{-1} \textstyle\sum_{j=0}^{n-1}(h_j(T^{n-j}\bar{\zeta}) - h(T^{n-j}\bar{\zeta})) \\ &\qquad \leqq n^{-1} \textstyle\sum_{j=0}^{L} |h_j(T^{n-j}\bar{\zeta}) - h(T^{n-j}\bar{\zeta})| + n^{-1} \textstyle\sum_{j=L+1}^{n} g_L(T^{n-j}\bar{\zeta}). \end{aligned}$$

On the basis of the Ergodic theorem we deduce that $\lim_{n\to\infty} n^{-1} \sum_{j=L+1}^{n} g_L(T^{n-j}\bar{\zeta}) = Eg_L(\bar{\zeta})$. Moreover, observe that

$$\lim_{n\to\infty} \frac{h_j(T^{n-j}\bar{\zeta})}{n} = 0 \qquad \text{a.s. for each } j.$$

Thus

$$\limsup_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} \left|\left(h_j(T^{n-j}\bar{\zeta}) - h(T^{n-j}\bar{\zeta})\right)\right| \leqq Eg_L(\bar{\zeta}) \qquad \text{a.s.}$$

Letting $L \to \infty$ we may conclude

$$\limsup_{n\to\infty} \left|n^{-1} \sum_{j=0}^{n-1} \left(h_j(T^{n-j}\bar{\zeta}) - h(T^{n-j}\bar{\zeta})\right)\right| = 0 \qquad \text{a.s.} \tag{48}$$

Again applying the Ergodic theorem we see that

$$\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} h(T^{n-j}\bar{\zeta}) = Eh(\bar{\zeta}) \qquad \text{a.s.}$$

The proof is now immediate from (48). ∎

Proof of Theorem 7. (i) Let $E(\log \varphi'_{\zeta_0}(1)) < 0$. Then by the mean value theorem

$$\pi_n(s, \bar{\zeta}) - \pi_n(0, \bar{\zeta}) \leqq \prod_{j=0}^{n-1} \varphi'_{\zeta_j}(1).$$

Since $\mu = E(\log \varphi'_{\zeta_0}(1)) < 0$, applying the Ergodic theorem yields the fact that $\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n} \log \varphi'_{\zeta_j}(1)$ exists and equals μ which is negative. Thus for almost every $\bar{\zeta}$, $\sum_{n=1}^{\infty} \left(\prod_{j=0}^{n-1} \varphi'_{\zeta_j}(1)\right)$ converges at a geometric rate.

(ii) Let $E(\log \varphi'_{\zeta_0}(1)) > 0$ and $E(-\log (1-\varphi'_{\zeta_0}(0)) < \infty$. Then according to Theorem 3 we know that $P(q(\bar{\zeta}) < 1) = 1$. By the mean value theorem, we have

$$\pi_n(s, \bar{\zeta}) - \pi_n(0, \bar{\zeta}) \leqq \prod_{j=0}^{n-1} \varphi'_{\zeta_j}(\varphi_{\zeta_{j+1}}(\cdots(\varphi_{\zeta_{n-1}}(s))\cdots)).$$

It suffices to show that

$$\limsup_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} f_{n-j}(T^j\bar{\zeta}, s) < 0 \tag{49}$$

where $f_n(\bar{\zeta}; s) = \log \varphi'_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{n-1}}(s)\cdots))$.
Decomposing f_n into its positive and negative parts, we note that

$$f_n^{+}(\bar{\zeta}) \leqq (\log \varphi'_{\zeta_0}(1))^{+}$$

and

$$\lim_{n\to\infty} f_n^{+}(\bar{\zeta}, s) \equiv f^{+}(\bar{\zeta}) \qquad \text{(by Theorem 6)}$$

where $f(\bar{\zeta}) = \log \varphi'_{\zeta_0}(q(T\bar{\zeta}))$.
Thus, by Lemma 6′, we get

$$\lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} f_j^{+}(T^{n-j}\bar{\zeta}) = Ef^{+}(\bar{\zeta}) \qquad \text{a.s.} \tag{50}$$

