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ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Annual meeting, Fort Collins, Colorado, August* 23–26, 1971. *Additional abstracts appeared in earlier issues.*)

131-31. On the growth rate of weighted averages of exponential random variables. R. J. TOMKINS, University of Saskatchewan.

For each $n \geqq 1$, let $Y_{n1}, Y_{n2}, \cdots, Y_{nn}$ be independent, identically distributed exponential random variables, and write $S_n = \sum_{k=1}^{n} Y_{nk}/k$. It is demonstrated that $\liminf S_n/\log n = 1$ a.e. and $1 \leqq \limsup S_n/\log n \leqq 2$ a.e. This result is shown to be the best possible in the sense that double sequences Y_{nk} exist for which $\limsup S_n/\log n = 2$. (Received June 8, 1971.)

131-32. Power of Cochran's test in Behrens–Fisher problems. G. NICHOLAS LAUER II AND CHIEN-PAI HAN, Iowa State University.

Suppose we have a random sample of size n_1 from $N(\mu_1, \sigma_1^2)$ and a second random sample of size n_2 from $N(\mu_2, \sigma_2^2)$. It is desired to test the hypothesis $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$ when no assumptions are made regarding σ_1^2 and σ_2^2. This is known as the Behrens–Fisher problem. This paper investigates an approximate test procedure suggested by Cochran (*Biometrics* **20** (1964) 191–195). A form of the distribution function of the Cochran's test statistic is written as a multiple integral and then transformed to an expression which facilitates numerical computations for specific values of n_1 and n_2. Size and power studies for several small sample combinations are then carried out to determine the behavior of the test for various values of $R = \sigma_1^2/\sigma_2^2$. It is found that the test is uniformly conservative. The Behrens–Fisher problem is also considered when a preliminary F-test of level α_0 is used for $\sigma_1^2 = \sigma_2^2$. Numerical computations of power are carried out for small samples and it is shown that for a proper choice of α_0 the preliminary test procedure achieves a higher power than the single Cochran's test of the same size. (Received June 8, 1971.)

131-33. Exact truncated sequential test for the variance. L. A. AROIAN, X. G. GORGE AND D. E. ROBISON, Union College, Schenectady and Union College, Poughkeepsie.

This paper shows how to find the exact results for the operating characteristic function and the average sample number of the truncated sequential test for the variance σ^2 of a normal random variable using the direct method. The case of testing $\sigma_0^2 = 1$ against $\sigma_1^2 = 4$ is given in detail with $\alpha = \beta = .025, .05$ and $.10$. Incidentally, a Wald test to infinity is evaluated exactly. It is concluded that the truncated sequential test is more economical over the complete range of σ^2 than previously believed. (Received June 9, 1971.)

131-34. A Bayes nonparametric selection procedure. JERRY FLORA AND MYLES HOLLANDER, The Florida State University.

For q a specified number between 0 and 1, a procedure is presented for selecting those treatment populations which have a q-quantile larger than that of a control population. The procedure is Bayes with respect to Dirichlet process priors of the form introduced by Ferguson (UCLA Technical Report (1969)). The procedure requires only partial prior information. The user must specify (Π_i, β_i), $i = 1, \cdots, k$ where Π_i is the prior probability that the q-quantile of population i exceeds the q-quantile of the control population, and β_i is a weight corresponding to the degree of belief in Π_i. The procedure is related to Ferguson's quantile estimate based on the Dirichlet process, to the sign test, and to a nonparametric selection procedure proposed by Rizvi, Sobel, and Woodworth (*Ann. Math. Statist.* **39** (1968) 2075–2093). In the case of a known control population with an indifference zone of length ε, the asymptotic Bayes risk efficiency of this procedure relative to a normal theory analogue has been obtained for several underlying distributions. This efficiency depends on q, ε, and the underlying distribution. If $q = \frac{1}{2}$ and the underlying population is normal, the efficiency tends to $(2/\pi)$ as ε tends to zero. The efficiency exceeds one in some cases when the underlying distribution is the double exponential. (Received June 9, 1971.)

131-35. Error bounds and approximations for Bayes decision functions for classifications. J. T. CHU, New York University.

Given two states of Nature. Let p_i and $f_i(x)$, $i = 1, 2$, be respectively the a priori probabilities and conditional pdfs. Let $\int |f_1 - f_2| = 2\delta$ and $|p_1 - p_2| = \varepsilon$. Several upper bounds, a lower bound, and approximations are derived for the average error probability P_e. For example, $(1-\delta)(1-\varepsilon)/2 \leqq P_e \leqq (1-\delta)(1+\varepsilon)/2$. Extensions are made to the case of m states which can be arranged into a hierarchy with two elements at each branching. Applications and numerical examples are given. (Received June 10, 1971.)

131-36. On the noncentral joint distributions of the largest and smallest latent roots of four random matrices. V. B. WAIKAR, Aerospace Research Laboratories and Miami University.

The joint distributions of the largest and smallest characteristic roots of certain random matrices are useful in the application of some test procedures in multivariate analysis. The above distributions have been derived by others in the null or central case only. The derivation of these distributions in the nonnull or noncentral case is important for the consideration of the power of the above mentioned tests. In this paper, the author has derived the joint distribution of the largest and the smallest characteristic roots in the *noncentral* case for the four random matrices namely (i) Wishart matrix, (ii) MANOVA matrix, (iii) Canonical correlation matrix and (iv) $S_1 S_2^{-1}$ where the matrices S_1 and S_2 have Wishart distribution with $E(S_1) \neq E(S_2)$. The expressions for these distributions are exact and are given in terms of zonal polynomials. (Received June 10, 1971.)

131-37. Estimation of models of autoregressive signal plus white noise (preliminary report). MARCELLO PAGANO AND EMANUEL PARZEN, State University of New York at Buffalo.

If $x(\cdot)$ is a time series which may be written as $x(t) = s(t)+n(t)$ where t is an integer, $s(\cdot)$ an autoregressive signal of order p and $n(\cdot)$ white noise, then the model has $p+2$ parameters. These are (i) the p autoregressive parameters, (ii) the residual variance of the autoregressive scheme and (iii) the variance of the white noise. A method is proposed to estimate the $p+2$ parameters. This method is based on analogies with regression theory and in the case of a normal series yields efficient estimators. (Received June 11, 1971.)

131-38. Bayes estimates of Dirichlet mixing parameters. R. M. KORWAR AND M. HOLLANDER, The Florida State University.

Antoniak (Ph.D. dissertation, UCLA (1969)) has introduced mixtures of Dirichlet processes. The following problem was included in those considered by Antoniak. Let $X_1, \cdots, X_n$ be a sample from the mixture of Dirichlet processes on (R, β) with the mixing distribution $H = N(\mu, 1)$ and the transition probability $\alpha_\theta = N(\theta, 1)$. With squared error loss, Antoniak shows that $\hat{\theta} = [\mu/(r+1)] + [\sum_{i=1}^r Y_i/(r+1)]$ is a Bayes estimator of θ (the mixing parameter), where $Y_1, \cdots, Y_r$ are the r distinct observations amongst $X_1, \cdots, X_n$. Consider a paired sample situation where $Z_i = (X_{1i}, X_{2i})$, $i = 1, \cdots, n$ is a sample of size n from a mixture of Dirichlet processes on (R^2, β^2) with mixing distribution H = Bivariate Normal $(\mu_1, \mu_2; 1, 1; \rho)$ and transition probability $\alpha_\theta = N(\theta_1, \theta_2; 1, 1; 0)$ for $\theta = (\theta_1, \theta_2)$. Then for sum of squared error loss, a Bayes estimator $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2)$ of θ is given by $\tilde{\theta}_i = \mu_i + \{[(r+\tau)H_i + \rho\tau H_{i'}]/[r^2+\tau(2r+1)]\}$ where (i, i') is a permutation of $(1, 2)$, $\tau = (1-\rho^2)^{-1}$, $H_i = \sum_{j=1}^r (W_{ij} - \mu_i)$, and $W_j = (W_{ij}, W_{2j}), j = 1, \cdots, r$ are the r distinct observations amongst $Z_1, \cdots, Z_n$. (The observations Z_s and Z_t are distinct if $X_{1s} \neq X_{1t}$ or $X_{2s} \neq X_{2t}$.) Consistency proofs for $\hat{\theta}$ and $\tilde{\theta}$ are given. It is shown that for a general Dirichlet process, if α is non-atomic then the distribution of r is generalized Bionomial and this is utilized in the derivations. (Received June 11, 1971.)

131-39. Stochastic integral representations of multiplicative operator functionals of a Wiener process. MARK A. PINSKY, Northwestern University.

Let M be a multiplicative operator functional of (X, L) where X is a d-dimensional Wiener process and L is a separable Hilbert space (for the definitions and elementary properties see the author's announcement, *Bull. Amer. Math. Soc.* **77** (1971)

377–380). Sufficient conditions are given in order that M be equivalent to a solution of the linear Itô equation $dM = \sum_{j=1}^{d} MB_j\, dx_j + MB_0\, dt$, where $B_0, \cdots, B_d$ are bounded operator functions on R^d. The conditions require that the equation $T(f) = E[M(t)f(x(t))]$ define a semigroup on $L^2(R^d)$ whose infinitesimal generator has a domain which contains all linear functions of the coordinates $(x_1, \cdots, x_d)$. The proof of this result depends upon a priori representation of the semigroup $T(t)$ in terms of the Wiener semigroup and a first order matrix differential operator. A second result characterizes solutions of the above Itô equation with $B_0 = 0$. A sufficient condition that M belong to this class is that $E_x[M(t)]$ be the identity operator on L and that $M(t)$ be invertible for each $t > 0$. The proof of this result uses the martingale stochastic integral introduced by H. Kunita and S. Watanabe. (Received June 11, 1971.)

131-40. Two-stage chi-square goodness of fit test. JOHN E. HEWETT AND R. K. TSUTAKAWA, University of Missouri–Columbia.