Next let us examine $n^{-1} \sum_{j=0}^{n-1} f_j^{-}(T^{n-j}\bar{\zeta})$. To estimate its limit we employ a truncation argument. Fix N and set $h_j^{(N)}(\bar{\zeta}) = \min (f_j^{-}(\bar{\zeta}), N)$, and $h^{(N)}(\bar{\zeta}) = \min (f^{-}(\bar{\zeta}), N)$. Noting $h_j^{(N)}(\bar{\zeta}) \to h^{(N)}(\bar{\zeta})$ and with the aid of Lemma 6′ and obvious inequalities we infer

$$\liminf_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} f_j^{-}(T^{n-j}\bar{\zeta}) \geqq \lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} h_j^{(N)}(T^{n-j}\bar{\zeta}) = Eh^{(N)}(\bar{\zeta}) \quad \text{a.s.}$$

Letting $N \to \infty$, we get

$$\liminf_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} f_j(T^{n-j}\bar{\zeta}) \geqq Ef^{-}(\bar{\zeta}) \qquad \text{a.s.} \tag{51}$$

It is clear from (50) and (51) that

$$\limsup_{n\to\infty} n^{-1}\sum_{j=0}^{n-1} f_j(\zeta^{n-j}\bar{\zeta}) \leqq Ef^+(\zeta)-Ef^-(\zeta)$$
$$= Ef(\zeta) = E\log\varphi'_{\zeta_0}(q(T\zeta)).$$

But by Theorem 1 we have the fact that

$$0 = E\left[\log\left(\frac{1-\varphi_{\zeta_0}(q(T\zeta))}{1-q(T\zeta)}\right)\right] \geqq E[\log\varphi'_{\zeta_0}(q(T\zeta))]$$

with equality holding only if ζ has a degenerate distribution, a case of no interest to us. Thus the proof of (49) is complete. □

The following corollary to the above theorem is a restatement of (18).

COROLLARY 1. *Let* $E(\log\varphi'_{\zeta_0}(1))^+ < \infty$ *and* $P(q(\zeta) < 1) = 1$. *Then,*

$$P\{Z_n \to \infty \mid \mathbb{F}(\zeta)\} = 1 - P\{Z_n \to 0 \mid \mathbb{F}(\zeta)\}. \tag{52}$$

PROOF. We need to show that for every finite K, Z_n stays in $[1, K]$ only finitely often a.s. By Borel–Cantelli it suffices to show $\sum_{n=1}^{\infty} P\{1 \leqq Z_n \leqq K \mid \mathbb{F}(\zeta)\} < \infty$. But for any $0 < s_0 < 1$,

$$P\{1 \leqq Z_n \leqq K \mid \mathbb{F}(\zeta)\} \leqq s_0^{-K}\sum_{j=1}^{K} P(Z_n = j \mid \mathbb{F}(\zeta))s_0^{\,j}$$
$$= s_0^{-K}(\pi_n(s_0,\zeta) - \pi_n(0,\zeta)).$$

Now use Theorem 7 to get (52). □

4. Extinction criteria for multi-type B.P.R.E. model. The reader is referred to Section 1 concerning notation and background material. Let ζ_t, $t \geqq 0$ be a stationary metrically transitive process and associated with ζ_0 a p-vector p.g.f. $\boldsymbol{\varphi}'_{\zeta_0}(\mathbf{s})$ (cf. Section 1) with mean matrix $A_{\zeta_0} = \|\partial\varphi^{(i)}_{\zeta_0}(\mathbf{1})/\partial s_j\|$. *We assume that w.p.*1

$$0 < C \leqq \frac{\partial\varphi^{(i)}_{\zeta_0}(\mathbf{1})}{\partial s_j} \leqq D < \infty \qquad i,j = 1,\cdots,p \tag{53}$$

$$0 < \frac{\partial^2\varphi^{(i)}_{\zeta_0}(1)}{\partial s_j\,\partial s_k} \leqq D < \infty \qquad i,j,k = 1,\ldots,p$$

and

$$E[-\log\langle \mathbf{1} - \boldsymbol{\varphi}_{\zeta_0}(0), \mathbf{1}\rangle] < \infty \tag{54}$$

where $\langle \mathbf{x}, \mathbf{y}\rangle$ designates the inner product of the indicated vectors and $\mathbf{1}$ represents the vector with all components one.

Subject to the restrictions (53) and (54) we recall from (24) the result of Furstenberg and Kesten (1960) asserting the existence of the limits

$$\lim_{n\to\infty} n^{-1}\log\|A_{\zeta_n}A_{\zeta_{n-1}}\cdots A_{\zeta_0}\| = \pi \qquad \text{w.p. } 1 \tag{55}$$

and

$$\lim_{n\to\infty} n^{-1} E[\log ||A_{\zeta_n} A_{\zeta_{n-1}} \cdots A_{\zeta_0}||] = \pi.$$

In our case because of (53) it is clear that π is finite.

The limit relation (55) can be expressed equivalently in the form

$$(56) \qquad \lim_{n\to\infty} n^{-1} \log \lambda_n(\zeta) = \lim_{n\to\infty} n^{-1} E \log \lambda_n(\zeta) = \pi$$

where $\lambda_n(\zeta)$ is the spectral radius of the matrix $\Gamma_n(\zeta) = A_{\zeta_{n-1}} A_{\zeta_{n-2}} \cdots A_{\zeta_0}$ and $\zeta = (\zeta_0, \zeta_1, \zeta_2, \cdots)$.

The confirmation of (56) relies on (55) and familiar characterizations of the spectral radius for positive matrices. Let

$$(57) \qquad \mathbf{q}(\zeta) = \lim_{q\to\infty} \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_n}(0)\cdots) = \lim_{n\to\infty} \mathbf{q}_n(\zeta)$$

denote the extinction probability vector. In terms of components we write

$$\mathbf{q}_n(\zeta) = (q_n^{(1)}(\zeta), q_n^{(2)}(\zeta), \cdots, q_n^{(p)}(\zeta)).$$

Manifestly $\mathbf{q}_n(\zeta) \ll \mathbf{q}_{n+1}(\zeta)$. (Recall that the notation $\mathbf{x} \ll \mathbf{y}$ signifies that every component of $\mathbf{y}-\mathbf{x}$ is positive.)

THEOREM 12. *Let the conditions* (53) *and* (54) *hold.*

(i) *If* $\pi < 0$ *then* $P(\mathbf{q}(\zeta) = \mathbf{1}) = 1$.

(ii) *If* $\pi > 0$ *then* $P(\mathbf{q}(\zeta) \ll \mathbf{1}) = 1$.

REMARK. The restrictions (53) and (54) can be considerably weakened with the validity of assertion (i) retained as the proof will amply demonstrate.

To ease the exposition of the proof of assertion (ii), we separate out two technical lemmas.

LEMMA A. *Assume the hypothesis of the theorem holds. Let*

$$\mathbf{v}_k(\zeta) = (v_k^{(1)}, v_k^{(2)}, \cdots, v_k^{(p)})$$

be the unique right eigenvector for the matrix $\Gamma_k(\zeta) = A_{\zeta_{k-1}} A_{\zeta_{k-2}} \cdots A_{\zeta_0}$ *normalized so that* $\langle \mathbf{v}_k(\zeta), \mathbf{1}\rangle = \sum_{i=1}^p v_k^{(i)} = 1$. *For each fixed k, the sequence of functions*

$$(58) \qquad \log\left[\frac{\langle \mathbf{1} - \mathbf{q}_{rk}(\zeta), \mathbf{v}_k(\zeta)\rangle}{\langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(T^k\zeta)\rangle}\right], \qquad r = 1, 2, 3, \cdots$$

is dominated by an integrable function depending on k but independent of r.

PROOF. Since $\Gamma_k(\zeta)$ is a positive matrix with elements bounded away from zero (condition (53)) we infer the existence of positive constants such that

$$(59) \qquad 0 < C_k' \leqq \min_{1\leqq i\leqq p} v_k^{(i)}(\zeta) \leqq \max_{1\leqq i\leqq p} v_k^{(i)}(\zeta) \leqq D_k' < \infty$$

and corresponding positive constants C_k'', D_k'' apply for the vector $\mathbf{v}_k(T^k\zeta)$.

Set $f_k(s) = \varphi_{\zeta_0}(\varphi_{\zeta_1}(\cdots \varphi_{\zeta_{k-1}}(s))\cdots) = (f_k^{(1)}(s), f_k^{(2)}(s), \cdots, f_k^{(p)}(s))$.

Now determine $\varepsilon_k > 0$ so that for

$$S_k = \frac{1}{C_k''} p \left[\max_{1 \leqq i,\nu,\mu \leqq p} \frac{\partial^2 f_k^{(i)}(\mathbf{1})}{\partial s_\nu \, \partial s_\mu} \right],$$

we have

$$\varepsilon_k S_k < \tilde{C}_k \tag{60}$$

where $\tilde{C}_k$ satisfies $\Gamma_k(\zeta)\mathbf{v}_k(\zeta) \geqq \tilde{C}_k \mathbf{v}_k(T_k\zeta)$ and this vector inequality is interpreted as valid for all components. Conditions (53) and (59) assure the existence of such $\tilde{C}_k > 0$.