Consider two independent random samples of sizes γn and $(1-\gamma)n$, $0 < \gamma < 1$, from the same distribution. Suppose each observation is classified into one of r mutually exclusive and exhaustive classes where $p_i > 0$ is the probability that a sample observation is an element of the ith class. Let Q_1 and Q be the Pearson χ^2 statistics for testing a simple hypothesis H_0 about the p_i's based on the first and combined samples, respectively. The asymptotic distribution of (Q_1, Q), as $n \to \infty$, is then bivariate χ^2 with $r-1$ df under H_0 and bivariate noncentral χ^2 with $r-1$ df under the sequence of alternatives considered by Cochran (*Ann. Math. Statist.* **23** (1952)) for the one sample problem. A two-stage test based on (Q_1, Q) is proposed and its advantage over the one sample test is illustrated in terms of its asymptotic relative efficiency. Under conditions similar to those used in the one sample problem, it is shown that the test may be applied to cases involving unknown parameters by reducing the df by one for each parameter estimated. (Received June 11, 1971.)

131-42. A strong law of large numbers in normed linear spaces. ROBERT LEE TAYLOR, University of South Carolina.

Some of the strong laws of large numbers for random variables can be extended to random elements (function-valued random variables) in separable normed linear spaces. In particular, a strong law of large numbers holds for independent random elements which are not necessarily identically distributed. Let $\{A_n\}$ be a sequence of real-valued random variables which are uniformly essentially bounded and let $\{V_n\}$ be a sequence of identically distributed random elements in a separable normed linear space such that $E||V_1||^2 < \infty$. If the random elements $\{A_n V_n\}$ are independent and if the Pettis integral $E(A_n V_n) = E(A_1 V_1)$ for each n, then $\|n^{-1} \sum_{k=1}^{n} A_k V_k - E(A_1 V_1)\| \to 0$ with probability one. A corollary to this result is the strong law of large numbers for independent, identically distributed random elements $\{V_n\}$ such that $E||V_1|| < \infty$. Also, the condition of uniformly essentially bounded random variables $\{A_n\}$ can be replaced by assuming that $\{E(A_n^4)\}$ is a bounded sequence and that $E||V_1||^4 < \infty$. (Received June 14, 1971.)

131-43. Regression optimality of principal components. R. L. Obenchain, Bell Telephone Laboratories, Inc.

Consider $p \geqq 2$ random variables, and let $A_1, \cdots, A_p$ denote the $(p-1)$ dimensional linear regression hyperplanes, one for each variable given the other $(p-1)$ variables. These hyperplanes do not usually coincide. Let A_0 denote the hyperplane which passes through the centroid of the distribution and is spanned by the direction vectors defining the first $(p-1)$ principal components—i.e. orthogonal to the last principal component direction. It follows from well-known optimality properties of principal components that A_0 expresses the single most important, unconditional, linear relationship among the p variables. However, a new optimality property of A_0 is also established; A_0 is the best single approximation to $A_1, \cdots, A_p$ when each regression hyperplane is given a certain weighting inversely proportional to the variability associated with its orientation and scaling. (Received June 15, 1971.)

131-44. Characterization of probability laws by linear functions. C. Radhakrishna Rao, Indian Statistical Institute, New Delhi.

Let x be a n-vector of independent random variables and $Z = Ax$ be a set of $p < n$ linear functions of x. We study the problem of characterizing the distribution of x given that of Z. The following propositions are established. (i) When $n = 3$ and $p = 2$, the distribution of Z determines that of x upto a change of location, which is a mild generalization of a result due to Kotlarski who chooses particular types of linear functions. (ii) When $n = 4$ and $p = 2$, the distribution of x is determined upto a convolution with independent normal variables. (iii) A general result is obtained for any n and p. (iv) Expressions are obtained for the smallest value of p for given n which determines the distribution of x upto change of location, convolution with a normal variable etc. (v) Multivariate generalizations are obtained. (Received June 16, 1971.)

131-45. Efficient techniques for unbiasing a Bernoulli generator. James A. Lechner, National Bureau of Standards.

Consider the problem of operating on a sequence of i.i.d. Bernoulli variables with unknown mean p to produce a sequence of symmetric Bernoulli variables. Define the efficiency of any proposed method to be the avarage number of output digits per input digit. The following results are proved: (A) No method exists having efficiency greater than $-p \log_2 p - q \log_2 q$, where $q = 1-p$. (B) Procedures do exist with efficiencies arbitrarily close to the bound in (A). Examples of the procedures of (B) are given, and compared with other methods in the literature. (Received June 16, 1971.)

131-46. Inequalities for characteristic roots of matrices with nonnegative elements. JULIAN KEILSON AND GEORGE P. H. STYAN, The University of Rochester and McGill University.

Let $\mathbf{A}$ be an $n \times n$ matrix with nonnegative elements, and let r be the dominant characteristic root. For $s > r$ the matrix $\mathbf{M}_s = s\mathbf{I} - \mathbf{A}$ is called an M-matrix (Ostrowski, *Comment. Math. Helvetici* **10** (1937) 69–96). We show that $\operatorname{tr}(\mathbf{M}_s^{-1}) \geqq ns^{n-1}/(s^n - r^n)$, with equality for $r > 0$ if and only if $\mathbf{A}/r$ is similar to an irreducible permutation matrix. Our proof uses a probabilistic argument. As corollary, we find that $\det(\mathbf{M}_s) \leqq s^n - r^n$. If $r > 0$, equality holds here if and only if equality holds above. We also show that $\prod_{j=2}^{n}(1-\lambda_j) \leqq n$, where $\lambda_1 = 1, \lambda_2, \cdots, \lambda_n$ are the characteristic roots of an $n \times n\ (\geqq 2)$ stochastic matrix $\mathbf{B}$ with equality if and only if $\mathbf{B}$ is an irreducible permutation matrix. This result also follows from $\det(\mathbf{M}_s) \leqq \prod_{i=1}^{n}(s - a_{ii})$, where $\{a_{ii}\}$ are the diagonal elements of $\mathbf{A}$, with equality if and only if there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P}$ is triangular (cf. Marcus, Minc, & Moyls, *J. Res. Nat. Bureau Standards Sect. B* **65** (1961) 205–209). (Received June 16, 1971.)

131-47. Some useful results on order statistics relevant to certain life testing experiments. SATYA D. DUBEY, New York University.

In this paper the following results are established which may be considered useful within the context of certain life testing experiments. Let t_{rn} and t_{rr} denote the rth failure time or the rth order statistic in a random sample of size n and r respectively $(1 \leqq r \leqq n)$ from any arbitrary continuous failure probability law. Then THEOREM 1. *t_{rn} is stochastically smaller than t_{rr}.* THEOREM 2. $Et_{rn}^k \leqq Et_{rr}^k$ only for odd $k(k = 1, 3, 5, \cdots)$. COROLLARY 1. *$Et_{rn}^k \leqq Et_{rr}^k$ for $k = 1, 2, \cdots$, when t_{rn} and t_{rr} are nonnegative failure times and also under certain other conditions.* COROLLARY 2. *Let $t_{rn}(\alpha)$ and $t_{rr}(\alpha)$ be the 100α percent probability limit for t_{rn} and t_{rr} respectively* $(0 < \alpha < 1)$. Then *$t_{rn}(\alpha) \leqq t_{rr}(\alpha)$ for a fixed α.* (Received June 17, 1971.)

131-48. Weak convergence of weighted empirical cumulatives based on ranks. HIRA LAL KOUL AND ROBERT G. STAUDTE, JR., Michigan State University.

For each $n \geqq 1$ let $R_{n1}, \cdots, R_{nn}$ and $R_{n1}^+, \cdots, R_{nn}^+$ denote the ranks and absolute ranks, respectively, of independent observations $X_{n1}, \cdots, X_{nn}$ having respective continuous distributions $F_{n1}, \cdots, F_{nn}$. Define the stochastic processes $S_n(v) = \sum_{i=1}^{n} c_{ni} I[R_{ni} \leqq (n+1)v]$ and $S_n^+(v) = \sum_{i=1}^{n} c_{ni} I[R_{ni}^+ \leqq (n+1)v] \operatorname{sgn}(X_{ni})$, $0 \leqq v \leqq 1$. The weak convergence of $\{S_n(v): 0 \leqq v \leqq 1\}$ and $\{S_n^+(v): 0 \leqq v \leqq 1\}$ to appropriate continuous Gaussian processes is proved. The results include as a special case a short proof of Theorem 1 of Dupac, V. and Hájek, J. [Asymptotic normality of linear rank statistics II, *Ann. Math. Statist.* **40** (1969) 1992–2017] and give an expression for the asymptotic mean which was not attempted there. The results also extend to general alternatives, the Theorems V.3.5.1 and VI.2.3.1 of the *Theory of Rank Tests* (1968) by Jaroslav Hájek. Moreover, they provide analogous theorems for generalized Smirnov type statistics for testing symmetry. (Received June 17, 1971.)

131-49. On limiting distributions of a random number of dependent random variables. DANIEL L. THOMAS, University of Connecticut.

Let $\{X_n, n \geqq 1\}$ be a sequence of random variables such that for suitably chosen constants $a_n > 0$ and b_n, $n \geqq 1$, $\{(X_n - b_n)/a_n\}$ converges in distribution to a nondegenerate random variable X. Let $\{N_m, m \geqq 1\}$ be a sequence of positive, integer-valued random variables distributed independently of the sequence $\{X_n\}$ and converging to infinity in probability as $m \to \infty$. If $\{a_n\}$ and $\{b_n\}$ are the normalizing constants computed from a cdf F which is in the domain of attraction of one of the extreme value distributions (Gnedenko, B. V. *Ann. of Math.* **44** (1943) 423–453) and if the cdf of X satisfies a condition determined by the domain of attraction to which F belongs, then conditions on the limiting distribution of $\{N_m/m\}$ are obtained which are necessary and sufficient for the convergence in distribution of the sequence $\{(X_{N_m} - b_m)/a_m\}$ to a nondegenerate random variable Y. The cdf of Y is either a location or a scale mixture of the cdf of X; and, the cdf F is often unrelated to the distribution of $\{X_n\}$. These results extend a theorem stated by Berman (*Ann. Math. Statist.* **33** (1962) 894–908); however, the method of proof is conceptually simpler. (Received June 17, 1971.)

131-50. Asymptotic distribution of a class of random variables derived from a linear stochastic process. K. C. CHANDA, University of Florida.