Note on the basis of (53) that $\alpha_k > 0$ exists fulfilling the inequality

$$\min_{1 \leqq i \leqq p} [1 - f_k^{(i)}(1, \cdots, 1, 1-\varepsilon_k, 1, \cdots, 1)] \geqq \alpha_k > 0 \tag{61}$$

where $f_k^{(i)}$ is evaluated at a vector point having one coordinate equal to $1-\varepsilon_k$ with the remaining coordinates equal to one.

We are now prepared to estimate (58). Specify E_k $(0 < E_k < \infty)$ such that $\Gamma_k(\zeta)\mathbf{v}_k(\zeta) \leqq E_k \mathbf{v}_k(\zeta)$ which is clearly feasible by virtue of (53). Convexity and this last inequality yield

$$\begin{aligned} \langle \mathbf{1} - \mathbf{q}_{rk}(\zeta), \mathbf{v}_k(\zeta) \rangle &= \langle \mathbf{1} - f_k(\mathbf{q}_{(r-1)k}(T^k\zeta)), \mathbf{v}_k(\zeta) \rangle \\ &\leqq \langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \Gamma_k(\zeta)\mathbf{v}_k(\zeta) \rangle \\ &\leqq E_k[\langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(\zeta) \rangle]. \end{aligned} \tag{62}$$

The desired upper bound becomes

$$\frac{\langle \mathbf{1} - \mathbf{q}_{rk}(\zeta), \mathbf{v}_k(\zeta) \rangle}{\langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(T^k\zeta) \rangle} \leqq E_k \frac{\langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(\zeta) \rangle}{\langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(T^k\zeta) \rangle} \leqq \frac{E_k D_k'}{C_k''}.$$

With view to establish a lower bound for (58) we define

$$R_k = \langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \Gamma_k(\zeta)\mathbf{v}_k(\zeta) \rangle - \langle \mathbf{1} - \mathbf{q}_{rk}(\zeta), \mathbf{v}_k(\zeta) \rangle.$$

which is manifestly nonnegative (see (62)).
Taylor's expansions of $\mathbf{1} - f_k(\mathbf{q}_{(r-1)k}(T^k\zeta))$ reveals that

$$R_k \leqq S_k \{\max_{1 \leqq i \leqq p} [1 - q^{(i)}_{(r-1)k}(T^k\zeta)]\} \langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(T^k\zeta) \rangle. \tag{63}$$

Now using this inequality on the set $I = \{\zeta; \max_{1 \leqq i \leqq p} [1 - q^{(i)}_{(r-1)k}(T^k\zeta)] \leqq \varepsilon_k\}$ and referring to (60), we obtain

$$\frac{\langle \mathbf{1} - \mathbf{q}_{rk}(\zeta), \mathbf{v}_k(\zeta) \rangle}{\langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(T^k\zeta) \rangle} = \frac{\langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \Gamma_k(\zeta)\mathbf{v}_k(\zeta) \rangle - R_k}{\langle \mathbf{1} - \mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(T^k\zeta) \rangle} \geqq C_k''' > 0. \tag{64}$$

For $\zeta \in I^c$ (complement of I) we have $\max_{1 \leqq i \leqq p}[1-q^{(i)}_{(r-1)k}(T^k\zeta)] > \varepsilon_k$. It follows by virtue of (61) and (59) that

$$\frac{\langle 1-q_{rk}(\zeta), v_k(\zeta)\rangle}{\langle 1-q_{(r-1)}(T^k\zeta), v_k(T^k\zeta)\rangle} = \frac{\langle 1-f_k(q_{(r-1)k}(T^k\zeta), v_k(\zeta)\rangle}{\langle 1-q_{(r-1)k}(\zeta), v_k(T^k\zeta)\rangle} \geqq \frac{\alpha_k C_k'}{\varepsilon_k D_k''} > 0. \tag{65}$$

The inequalities (64) and (65) together exhibit a lower bound for (58) independent of r. The proof of Lemma A is complete.

LEMMA B. *For every i, j satisfying $1 \leqq i, j \leqq p$, we have*

$$\infty > \beta' \geqq \frac{1-q_n^{(i)}(\zeta)}{1-q_n^{(j)}(\zeta)} \geqq \beta > 0 \tag{66}$$

where β and β' are constants independent of ζ and n.

PROOF. We develop the proof for the lower bound since the discussion for the upper bound is analogous. Start with the obvious inequality

$$\frac{1-q_n^{(i)}(\zeta)}{1-q_n^{(j)}(\zeta)} \geqq \frac{\sum_{\nu=1}^{n} \frac{\partial\varphi^{(i)}_{\zeta_0}(1)}{\partial s_\nu}[1-q^{(\nu)}_{n-1}(T\zeta)]-R}{\sum_{\mu=1}^{p} \frac{\partial\varphi^{(j)}_{\zeta_0}(1)}{\partial s_\mu}[1-q^{(\mu)}_{n-1}(T\zeta)]}$$

where by definition

$$\sum_{\nu=1}^{p} \frac{\partial\varphi^{(i)}_{\zeta_0}(1)}{\partial s_\nu}[1-q^{(\nu)}_{n-1}(T\zeta)]-(1-q_n^{(i)}(\zeta) = R \geqq 0.$$

Determine δ such that $\delta pD < C$. On the set $\{\zeta; \max_{1 \leqq \nu \leqq p}[1-q^{(\nu)}_{n-1}(T\zeta)] \leqq \delta\}$ we secure (66) as in Lemma A. On the complementary set, we have

$$\frac{1-q_n^{(i)}(\zeta)}{1-q_n^{(j)}(\zeta)} = \frac{1-\varphi^{(i)}_{\zeta_0}(\mathbf{q}_{n-1}(T\zeta)}{1-\varphi^{(j)}_{\zeta_0}(\mathbf{q}_{n-1}(T\zeta))} \geqq \frac{1-\varphi^{(i)}_{\zeta_0}(1,\cdots,1,1-\delta,1,\cdots,1)}{1-\varphi^{(j)}_{\zeta_0}(0)} \geqq \beta > 0.$$

The proof is complete.

With the aid of this lemma, and since $\mathbf{v}_k(\zeta)$ and $\mathbf{v}_k(T^k\zeta)$ are normalized vectors, we deduce

COROLLARY A. *For all positive integers r and k,*

$$\infty > \tilde{\beta}' \geqq \frac{\langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(\zeta)\rangle}{\langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(T^k\zeta)\rangle} \geqq \beta > 0.$$

PROOF OF THEOREM 12. (i) Suppose $\pi < 0$. On the basis of (55) we see that

$$\lim_{n\to\infty} \log ||A_{\zeta_0} \cdots A_{\zeta_n}|| = \lim_{n\to\infty} \log ||A'_{\zeta_n} \cdots A'_{\zeta_\infty}|| = -\infty.$$

Obviously

$$\log \langle \mathbf{1}-\mathbf{q}_n(\bar{\zeta}), \mathbf{1}\rangle \leqq \log \|A_{\zeta_0} \cdots A_{\zeta_n}\| + \log p,$$

and the right side goes to $-\infty$ w.p.1. Whence

$$\lim_{n\to\infty} \mathbf{q}_n(\zeta) = \mathbf{1} \qquad \text{w.p. 1.}$$

(ii) Let $\pi > 0$. Assume to the contrary that $P(\bar{\zeta}; \mathbf{q}(\bar{\zeta}) = 1) > 0$ and so by the zero one law $P(\bar{\zeta}; \mathbf{q}(\bar{\zeta}) = \mathbf{1}) = 1$. Therefore $\lim_{r\to\infty} \mathbf{q}_{rk}(\bar{\zeta}) = 1$ w.p.1. Since $\pi > 0$ we know that $En^{-1} \log \lambda_n(\bar{\zeta}) \to \pi$ and consequently $E \log \lambda_n(\bar{\zeta}) \to \infty$. This fact in conjunction with the conclusion of Corollary A for k large enough implies

$$E\{\log \lambda_k(\bar{\zeta})\} - \{\max |\log \beta|, |\log \beta'|\} = \gamma > 0.$$