Let $\{Z_t, t = 0, \pm 1, \cdots\}$ be a pure white noise process with a common cdf F. Assume that the ch.f. φ of F is integrable and $E\{|Z_1|^\delta\} < \infty$ for some $\delta > 0$. Let $\{g_j, j \geqq 1\}$ be a sequence of real numbers such that $\sum_{j=0}^{\infty} j^2 |g_j|^\lambda < \infty$, $\lambda = \delta/(2+\delta)$. Assume that $\sum_{j=0}^{\infty} \log_e \varphi(g_j u)$ is convergent for all real values of u, so that $X_t = \sum_{j=0}^{\infty} g_j Z_{t-j}$ is a random variable with ch.f. $\Psi(u)$ given by the relation $\log \Psi(u) = \sum_{j=0}^{\infty} \log \varphi(g_j u)$. Then it is shown that if h is a real step function on R^1, and we define $S_n = \sum_{t=1}^{n} [h(X_t) - \mu]$ where $\mu = E\{h(X_1)\}$, (i) $S_n/n^\alpha \to 0$ a.e. whenever $\alpha > \frac{3}{4}$, and (ii) $\mathscr{L}(S_n/n^{\frac{1}{2}}) \to N(0, \sigma^2)$ where σ is a finite positive constant. (Received June 18, 1971.)

131-51. A functional characterization of a class of covariance inequalities. GEORGE KIMELDORF AND ALLAN SAMPSON, Florida State University.

This paper considers the class of covariance inequalities of the form $\operatorname{Cov}(X, Y) \leqq \operatorname{Var} g(X, Y)$. We present a functional inequality and the partial differential inequality $(\partial g/\partial x)(\partial g/\partial y) \geqq \frac{1}{4}$, each of which is a necessary and sufficient condition on g for the covariance inequality to be satisfied for all random variables X, Y jointly taking values in a given region. (Received June 21, 1971.)

131-52. On problems of estimation and prediction for stochastic growth processes. JAMES T. MCCLAVE, University of Florida.

This paper considers the problems of estimation and forecasting for stochastic growth processes. The processes have wide application in describing animal populations' growth, economic growth patterns, etc. Until now most of the research on estimation of the parameters of the process mean function has been carried out without assigning a probabilistic structure to the process, or at best assigning an unrealistic one. We treat these growth processes under the auspices of the pure-birth probability structure. It is shown that under certain minor regularity conditions one can obtain maximum likelihood estimators for the parameters of the "growth" rate (or, more commonly, "birth" rate) for the process. These methods are then applied to the logistic growth process. Several simulated examples are worked, and finally we attempt to fit the logistic model to the U.S. Census Bureau figures for the population of conterminous United States at ten year intervals from 1800–1970. In all examples we compare our methods with those popularly used (with the logistic growth process) for estimation and forecasting. (Received June 21, 1971.)

131-53. The distribution of the MLE of the uniform correlation coefficient in the multivariate normal population. S. A. PATIL, J. L. KOVNER AND D. C. PATEL, Rocky Mountain Forest and Range Experimental Station.

The distribution of the maximum likelihood estimator (MLE) of the correlation coefficient when the sample is taken from a multivariate normal population with uniform covariance matrix is determined. The expressions for the distribution function and rth moment of the statistic are obtained in terms of the hypergeometric function. The tables for the distribution function for some values of ρ and N are given. The distribution function is used to test the hypothesis $\rho = \rho_0$ and in setting the confidence interval for ρ. The curves for getting confidence interval estimates are obtained. The curve for the power function of the test is given. The likelihood ratio criterion is considered. The moments of the MLE are used in estimating the linear combination of the means of the trivariate population for incomplete data. (Received June 21, 1971.)

131-54. Some inference problems related to marginal lumpability in vector-valued Markov chains. LUIS B. BOZA, Bell Telephone Laboratories, Inc.

Let S_x and S_y be finite sets and $S = S_x \times S_y$. Let $\{\mathbf{Z}_t = (X_t, Y_t); t = 0, 1, 2, \cdots\}$ be a Markov chain on S with time-homogeneous transition probability matrix

$P = (p_{ij,kl})$. The two-dimensional chain is said to be X-strongly lumpable if for any initial distribution the marginal process $(X_t; t = 0, 1, 2, \cdots)$ is a Markov chain on S_x, or alternatively if the transition probabilities $p_{ij,kl}$ admit the factorization $p_{ij,kl} = p_{ik}q_{jl}^{ik}$, where $p_{ik} \geqq 0$, $q_{jl}^{ik} \geqq 0$, $\sum_{k \in S_x} p_{ik} = 1$ for any $i \in S_x$, $\sum_{l \in S_y} q_{jl}^{ik} = 1$ for any $(i, j) \in S$, $k \in S_x$. Estimators of the transition probabilities are produced under the various lumpability hypotheses and other subhypotheses of interest. On the basis of observations $\mathbf{Z}_0, \mathbf{Z}_1, \cdots, \mathbf{Z}_T$ on a single path of length T, test statistics for these hypotheses are also constructed. Asymptotic results are obtained as $T \to \infty$. The case where the observation is a set of N replicates of fixed finite length T_0 is briefly described and some practical applications sketched. The theory can be extended to higher dimensional state spaces; it allows the analyses of probabilistic relationships between the coordinate processes. (Received June 21, 1971.)

131-55. Weak limits of residual life times (preliminary report). A. A. BALKEMA AND L. DE HAAN, University of Amsterdam and Michigan State University.

Let F be the probability distribution of the random life time X ($\geqq 0$ and unbounded). For $t \geqq 0$ we define X_t, the residual life time at t, to have the distribution $F_t(x) = \max\{0, 1-(1-F(t))^{-1}(1-F(t+x))\}$. F belongs to the domain of attraction of the distribution G if there exist $a(t) > 0$ and $b(t)$ such that $P\{(a(t))^{-1}(X_t - b(t)) \leqq x\}$ converges weakly to $G(x)$ for $t \to \infty$. The limit functions G are either continuous or discrete. The class of continuous distributions G_γ is closely related to the extreme value distributions. The class of discrete distributions is essentially $G_\gamma([x])$ (where $[x]$ is the largest integer not exceeding x). Domain of attraction criteria are given. For continuous limit distributions weak convergence is equivalent to the convergence of some moment. This result gives a probabilistic interpretation of the fundamental properties of regularly varying functions. (Received June 21, 1971.)

131-56. Population mixture models and clustering algorithms (preliminary report). STANLEY L. SCLOVE, Stanford University and Carnegie-Mellon University.

A modification of the usual approach to population mixtures is employed. As usual, a parametric family of distributions is considered, a set of parameter values being associated with each population. In addition, with each observation is associated a set of parameters which indicate from which population the observation arose. The resulting likelihood function is interpreted in terms of the conditional probability density of a sample from a mixture of populations, given the population identifications of each observation. Clustering algorithms are obtained by applying a method of iterated maximum likelihood to this likelihood function. (Received June 21, 1971.)

131-57. On maxima of Gaussian sequences. CHANDRAKANT M. DEO, University of Oregon.

Let $\{X_n\}$ be an arbitrary sequence of Gaussian random variables with mean zero and unit variance. Let $r(i,j) = E(X_iX_j)$ and $\delta_n = \sup_{|i-j| \geqq n} |r(i,j)|$. Following theorems about the asymptotic behavior of $Z_n = \max_{1 \leqq j \leqq n} X_j$ are obtained. THEOREM 1. *If* $\limsup \delta_n = 0$ *then* $Z_n/(2 \log n)^{\frac{1}{2}}$ *converges to* 1 *almost surely*. THEOREM 2. *If* $\delta_n \to 0$ *and* $\exists\, \gamma > 0: \sum_{1 \leqq i < j \leqq n} |r(i,j)| = O(n^{2-\gamma})$ *then* $Z_n - (2 \log n)^{\frac{1}{2}} \to 0$ a.s. THEOREM 3. *If* $\exists\, \alpha > 0: n^{\alpha}\delta_n \to 0$ *then* $Z_n - (2 \log n)^{\frac{1}{2}} \to 0$ a.s. THEOREM 4. *Let* $\delta_1 < 1$. *Then if either* $\delta_n \log n \to 0$ or *hypothesis of Theorem* 2 *holds we have* $\Pr\{(Z_n - b_n)/a_n \leqq x\} \to \exp(-e^{-x})$, $-\infty < x < \infty$ *where* $a_n = (2 \log n)^{-\frac{1}{2}}$ *and* $b_n = (2 \log n)^{\frac{1}{2}} - \frac{1}{2}(\log \log n + 4\pi)/(2 \log n)^{\frac{1}{2}}$. These theorems extend some of the results of (J. Pickands III: *Z. Wahrsheinlichkeitstheorie und Verw. Gebiete* **7** (1967)) and (S. Berman: *Ann. Math. Statist.* **35** (1964)). (Received June 21, 1971.)

131-58. On mixture, quasi-mixture and nearly normal random processes. RICHARD D. MARTIN AND STUART C. SCHWARTZ, University of Washington and Princeton University.

There is a considerable need for viable models of nearly Gaussian random processes. The robustness studies of Huber (*Ann. Math. Statist.* **35** 73–101) and Tukey (*Contributions to Probability and Statistics* (1960) 448) for the case of i.i.d. random variables indicate one of the many possible uses of such models. We consider the following two nearly normal processes: a mixture process, $\hat{x}_t$, specified by the multi-dimensional simple mixtures, $\hat{F}(x_1, x_2, \cdots, x_n) = a_1F_1(x_1, x_2, \cdots, x_n) + a_2F_2(x_1, x_2, \cdots, x_n)$, $a_1 + a_2 = 1$, and a quasi-mixture process, $\tilde{x}_t$, specified by the multi-dimensional distributions $\tilde{F}(x_1, x_2, \cdots, x_n) = \sum_{k=0}^{n} \sum_{\Delta_k{}^n} a_1{}^k a_2{}^{n-k} F_1(x_{i_1}, x_{i_2}, \cdots, x_{i_k}) F_2(x_{i_{k+1}}, \cdots, x_{i_n})$. It is shown that if the set of random variables of the component cdf's are independent then the random variables of the resulting mixture are independent if and only if the mixture cdf $\hat{F}$ is degenerate. The quasi-mixture process, on the other hand, does have the property that factorization of the component cdf's implies factorization of the resulting mixture cdf. Specializing to the case of Gaussian cdf's, it is further shown that the GMP (Gaussian Mixture Process) never satisfies the strong mixing condition, while with reasonable assumptions concerning the component correlation functions the GQMP (Gaussian Quasi-Mixture Process) does satisfy the strong mixing condition. Specific methods of generating GMP and GQMP's are given. The processes are easily simulated on the computer. (Received June 21, 1971.)