Next, write the identity

$$(67) \qquad \log \langle \mathbf{1}-\mathbf{q}_{mk}(\bar{\zeta}), \mathbf{v}_k(\bar{\zeta})\rangle$$
$$= \sum_{r=1}^{m} \log \left\{ \frac{\langle \mathbf{1}-\mathbf{q}_{rk}(T^{(m-r)k}\bar{\zeta}), \mathbf{v}_k(T^{(m-r)k}\bar{\zeta})\rangle}{\langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^{(m-r+1)k}\bar{\zeta}), \mathbf{v}_k(T^{(m-r+1)k}\bar{\zeta})\rangle} \right\}$$
$$+ \log \langle \mathbf{1}-\mathbf{q}_k(T^{mk}\bar{\zeta}), \mathbf{v}_k(T^{mk}\bar{\zeta})\rangle.$$

Because of (54) we may take expectations in (67). Invoking stationarity leads to the identity

$$(68) \quad \mu_m = E[\log \langle \mathbf{1}-\mathbf{q}_{mk}(\bar{\zeta}), \mathbf{v}_k(\bar{\zeta})\rangle]$$
$$= \sum_{r=1}^{m} E\left[\log \left\{ \frac{\langle \mathbf{1}-\mathbf{q}_{rk}(\bar{\zeta}), \mathbf{v}_k(\bar{\zeta})\rangle}{\langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^k\bar{\zeta}), \mathbf{v}_k(T^k\bar{\zeta})\rangle} \right]\right\} + E[\log \langle \mathbf{1}-\mathbf{q}_k(\bar{\zeta}), \mathbf{v}_k(\bar{\zeta})\rangle]$$
$$= \sum_{r=1}^{m} \theta_r + \mu_1.$$

But

$$\theta_r = E \log \left\{ \frac{\langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^k\bar{\zeta}), A_{\zeta_{k-1}} A_{\zeta_{k-2}} \cdots A_{\zeta_0} \mathbf{v}_k(\bar{\zeta})\rangle - R_{r,k}}{\langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^k\bar{\zeta}), \mathbf{v}_k(T^k\bar{\zeta})\rangle} \right\}$$

where

$$R_{r,k} = \langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^k\bar{\zeta}), \Gamma_k(\bar{\zeta})\, \mathbf{v}_k(\bar{\zeta})\rangle - \langle \mathbf{1}-\mathbf{q}_{rk}(\bar{\zeta}), \mathbf{v}_k(\bar{\zeta})\rangle.$$

Since $\mathbf{q}_{rk}(T_k\bar{\zeta}) \to 1$ w.p.1 and $R_{r,k}$ is bounded by the quadratic form

$$|R_{r,k}| \leqq E_k \textstyle\sum_{i,j=1}^{p} (1 - q_{(r-1)k}^{(i)}(T^k\bar{\zeta}))((1 - q_{(r-1)k}^{(j)}(T^k\bar{\zeta}))$$

we may infer that

$$(69) \qquad \frac{R_{r,k}}{\langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^k\bar{\zeta}), \mathbf{v}_k(T^k\bar{\zeta})\rangle} \to 0 \qquad \text{w.p. 1.}$$

Now Lemma A tells us that the integrand in the definition of θ_r is uniformly dominated by an integrable function and therefore in view of (69) we have

$$\text{(70)} \qquad \liminf_{r\to\infty} \theta_r \geqq E\left\{\liminf_{r\to\infty} \log\left[\frac{\langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^k\zeta), \Gamma_k(\zeta)\mathbf{v}_k(\zeta)\rangle}{\langle \mathbf{1}-\mathbf{q}_{(r-1)k}(T^k\zeta), \mathbf{v}_k(T^k\zeta)\rangle}\right]\right\}$$

$$\geqq E\{\log \lambda_k(\zeta)\} - \max\{|\log \beta|, |\log \beta'|\}$$

the last inequality resulting by virtue of Corollary A and the fact $\Gamma_k(\zeta)v_k(\zeta) = A_{\zeta_{k-1}} \cdots A_{\zeta_1} A_{\zeta_0} v_k(\zeta) = \lambda_k(\zeta)v_k(\zeta)$. But (70) compared to (68) clearly implies

$$\lim_{m\to\infty} \{\textstyle\sum_{r=1}^m \theta_r + \mu_0\} \to +\infty.$$

On the other hand, since $\mathbf{q}_m(\zeta)$ increases to $\mathbf{1}$ we see that $\mu_m \to -\infty$. These evaluations manifest a contradiction and the proof of (ii) is complete by reductio ad absurdum.

REMARK. The conditions (53) and (54) are more stringent than necessary. They served in a technical capacity in order to justify suitable interchanges of limits with integral.

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