131-59. Judging a decision maker by his actions (preliminary report). GEORGE T. DUNCAN, University of California, Davis.

Quality of a binary decision made under 0–1 loss is based on the subjective probability p assigned to the correct state. Suppose only the decision maker's action and the correct state becomes known to his judge. A one-parameter beta family, $\alpha p^{\alpha-1}$, is used as a prior on p. The moment generating function of the posterior and Bayes estimators are obtained. Models for a sequence of decision problems are considered in which nature chooses the first state with probability π_n at the nth trial and the decision maker arrives at p according to the pdf $\alpha p^{\alpha-1}$. The compound decision problem is explored. The importance of assumptions about stationarity and independence in the state sequence is emphasized. An empirical Bayes procedure is used to estimate the prior parameter α. Methods for combining judgements of decision maker quality are examined under various models. (Received June 21, 1971.)

131-60. Asymptotic comparisons of estimators of the slope parameter of a linear functional relationship. M. W. J. LAYARD, University of California, Davis.

It is well known that the least-squares estimator of the slope parameter of a straight line regression relationship is biased and not consistent if the regressor is subject to random observation error. If the ratio of the variance of this observation error and the variance of the regression "error" is known, and both errors are assumed to be normally distributed, a consistent maximum-likelihood estimator exists. The asymptotic distributions of this maximum-likelihood estimator and of the least-squares estimator are derived, and used to make some approximate mean-square-error comparisons of the two estimators. Approximate comparisons of these estimators with Wald's grouping estimator are also made. These comparisons are supplemented by means of some simulated sampling experiments. (Received June 21, 1971.)

131-61. Maximum likelihood estimation of distributions with monotone failure rates over finite intervals from a renewal testing procedure. I. N. SHIMI AND L. H. CROW, Florida State University.

Let T be a fixed positive real number. A cdf F, $F(0) = 0$, is said to be IFR (Increasing Failure Rate) on $[0, T]$ iff it satisfies one of the following conditions: (i) $-\log[1-F(x)]$ is convex on $[\alpha_F, \beta_F)$, $0 \leqq \alpha_F \leqq \beta_F \leqq T$ and $F(\beta_F) = 1$ if $\beta_F < T$; or (ii) the part of the support of F in $[0, T]$ is empty. Let $\mathscr{F} = \{F: F \text{ is IFR on } [0, T]\}$. This class includes the usual class of IFR distributions. Furthermore, there exists no sigma-finite measure relative to which all the distributions in $\mathscr{F}$ are absolutely continuous. In this paper the general definition of MLE due to Kiefer and Wolfowitz (*Ann. Math. Statist.* **27** (1956) 887–906) is used to determine the MLE of the life-time distribution F over $[0, T)$, where $F \in \mathscr{F}$. The following renewal type testing plane is used. At time zero, the beginning of the testing, n new items from a population with a cdf $F(\in \mathscr{F})$ are put on test. When an item fails it is instantaneously replaced with another new item from the same population and at time T all the testing is stopped. It is also shown that the MLE of F over $[0, T)$ is strongly consistent as n tends to infinity. (Received June 21, 1971.)

131-62. Some results on robust estimation based on Kolmogorov–Smirnov statistics. P. V. RAO AND R. C. LITTELL, University of Florida.

Robust estimation of shift parameters based on the two-sample Kolmogorov–Smirnov statistic is discussed. Computational procedures are presented and it is shown that the estimators of this paper perform well when the underlying distribution has heavy tails, according to an empirical study and the asymptotic length of confidence intervals. (Received June 21, 1971.)

131-63. Properties of a nonsubstitutable observation (preliminary report). MIKISO MIZUKI, ITT Federal Electric Corporation.

In specifying $(\Omega, \mathfrak{F}, P)$ two distinct interpretations for $P(\omega \in A)$ are possible, i.e., ω is varied over Ω when A is fixed, or in others ω is treated as a fixed unspecified element of Ω when A is varied over $\mathfrak{F}$. The latter applies to a special class of problems of nonrepeatable observations, for which it has been known that sometimes P cannot be assigned subjectively in a manner consistent with the utility theory. E.g., the decision problem of a mountain climber risking his life cannot be formulated using the utility concept due to the nonconstant values of "to be alive" under different contingencies. To analyze such problems, the familiar set theory is extended by adding the notions of (i) a substitutable (or nonsubstitutable) observation by treating an unspecified element variable (or fixed), and (ii) specifications for an element to belong to a specified set. For a nonsubstitutable observation, the specifications for a fixed element belonging to various sets must be examined. The class of sets for which a fixed element meets specifications does not form a field. Consequently no probability measures can be assigned to this class of sets to represent the uncertainty of a nonsubstitutable observation. A weight function weaker than P is defined and analyzed. (Received June 21, 1971.)

131-64. Augmenting incomplete multidimensional designs (preliminary report). J. T. SENNETTI AND K. HINKELMANN, Virginia Polytechnic Institute and State University.

Many incomplete multidimensional designs are constructed under the assumption that no interactions between the factors are present. In this paper a method is considered for adding the minimum number of points necessary to such incomplete design in order to estimate first order interaction contrasts between certain factors. The method is based on an extension of the concept of connectedness as discussed by Srivastava and Anderson [*Ann. Math. Statist.* **41** (1970) 1438–1445]. It is shown how this method can be used to construct resolution IV designs and other irregular asymmetrical factorial fractions. Properties of this class of minimal designs are discussed. (Received June 21, 1971.)

131-65. A criterion of estimation based on measures of information (preliminary report). TAKIS PAPAIOANNOU, University of Georgia.

Measures of information may be used to introduce the following criterion of estimation. Let $\{P_\theta, \theta \in \Theta\}$ be a family of probability measures and $I_x(\theta, \theta^*)$ a measure of information defined for all $(\theta, \theta^*) \in \Theta \times \Theta$. An estimator $T(x)$ is said to be information optimum (I-optimum) if $E_\theta[I_x(\theta, T(x))]$ is minimum for all estimators belonging to a certain class and uniformly in θ. This criterion when applied with the Kullback–Leibler and Bhattacharyya measures of information leads to various interesting results. If the family of distributions is normal with known variance, the I-Kullback–Leibler optimum unbiased estimators are identical with the UMVU's. I-optimum estimators have been investigated for various families of distributions by means of decision theory which regards $I_x(\theta, \theta^*)$ as a loss function. Results similar to those obtained by Rao–Blackwell and completeness theorems are presented for those families of distributions for which $I_x(\theta, \theta^*)$ is convex in θ^* for all θ. Nonregular families of distributions for which the Kullback–Leibler measure is not defined may be treated using the Bhattacharyya loss function. Confidence intervals are developed through a similar approach. (Received June 21, 1971.)

131-66. Optimal predicitive linear discriminants. PETER ENIS AND SEYMOUR GEISSER, State University of New York at Buffalo.

We are given an observation Z which has arisen (with known prior probabilities) from one of two p-variate nonsingular normal populations with known parameters. The discriminant, say U, which minimizes the (total) probability of misclassification is based on the ratio of the densities of the two populations. When the parameters are unknown, the "classical" procedure has been to substitute sample estimates for the unknown parameters in U and then use this sample discriminant, say V, as a basis for classifying observations. This procedure does not enjoy the property of minimizing the probability of misclassification and has been justified, from the classical point of view, almost entirely on the grounds that it seems intuitively reasonable. When the covariance matrices of the two normal populations are equal, U is a linear function of the observation vector Z. The fact, as mentioned before, that U minimizes the probability of misclassification does not imply that V will. Further, although U is linear, the sample discriminant which minimizes the probability of misclassification will, in general, not be linear. Hence, we obtain here from amongst the class of linear sample discriminants that sample discriminant which minimizes the probability of misclassification. (Received June 21, 1971.)

131-67. Effects of restricting nuisance parameters on the power of the generalized likelihood ratio criterion under fixed noncentralities. J. W. L. COLE, University of Iowa.

We consider the effects, in large samples, of restrictions on nuisance parameters ($\boldsymbol{\theta}_2$) on the power of the generalized likelihood ratio criterion under fixed noncentral values of the parameters tested ($\boldsymbol{\theta}_1$). The context is a situation in which we have random samples of n continuous random vectors $\mathbf{X}$ from a population with parameters $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. In addition to certain regularity conditions on the likelihood function we assume that the nuisance parameters $\boldsymbol{\theta}_2$ conform to a restriction R confining them to a subset of the set admitted under the model. Thus the standard likelihood ratio λ_0 defined by the model and hypothesis can be compared with an alternative criterion λ_1 defined by the additional assumption of the restriction R. Under these assumptions it is found that two conditions, C^+ and C^-, can be defined for values of $\boldsymbol{\theta}$ violating the hypothesis being tested. If $\boldsymbol{\theta}$ satisfies C^+, then λ_1 (assuming R) has more power for large n than λ_0 (ignoring R), while there is no such gain for λ_1 over λ_0 if $\boldsymbol{\theta}$ satisfies C^-. Examples are found in univariate and multivariate normal testing problems to illustrate the conditions C^+ and C^-. (Received June 22, 1971.)

131-68. Certain weak and strong convergence results for sequences of ϕ-mixing random variables. P. K. SEN, University of North Carolina.

Two theorems of Billingsley [*Convergence of Probability Measures*, Wiley, New York (1968) 195–205] on the weak convergence of empirical processes to Gaussian processes are proved here under comparatively weaker regularity conditions. Also, a functional central limit theorem for sample quantiles and the law of iterated logarithm for sample quantiles for ϕ-mixing processes are established here through an extension of an elegant result of Bahadur [*Ann. Math. Statist.* **37** (1966) 577–580]. (Received June 23, 1971.)

131-69. Laws of the iterated logarithm for maximums of absolute values of partial sums of permuted random variables. G. G. MAKOWSKI, Marquette University.

This paper deals with laws of the iterated logarithm for maximums of absolute values of partial sums of permuted random variables (the permutations are of a particular type called order-preserving), and their application to the estimation of integral regression functions and nondecreasing regression functions. First, laws of the iterated logarithm are derived under the assumptions used by Kolmogorov and Hartman and Wintner. Then similar results are obtained under a strengthened version of a normal convergence condition used by Petrov. Next it is shown that these maximums of absolute values of partial sums of permuted random variables and related statistics are semimartingales. A law of the iterated logarithm is derived using a law of the iterated logarithm for semimartingales due to Csaki. Finally, it is shown how the maximum absolute difference of an integral regression function and an estimator (due to Brunk) can be represented as a multiple of a maximum of absolute values of partial sums of certain permuted random variables. Laws of the iterated logarithm derived earlier are then applied to yield convergence rates for this maximum absolute difference. A similar approach yields convergence rates for a well-known estimator of an increasing regression function at the observation points. (Received June 23, 1971.)

131-70. Two limiting distributions for two-sample Kolmogorov-Smirnov type statistics (preliminary report). NANCY L. GELLER, University of Rochester.

Let $\xi_1, \xi_2, \cdots, \xi_m$ and $\eta_1, \eta_2, \cdots, \eta_n$ be two independent random samples having the same underlying continuous distribution function. Let $\delta^+(m, n)$ denote the one-sided Kolmogorov–Smirnov statistic; $\delta(m, n)$ denote the two-sided Kolmogorov–Smirnov statistic; and $\tau(m, n)$, the index of the η-order statistic at which $\delta^+(m, n)$ is attained, a statistic introduced by Tākacs [*Nonparametric Techniques in Statistical Inference* (1970) 359–384]; and $\tau^*(m, n)$ the index of the η-order statistic at which $\delta(m, n)$ is attained. The joint limiting distributions for $\delta^+(m, n)$ and $\tau(m, n)$ and for $\delta(m, n)$ and $\tau^*(m, n)$ are obtained when n and m tend to infinity in such a way that n/m tends to a positive constant. This generalizes a result of Vincze [*Publ. Math. Inst. Hungar. Acad. Sci.* **2** (1957) 183–203] who obtained the same results for equivalent statistics in the case $n = m$. The proofs employ weak convergence of sample distribution functions as suggested by Doob [*Ann. Math. Statist.* **20** (1949) 393–403] and Donsker [*Ann. Math. Statist.* **23** (1952) 277–281]. (Received June 23, 1971.)

131-71. A sequential test for a multivariate mean with unknown covariance matrix (preliminary report). W. J. HALL AND GOVIND S. MUDHOLKAR, The University of Rochester.

Suppose X is a p-variate $N(\mu, \Sigma)$ random vector and H_i is the hypothesis that $\mu = \mu_i$ for $i = 0$ and 1; Σ is unspecified and unknown. A sequential test procedure is developed for testing H_0 vs. H_1, with specified bounds on the two error probabilities. The method is based on an extension of a technique of A. G. Baker and Hall [*Biometrika* **49** (1962) 367–378] for the univariate case, and on an inequality for the minimal characteristic root of $\Sigma^{-1}S$, S being the unbiased estimate of Σ based on an initial sample of size n_0. The test is based on the log likelihood ratio statistic, with Σ estimated by S, with two constant termination boundaries. Tables for the boundary constants, as functions of n_0 and the two nominal error probabilities, are provided for small values of p, and asymptotic (valid for large n_0) formulas are provided for arbitrary p. The special case in which Σ is assumed to be diagonal is treated separately. The analogous problem, with Σ assumed to be $\lambda\Sigma_0$ (Σ_0 known, λ unknown) has been investigated by C. B. Read [*Ann. Math. Statist.* **41** (1970) 1133]. (Received June 23, 1971.)

131-72. Ratio and regression estimators with distinct units. PODURI S. R. S. RAO, The University of Rochester.

Suppose that from a finite population of N units, samples of sizes n' and $n(n' \geqq n)$ are drawn without replacement and independent of each other. It is well known that the sample mean based on the v distinct units is a sufficient statistic for the population mean. In this paper, we show that the Ratio and Regression estimators based on the distinct units have smaller biases and variances than the corresponding estimators given in the literature, for instance in Cochran [*Sampling Techniques* (1963) 334–341]. The variances of the above estimators depend on $E(1/v)$, which is cumbersome to compute. We obtain two expressions to this quantity; the first one has only $(n+1)$ terms in it and the second has $(N-n'+1)$ terms. The second expression is obtained by an extension of the method of Korwar and Serfling [*Ann. Math. Statist.* **41** (1970) 2132–2134] and reduces to that of Pathak [*Ann. Math. Statist.* **35** (1964) 795–808] when $n = n'$. We also consider a different procedure where the number of distinct units d is fixed in advance. In this case, the sample size clearly is a random variable. We discuss the estimation procedure for this situation. (Received June 23, 1971.)

131-73. Asymptotic distribution of the log-likelihood ratio based on ranks in the two-sample problem II (preliminary report). DOUGLAS H. JONES AND J. SETHURAMAN, Florida State University.

Continuing the work of Savage and Sethuraman with the above name in the Sixth Berkeley Symposium we establish the asymptotic normality of l_N (defined below) under the alternative hypothesis under weaker conditions. Let $X_1, \cdots, X_m(Y_1, \cdots, Y_n)$ be independently distributed with a common distribution function $F_1^*(F_2^*)$. Let $N = m+n$ and $\mathbf{R}_N$ be the vector of ranks of the combined sample. Let $l_N = \log \{P(\mathbf{R}_N = \mathbf{r} \mid F_1^* = F_1, \ F_2^* = F_2)/P(\mathbf{R}_N = \mathbf{r} \mid F_1^* = F_2^*)\}$. Let $\lambda_N = n/N$ and $\lambda_N \to \lambda$, $0 < \lambda < 1$. Let $H = \lambda F_1 + (1-\lambda)F_2$ and $U_j = F_j H^{-1}$ have a probability density function u_j, $j = 1, 2$. Let $\phi(t) = \log \{u_1(t)/u_2(t)\}$. We assume that $\phi(t)$ is strictly increasing and absolutely continuous on (0, 1) and $|\phi^{(i)}(t)| \leqq K[t(1-t)]^{-i-\frac{1}{2}+\delta}$ for $i = 0, 1$ with $\delta > 0$ and $K < \infty$. Then, if $F_1^* = F_1$, and $F_2^* = F_2$, $[l_N - N\lambda \int u_1 \log u_1 \, dt - N(1-\lambda) \int u_2 \log u_2 \, dt]/N^{\frac{1}{2}}$ has a limiting normal distribution with mean zero and finite variance. (Received June, 24, 1971.)

131-74. Asymptotic results for goodness-of-fit statistics when parameters are not known. MICHAEL A. STEPHENS, McGill University and Nottingham University.

The goodness-of-fit statistics associated with the names of Cramér–von Mises, Watson, and Anderson–Darling, and called W^2, U^2 and A respectively, cannot be used with the usual tables when parameters of the tested distribution must be estimated from the data. The asymptotic distributions are investigated in this situation for the cases when the distribution tested is the normal, with either mean or variance or both to be estimated, and the exponential, when the scale parameter must be estimated. Significance points are given and modified forms for use with small samples. For these cases the tests are therefore fully operational, and are found to have very good power properties. (Received June 24, 1971.)

131-75. Economical second order designs in sequences of partially balanced factorial fractions of 3^n (preliminary report). ALBERT HOKE, Columbia University.

We present in this paper some new designs in the 3^n series which allow the fitting of a second order model in stages for any number of factors $n \geqq 3$. At stage (i) the model includes only the mean and linear components of main effects. At stage (ii) quadratic components are introduced and at stage (iii) linear x linear two-factor interactions. The designs are multidimensional partially balanced arrays in the sense of Bose and Srivastava (*Sankhyā Ser. A* **26** 145–168) and require fewer runs than other designs in the literature. Indeed we produce a saturated version of the designs at each stage. When inverting the information matrix $X'X$, it has not been found feasible to use the Bose–Srivastava algorithm. Instead we develop a technique for deriving the Moore–Penrose inverse X^+ for any n. From this the variance–covariance and aliasing structures follow. Our best design estimates the $p(n) = (n+1)(n+2)/2$ parameters in the second order model in $p(n)+n$ runs. The trace of the covariance matrix of the estimates compares well with that for the designs of Box and Behnken (*Technometrics* **2** (1960) 455–475). E.g., when the definition of effects is the same, our $3^7//43$ has trace 1.75, their $3^7//62$ a trace of 3.07. (Received June 24, 1971.)

131-76. Some results concerning restricted linear models. GEORGE A. MILLIKEN, Kansas State University.

The restricted linear model with fixed effects can be defined as $\mathbf{y} = \mathbf{X}\boldsymbol{\beta}+\mathbf{e}$ with restrictions $\mathbf{H}\boldsymbol{\beta} = \mathbf{b}$ where $\mathbf{y}$ is a $n\times 1$ random observation vector, $\mathbf{X}$ is a $n\times p$ matrix of known constants of rank q with $q \leqq p$, $\mathbf{H}$ is a $k\times p$ matrix of known constants of rank h with $h \leqq k$, $\boldsymbol{\beta}$ is a $p\times 1$ vector of unknown parameters, $\mathbf{b}$ is a $k\times 1$ vector of constants such that the system of equations $\mathbf{H}\boldsymbol{\beta} = \mathbf{b}$ are consistent, $\mathbf{e}$ is a $n\times 1$ unobserved random vector with $\mathbf{e} \sim N(\mathbf{0}, \sigma^2\mathbf{I}_n)$ where σ^2 is an unknown parameter and $\mathbf{I}_n$ is a $n\times n$ identity matrix. The restricted linear model is first expressed as the new linear model $\mathbf{y}-\mathbf{XH}^-\mathbf{b} = \mathbf{X}(\mathbf{I} = \mathbf{H}^-\mathbf{H})\boldsymbol{\beta}+\mathbf{e}$ where the matrix $\mathbf{H}^-$ denotes the Moore–Penrose generalized inverse of the matrix $\mathbf{H}$ and $\mathbf{X}(\mathbf{I}-\mathbf{H}^-\mathbf{H})$ is a matrix of rank r. The concept of estimability for the restricted linear model is then defined in terms of this new linear model. It is proved that the linear combinations $\mathbf{A}\boldsymbol{\beta}$ where $\mathbf{A}$ is a $m\times p$ matrix of known constants of rank m are estimable if and only if the trace of $\mathbf{X}(\mathbf{I}-\mathbf{R}^-\mathbf{R})(\mathbf{I}-\mathbf{A}^-\mathbf{A})[\mathbf{X}(\mathbf{I}-\mathbf{R}^-\mathbf{R})(\mathbf{I}-\mathbf{A}^-)\mathbf{A}]^-$ is $r-m$. A hypothesis, say $\mathbf{D}\boldsymbol{\beta} = \mathbf{d}$ against the alternative hypothesis $\mathbf{D}\boldsymbol{\beta} \neq \mathbf{d}$, where $\mathbf{D}$ is a

$m \times p$ matrix of known constants of rank m and $\mathbf{d}$ is a $m \times 1$ vector of known constants such that the system of equations $\mathbf{D}\boldsymbol{\beta} = \mathbf{d}$ are consistent, is defined to be testable if and only if the linear combinations $\mathbf{D}\boldsymbol{\beta}$ are also estimable for the new linear model. The sum of squares corresponding to such a testable hypothesis is obtained. The results are very applicable to computer programming and an example is presented with results from a computer program. (Received June 24, 1971.)

131-77. Asymptotic results for censored samples. RICHARD A. JOHNSON, University of Wisconsin.

We consider the life testing problem where a random sample from a population with pdf $f(x; \theta)$ is censored at the rth order statistic. Under the mild conditions for quadratic mean differentiability of a function of $f(x; \theta)$, we obtain an approximation to the likelihood which leads to the representation for limit distributions of regular estimators given by Hájek [*Z. Warhrscheinlichkeitstheorie und Verw. Gebiete* (1970) 323–330]. Thus, we obtain asymptotic lower bounds on the concentration of point estimates of θ. A family of distributions, based on a sum of functions of the first r order statistics and the hazard rate evaluated at the rth order statistic, is constructed which is asymptotically sufficient. From this we obtain asymptotically optimal tests. Namely, when θ is real valued, we obtain asymptotically most powerful tests of $H_0: \theta = \theta_0$ against one-sided alternatives. Asymptotically most powerful unbiased tests are found for two-sided alternatives. Other optimality criteria are met in the multiparameter case. (Received June 24, 1971.)

131-78. A recursion relation for the joint density of the sample mean and variance. R. C. SANSING, National Bureau of Standards.

The density of the t-statistic when samples are drawn from nonnormal parent populations has been in question for a very long time. The purpose of this work is to provide tools for answering those questions. Recursion relations for the sample mean and variance are used to derive a recursion relation for the joint density of the sample mean and variance. This recursion relation is restricted to independent observations from a population with a density function that is positive almost everywhere. The joint density is given for sample size 2 for use with the recursion relation and the recursion relation is used to display the joint density for sample size 3. The recursion relation and the joint density for sample size 2 are used to prove by induction that the density of the t-statistic is symmetric about zero whenever the underlying population density is symmetric about zero. (Received June 24, 1971.)

131-79. On a representation of the Hilbert space of a process with applications to Kalman filtering. WILLIAM TUCKER, Southern Methodist University.

In a recent paper Kailath (*IEEE Trans. Auto. Cont.* **AC-13** (1968) 646–655) has employed an innovations theorem and certain integral representations of the elements in the Hilbert Space of a process in solving the linear least-squares prediction problem. The proof of the innovations theorem employed by Kailath is based on the best least-squares predictor (Kailath, *IEEE Trans. Information Theory* **IT-15** (1969) 350–369), that is, on the conditional expectation. Thus Kailath's approach applies only when the conditional expectation is linear. By considering integral representations of processes with orthogonal increments it is possible to generalize the stochastic integrals employed by Kailath and to prove a version of an innovations theorem which applies in linear least-squares prediction. It is also possible to generalize the Kalman filtering equations. (Received June 24, 1971.)

131-80. Some bounds on the distribution functions of linear combinations. GOVIND S. MUDHOLKAR AND SIDDHARTHA R. DALAL, The University of Rochester.

Let $X_1, X_2, \cdots, X_n$ be n nonnegative, independent, identically distributed random variables. Although it is often easy to obtain the distribution of $X_1+X_2+\cdots+X_n$ it is not so easy to obtain that of $\lambda_1X_1+\lambda_2X_2+\cdots+\lambda_nX_n$, $\lambda_i > 0$, $i = 1, 2, \cdots, n$; e.g. X's may be chi-square random variables. The primary purpose of this article is to present some bounds on the distribution function of $\lambda_1X_1+\lambda_2X_2+\cdots+\lambda_nX_n$ in terms of the distribution function of $X_1+X_2+\cdots+X_n$. However we have proved a more general result. The bounds for the linear combinations are derived as particular cases of inequalities on distribution functions of a variety of functions $\psi(\lambda_1X_1, \lambda_2X_2, \cdots, \lambda_nX_n)$ in terms of the distribution functions of $\psi(X_1, X_2, \cdots, X_n)$, where the functions ψ satisfy some symmetry and convexity or concavity-type conditions. In particular it is shown that, for independently distributed chi-square random variables $\Pr\{G^{n/k}\chi_k{}^2+\chi_{m-k}^2 \leqq c\} \leqq \Pr\{\sum_{i=1}^n \lambda_i\chi_{b_i}^2 \leqq c\} \leqq \Pr\{G\chi_m{}^2 \leqq c)$, where $m = \sum_{i=1}^n b_i$, $G = (\prod_{i=1}^n \lambda_i{}^{b_i})^{1/m}$ and $1 \leqq k \leqq (n \log G)/\max_i \log \lambda_i$. (Received June 24, 1971.)

131-81. Asymptotic Bayes risk for infinite dimensional two-sided test of location (preliminary report). P. L. COHEN, Purdue University.

Let X_1, X_2—be independent normal with mean vector θ and covariance matrix I. Let the null hypothesis be $\theta = 0$ and the alternative $\theta \neq 0$. Let the prior also be normal with zero mean and covariance matrix Σ. Quadratic loss ($\theta' A\theta$) and constant loss are considered. Rubin and Sethuraman (*Sankhyā Ser. A* **27** (1965) 347–356) obtained order results for the above test in the case of a finite dimensional parameter space. New results have been obtained for the case in which the parameter space is infinite dimensional. This development is motivated by the need to extend the Bayes risk efficiency analysis to time series problems and to problems in which the alternative is a function space. For the case of constant loss, exact results have been obtained if the characteristic roots of Σ are pairs of $1/i$ or $1/i^2$. Order results have been obtained for the characteristic roots $\lambda_i = i^{-\rho}$, $\rho > .5$, or $\lambda_i = a^i$, $0 < a < 1$. In the latter case one still obtains the Rubin–Sethuraman result that the type 2 risk, R_2, is asymptotic to $R_1 \log n$ where n is the sample size. In the former case it can be shown that the risks are asymptotically proportional. A generalized expression for the proportionality constant is obtained. Similar order results are obtained for quadratic loss. (Received June 24, 1971.)

131-82. A generalization of Dynkin's identity. KRISHNA B. ATHREYA AND THOMAS G. KURTZ, University of Wisconsin.

Let $\{T_t : t \geqq 0\}$ be a semigroup of contraction operators on a Banach space B with A as its infinitesimal generator. Let $f \in \bigcap_{j=1}^{n} D(A^j)$ where $D(A^r)$ is the domain of the operator A^r. Then,

$$f = \sum_{k=0}^{n-1} (-1)^k \frac{t^k}{k!} T_t A^k f + \frac{(-1)^n}{(n-1)!} \int_0^t u^{n-1} T_u A^n f \, du.$$

If $\{T_t\}$ corresponds to a standard Markov process $\{X_t; t \geqq 0\}$ and τ is a stopping time for this process then $E_x \tau < \infty$ implies

$$E_x f(X(\tau)) = f(x) + \sum_{k=1}^{n} \frac{(-1)^k}{k!} E_x(\tau^k A^k f(X(\tau)) + \frac{(-1)^n}{(n-1)!} E_x\left(\int_0^\tau u^{n-1} A^n f(X(u)) \, du\right).$$

This generalization of Dynkin's identity is used to compute moments of stopped Markov process. Applications to processes with independent increments yield known results in a simple manner. (Received June 24, 1971.)

131-83. Some results on extreme-value theory for dependent variates. ROY E. WELSCH, Massachusetts Institute of Technology.

Sufficient conditions are given for the weak convergence in the Skorohod space $D^d[a, b]$ of the processes $\{(Y_{1,[nt]} - b_n)/a_n, (Y_{2,[nt]} - b_n)/a_n, \cdots, (Y_{d,[nt]} - b_n)/a_n\}$, $0 < a \leqq t \leqq b$, where $Y_{i,n}$ is the ith largest among $\{X_1, X_2, \cdots, X_n\}$, a_n and b_n are normalizing constants, and $\langle X_n : n \geqq 1 \rangle$ is stationary and strong-mixing or Gaussian or satisfies Berman's mixing condition (*Ann. Math. Statist.* **35** 503). Possible limit laws for a sequence of normalized extreme order statistics (maximum, second maximum, etc.) from a stationary strong-mixing sequence are described.

The results show that the possible limit laws for the kth maximum process ($k > 1$) from a strong-mixing sequence form a larger class than can occur in the independent case. This extends the work of Loynes (*Ann. Math. Statist.* **36** 993–999) who treated only the maximum process. (Received June 24, 1971.)

131-84. Best invariant confidence bands for a continuous cumulative distribution function (preliminary report). E. G. PHADIA. Ohio State University.

Let $\mathbf{X} = (X_1, X_2 \cdots, X_n)$ represent a random sample of size n from a continuous cumulative distribution function F, and let $\varphi\ (X, t)$ be a right continuous, non-decreasing function on the real line R^1 such that $\varphi(\mathbf{X}, -\infty) = 0$, $\varphi(\mathbf{X}, \infty) \leqq 1$. We define $\varphi(\mathbf{X}, t)$ to be a α-level best invariant lower confidence band for a continuous cumulative distribution function F, if $P(F(t) \geqq \varphi(\mathbf{X}, t), \forall\, t \in R^1) = \alpha$ and $\varphi(\mathbf{X}, t)$ is best among invariant rules under a loss function L and group of transformation $\mathscr{G}$. In this paper we take averaged squared error loss function and the group of all one-to-one monotone transformations of the R^1 onto themselves which leave the sample values invariant and find lower confidence bands. These bands are essentially step functions. In an analogous way we define two-sided α-level asymmetric best invariant confidence bands and obtain them under certain loss functions and the group described above. (Received June 25, 1971.)

131.85. A new approach to the construction of confounded asymmetrical factorial designs (preliminary report). JAGDISH N. SRIVASTAVA, Colorado State University.

In this paper, an entirely new approach to confounding theory is developed. For the construction of an $s_1 \times s_2 \times \cdots \times s_m$ confounded design, this approach essentially uses certain appropriate $(m+1)$ association schemes. (A similar technique has been used by Srivastava and Anderson (*Ann. Math. Statist.* (1971)) for the construction of MDPB designs, which are essentially certain kinds of main effect plans for asymmetrical factorials). Using this approach, a large class of new confounded plans have been obtained for both symmetrical and asymmetrical factorials. For illustration, we mention a few designs with $m = 2$. (Below, k denotes block size.) The values of $(s_1, s_2; k)$ for these designs are (2, 5; 2), (2, 7; 2), (3, 5; 3), (3, 6; 3), (4, 6; 4), (4, 6; 3), (5, 5; 3), (5, 10; 4), (6, 6; 5), (6, 6; 4), (6, 9; 3), (8, 8; 5), (9, 9; 5), (10, 15; 3), (5, 10; 10), (3, 10; 6), (2, 21; 6). We stress that this is only a small subset of the 2-factor designs obtained. In addition, designs with 3 and more factors are also constructed. Moreover, the problem of balancing is studied, and designs with a relatively small number of replications obtained. This approach seems to generate confounded designs with structure different from that of the designs obtained using the standard Kishen–Srivastava (*J. Indian Soc. Agric. Statist.* (1960)) theory, and also designs discovered by various other authors. (Received June 30, 1971.)

131-86. Some Monte Carlo studies on estimating the dispersion matrix of a bivariate normal population using incomplete samples (preliminary report).
J. N. SRIVASTAVA AND M. K. ZAATAR, Colorado State University.

Let (X_1, X_2) be distributed as a bivariate normal with mean zero and dispersion matrix $\Sigma = ((\sigma_{ij}))$. Suppose that a sample S_{12} of size n_{12} from this bivariate population is given, along with a sample $S_i (i = 1, 2)$ of size n_i from the marginal of X_i. In this paper, we compare four methods of estimating Σ by using S_1, S_2 and S_{12}. (I) *Wilks estimator.* The estimator $\hat{\sigma}_{ii}(i = 1, 2)$ of σ_{ii} is the usual one obtained by using all the observations on X_i alone, and ignoring the other response. Also, σ_{12} is estimated using S_{12} alone. (II) *An estimator proposed by one of the authors (Srivastava).* Here, σ_{ii} is estimated as in Wilks case, and the estimate of σ_{12} is $n(\sigma_{11}\sigma_{22})^{\frac{1}{2}}$, when r is the estimate of the correlation, using S_{12}. (III) *Hocking–Smith estimator.* (See *J. Amer. Statist. Assoc.* (1968)). (IV) *Maximum likelihood estimator.* Note that when n_1 and n_2 are zero, methods (III) and (IV) coincide. Let $\rho = \sigma_{12}(\sigma_{11}\sigma_{22})^{-\frac{1}{2}}$. In the Monte Carlo studies, we used samples of size 500, and considered the following sets of values of (n_1, n_2, n_{12}): (6, 14, 10), (14, 4, 10), (4, 4, 16), (20, 0, 10), (0, 20, 10), (3, 7, 5), (7, 3, 5) and (2, 2, 8). For each set, we took $\sigma_{11} = 1$, $\sigma_{22} = 1, 2, 5, 10$, and $\rho = -0.7, -0.3, 0.1, 0.5, 0.9$. The results indicate that Method III is the worst (with respect to the criteria of the trace and the determinant of the variance matrix of the estimators) almost all the time, and II is best in almost 70 per cent cases. Method IV is almost always best when $\rho = 0.9$, and occasionally also when $\rho = -0.7$. The variation of σ_{22} did not affect the trace or determinant (for I, II and IV), apart from the change in scale. (Received June 30, 1971.)

131-87. Minimum variance unbiased estimation for truncated Poisson and binomial distributions. T. CACOULLOS AND CH. CHARALAMBIDES, University of Athens.

The Poincare formula and certain combinatorial results are employed for the derivation of the distribution of (a) the sum of independent but not necessarily identically distributed Poisson random variables truncated away from the set $A = \{0, 1, \cdots, c\}$ for some integer c, (b) the sum of independent and identically distributed binomial variables truncated away from $A = \{0, 1, \cdots, c\}$. The result in (a) gives as a special case the n-fold convolution of the truncated Poisson distribution, first obtained by Tate and Goen (*Ann. Math. Statist* (1958) 755–765) by the use of characteristic functions. The neater expressions obtained here led to an expression for the MVU estimator $\tilde{\lambda}c$ of λ in terms of the Stirling numbers of the second kind. This makes possible the computation of $\tilde{\lambda}c$ by using the tables of Tate and Goen also for the general case of $c > 0$. In the binomial case the distribution of the sufficient statistic was also used to provide the MVU estimator $\tilde{\theta}_c$ of the odds ratio $\theta = p/(1-p)$. The n-fold convolution of the truncated binomial and the expression for $\tilde{\theta}_c$ motivated the study of certain numbers, analogues of the Stirling numbers for the Poisson case. (Received July 6, 1971.)

(Abstracts contributed by title)

71T-63. Empirical Bayes estimation via the Dirichlet process. R. M. KORWAR AND M. HOLLANDER. The Florida State University.

Let (P_i, X_i), $i = 1, 2, \cdots$ be a sequence of independent pairs of random variables, the P_i having a common prior distribution given by a Dirichlet process (Ferguson, UCLA Technical Report (1969)) on (R, β) with a parameter α (assumed σ-additive), and given $P_i = P$, $X_i = (X_{i1}, \cdots, X_{im})$ is a sample of size m from P. Consider the problem of estimating $F_{n+1}(t) = P_{n+1}\{(-\infty, t]\}$ on the basis of $X_1, \cdots, X_{n+1}$. Assume α is non-atomic, $\alpha(R)$ is known, and take the loss in estimating $F(t)$ by $a(t)$ to be $\int (F(t)-a(t))^2\, dW(t)$ where W is a weight function. Define $G_n(t) = p_m(\sum_{i=1}^n \hat{F}_i(t)/n)+(1-p_m)\hat{F}_{n+1}(t)$ where $p_m = [\alpha(R)/(\alpha(R)+m)]$ and $\hat{F}_i$ is the empirical distribution function for the sample $X_{i1}, \cdots, X_{im}$. Then the sequence $\{G_n\}$ is shown to be asymptotically optimal in the sense of Robbins (cf. *Ann. Math. Statist.* **35** (1964) 1–20). The proof utilizes a result giving the joint distribution of a sample from a Dirichlet process. The empirical Bayes approach is applied to other problems, including estimation of a mean and estimation of $P(X < Y)$. (Received June 11, 1971.)

71T-64. Convolution of independent left-truncated negative binomial variables and limiting distributions. J. C. AHUJA AND E. A. ENNEKING, Portland State University.

Let X_i $(i = 1, 2, \cdots, n)$ be n independent and identically distributed random variables having the left-truncated negative binomial distribution $f_c(x; k, \theta) = \binom{x+k-1}{x}\theta^x/g_c(k,\theta)$, $x \in T$, where $0 < \theta < 1$, $k > 0$, $g_c(k, \theta) = \sum_{x=c+1}^{\infty} \binom{x+k-1}{x}\theta^x$, and $T = \{c+1, c+2, \cdots, \infty\}$. Let Z denote their sum. The distribution of Z for $c = 0$ has been recently obtained by the author [*Ann. Math. Statist.* **42** (1971) 383–384] which we call the associated Lah distribution. In this paper we derive the distribution of Z for the general case in terms of the generalized Lah numbers which we name the generalized Lah distribution. A recurrence relation is provided for the generalized Lah numbers and is utilized to obtain a recurrence formula for the probability function of Z. The distribution function of Z is found in an explicit form in terms of a linear combination of the incomplete beta functions. It is shown further that, under certain limiting conditions, the generalized Lah distribution approaches the generalized Stirling distribution of the first kind and the generalized Stirling distribution of the second kind obtained by the author [*Ann. Math. Statist.* **42** (1971) 846] and Tate and Goen [*Ann. Math. Statist.* **29** (1958) 755–765] respectively. (Received June 14, 1971.)

71T-65. Minimum bias approximation of a general regression model by a relatively simpler one. ROBERT J. HADER, ALLISON R. MANSON AND ROBERT COTE, North Carolina State University.

It is assumed that the true response to be approximated is of the form $\eta(x) = \sum_{j=1}^{n} \theta_j f_j(x)$ with $x \in \mathscr{X}$ where $\mathscr{X}$ is a compact set; the f_j are linearly independent, real-valued, continuous functions defined on $\mathscr{X}$ and belonging to $L_2(\mathscr{X}, \mathscr{A}, \mu)$; and $(\mathscr{X}, \mathscr{A}, \mu)$ is a finite measure space. A class of simpler models $\eta_0(x)$ is defined such that $\mathscr{C} = \{\eta_0(x): \eta_0(x) = \sum_{j=1}^{s} \alpha_j g_j(x), s \leqq n\}$. A model is chosen from this class in order to minimize the averaged squared bias (B) over $\mathscr{X}$. The chosen model is fitted by the method of least squares; with $N \geqq s$ observations $Y(x_k)$, $k = 1, 2, \cdots, N$; with $E(Y(x_k)) = \eta(x_k)$; and with known correlation apart from a constant σ^2. Subject to achieving minimum B this method, which is a generalization of the Karson, Manson, Hader [*Technometrics* **11** (1969) 461–475] approach, offers a great deal of design flexibility which may be used to minimize the averaged variance of $\hat{\eta}_0(x)$ over $\mathscr{X}$. The method developed is applied to fit the ratio of polynomials by simple polynomials. Some a priori knowledge of certain parameters is assumed. For example, $\eta(x) = (\theta_0 + \theta_1 x + \theta_2 x^2)/(\gamma + x)$ is approximated both by linear and by quadratic polynomials $\hat{\eta}_0(x)$. Symmetric designs in the variate x, satisfying both the bias and variance criterian, are obtained. These designs depend on the particular values of γ for which minimum B is attained. (Received June 15, 1971.)

71T-66. Minimum bias estimation and designs for 3 and 4 component mixtures. GEORGE A. PAKU, A. R. MANSON, AND L. A. NELSON, North Carolina State University.

Draper and Lawrence [*J. Roy. Statist. Soc. Ser. B.* **27** (1965) 450–465 and 473–478], using procedures developed by Box and Draper, obtained for 3- and 4-component mixtures, designs which compensate for model inadequacies. Using the criterion of minimizing the mean square error integrated over an appropriate region, R, they investigated situations where the true model is a polynomial of degree $d(2, 3)$ with the true response $\boldsymbol{\eta} = X_1\boldsymbol{\beta}_1 + X_2\boldsymbol{\beta}_2$. This response was fitted by a polynomial of degree $d-1$, $\hat{\mathbf{Y}} = X_1\mathbf{b}_1{}^*$, using standard least squares methods. An alternative approach was introduced by Karson, Manson and Hader [*Technometrics* **11** (1969) 461–475]. First by choice of an estimator $\mathbf{b}_1 = T'\mathbf{Y}$, we directly minimize the bias contribution to integrated mean square error (IMSE). The estimator obtained gives the same minimum bias for any design within a very large class of designs which satisfy a simple estimability condition. A design is then obtained from this class which minimizes the variance contribution to IMSE. Designs are given for the model situations mentioned above, up to inadequacy of degree two. The designs, in most cases, turn out to be contractions of Scheffé's simplex-lattice and simplex-centroid designs. These designs will have smaller IMSE than those designs developed by Draper and Lawrence which minimize the bias contribution to IMSE. (Received June 15, 1971.)

71T-67. Premium and protection of several procedures for dealing with outliers when sample sizes are moderate to large. IRWIN GUTTMAN, Université de Montréal.

We use the method of adjusted residuals of Tiao and Guttman (*Technometrics* **9** (1967) 541–559) to derive expressions for rules of the Winsorization and Semi-Winsorization type (see Guttman and Smith, *Technometrics* **11** (1969) 527–550 and **13** (1971) 101–112, respectively for the definitions of Winsorization and Semi-Winsorization rules). We discuss the cases of guarding against the possibility of 1 and 2 spurious observations, as well as the cases σ known and unknown, when sampling is thought to be from $N(\mu, \sigma^2)$. If spuriosity occurs, we assume it is from $N(\mu+a\sigma, \sigma^2)$. The rules contain a generic constant C, and computations are in progress to calculate the C's necessary to give the rules premium of .005, .01(.01).05, .10 for various $n \geqq 15$. The protections are compared. Included in this comparison is the protection of Anscombe's rule, found in Tiao and Guttman (cited above). (Received June 17, 1971.)

71T-68. Limit theorems for partial sums with drift (preliminary report). J. G. WENDEL, University of Michigan.

Let $\{X_n\}$ be i.i.d. with zero means and attracted to a stable law of index $\alpha \in (1, 2]$. For $\varepsilon > 0$ and the partial sums S_n define $M(\varepsilon) = \max\{S_n - n\varepsilon : n \geqq 0\}$, $N(\varepsilon) = \#\{n : S_n > n\varepsilon\}$, $T(\varepsilon) = \max\{n : S_n > n\varepsilon\}$ and $R_k(\varepsilon) = k$th largest difference $S_n - n\varepsilon$. Then, as $\varepsilon \to 0$ these rv's converge in law to functionals of the appropriate stable process $X(t)$. More precisely, let $V(x) = E(X_n^2; |X_n| \leqq x)$ and choose a_m, b_m to satisfy $V(a_m)/a_m^2 \sim V(b_m)/b_m \sim 1/m$ as $m \to \infty$. Classically $S_n/a_n \to X(1)$ in law. Let $\varepsilon = 1/m$ and write $M_m, \cdots$ for $M(\varepsilon), \cdots$. We show that M_m/b_m and N_m/mb_m converge respectively in law to $\max\{X(t)-t : t \geqq 0\}$ and to meas $\{t : X(t) > t\}$, with a similar result for T_m/mb_m; these generalize results of Robbins, Siegmund and Wendel (*Proc. Nat. Acad. Sci.* **61** (1968) 1228–1230) for the case $\alpha = 2$, $V(x) \to 1$. For the order statistic $R_{k,m}$ let $k/mb_m \to$ const.; then $R_{k,m}/b_m$ has a limit law. The proofs rest on known formulas for the maximum of partial sums and the like. (Received June 25, 1971.)

71T-69. Maximum likelihood estimation of parameters in renewal models. I. V. BASAWA, University of Sheffield.

Let $\{X(t), t \geqq 0\}$ be a renewal point process with interarrival time density $f(t; \theta_1, \cdots, \theta_k)$ which depends on k unknown real parameters. If a sample of, say, n between-event intervals is available, the problem of finding maximum likelihood (ML) estimates of $(\theta_1, \cdots, \theta_k)$ does not present any new difficulties. If, however, these intervals are unobservable one may be forced to use the realization of counts of events in certain predetermined intervals. But, except in the case of Poisson process, the distributions of counts of events in fixed intervals are not simple and consequently the likelihood function based on such a sample is usually difficult to handle. Kingman (*Ann. Math. Statist.* **34** (1963) 1217–1232) has studied some properties of randomized count processes derived from renewal processes. In this paper similar methods are used in obtaining a suitable realization of counts of events in random intervals (with exponential density). An inverse sampling plan is then used to construct the likelihood function. The suggested estimation procedure does not require the knowledge of $f(t; \theta_1, \cdots, \theta_k)$ explicitly; but its Laplace transform $f^*(s, \theta_1, \cdots, \theta_k)$ would suffice. (Received June 28, 1971.)

71T-70. Estimation of the autocorrelation coefficient in simple Markov chains. I. V. BASAWA, University of Sheffield.

Let $\{X_k\}$, $k = 1, 2, \cdots$ be a sequence of random variables forming a stationary ergodic Markov chain defined on a finite state-space. Many interesting aspects of dependence in such sequences can be studied in terms of the autocorrelation, $R = \text{Cov}\,(X_k, X_{k+1})/V(X_k)$. This paper is concerned with the problem of estimation of R. Four estimates based on the serial correlation are suggested, and their asymptotic normality established. Exact distributions of two of the estimates are given in terms of their moment generating functions. Also, maximum likelihood estimation is discussed for the case when the transition probabilities are functions of an unknown parameter. Finally, the efficiencies of the estimates are compared with the help of an example. (Received June 28, 1971.)

71T-71. A note on the equivalence of Gauss's principle and the principle of the minimum discrimination information. M. RAGHAVACHARI, Carnegie-Mellon University.

Recently L. L. Campbell in *Ann. Math. Statist.* **41** 1011–1015 has shown that in a context of choosing an unknown distribution function, the Gauss principle and the principle of the minimum discrimination information lead to the same choice. In this note this result is shown by relaxing many of the required assumptions and by making explicit some of the conditions required for the validity of results in Campbell's paper. The proof of the equivalence of the two principles is also shorter and more direct than that of Campbell. (Received July 1, 1971.)

71T-72. Asymptotic formulas for the non-null distributions of three statistics for multivariate linear hypothesis in the case of $n_e = n\rho_e$, $n_h = n\rho_h$ and $\Omega = O(n)$. YASUNORI FUJIKOSHI, Kōbe University.

Let S_e and S_h be the independent $p \times p$ matrices with central Wishart distribution $W_p(n_e, \Sigma)$ and noncentral Wishart distribution $W_p(n_h, \Sigma, \Omega)$. In the canonical form of multivariate linear hypothesis we can express the likelihood ratio statistic by $\lambda = |S_e(S_e+S_h)^{-1}|$, Hotelling's statistic by $T_0{}^2 = n_e \operatorname{tr} S_h S_e{}^{-1}$ and Pillai's statistic by $V = (n_e+n_h) \operatorname{tr} S_h(S_h+S_e)^{-1}$. Then n_e and n_h mean the degrees of freedom for the error and for the hypothesis respectively. Under the assumption that n_h is fixed, n_e tends to infinity and $\Omega = o(1)$, asymptotic expansions of the distributions of λ, T_0^2 and V have been obtained in terms of noncentral χ^2-distributions by several authors. In this paper we derive asymptotic expansions of the distributions of three statistics mentioned above, in the case of $n_e = n\rho_e$, $n_h = n\rho_h$, ($\rho_e > 0$, $\rho_h > 0$, $\rho_e + \rho_h = 1$) and $\Omega = O(n)$. New asymptotic formulas are given in terms of normal distribution and its derivatives up to order $O(n^{-1})$. (Received July 1, 1971.)

71T-73. Asymptotic formulas for the distributions of the determinant and the trace of a noncentral beta matrix. YASUNORI FUJIKOSHI, Kōbe University.

Let S_e and S_h be independently distributed as central Wishart distribution $W_p(r, \Sigma)$ and noncentral Wishart distribution $W_p(q, \Sigma, \Omega)$, respectively. Then a noncentral beta matrix is defined by $B = (S_e+S_h)^{-\frac{1}{2}}S_h(S_e+S_h)^{-\frac{1}{2}}$. It is the purpose of this paper to study the distributions of $|B|$ and tr B, which are Wilks' statistic and Pillai's statistic for testing a multivariate linear hypothesis, in the case when r is fixed and q tends to infinity. Under this assumption asymptotic expansions of the null and non-null distributions of $|B|$ and tr B are derived up to order q^{-3} and q^{-2}, respectively. The asymptotic expansions for these statistics are given in terms of central χ^2-distributions. (Received July 1, 1971.)

71T-74. Asymptotic expansions of the non-null distributions of two criteria for the linear hypothesis concerning complex multivariate normal populations. YASUNORI FUJIKOSHI, Kōbe University.

It is known that some distributions in the case of complex normal distributions can be obtained in the same manner as those in the case of real multivariate normal distributions. Such examples have been seen in the papers, e.g. Goodman [*Ann. Math. Statist.* **34** (1963) 152–177] and James [*Ann. Math. Statist.* **35** (1964) 475–501]. Recently, asymptotic expansions of the distributions of some tests in multivariate analysis have been obtained by the author [*J. Sci. Hiroshima Univ. Ser. A* **I** (**32**) 293–299, (**34**) 73–144] and Sugiura and Fujikoshi [*Ann. Math. Statist.* **40** (1969) 942–952], based on the hypergeometric function of matrix argument. In this paper we derive asymptotic expansions of the distributions of Hotelling's and Pillai's criteria for the linear hypothesis concerning complex normal populations which are extensions of some results due to the author to complex variates. For the purpose, we give the needed results on complex variates in a way parallel to one method of obtaining the asymptotic distributions of real multivariate analysis. (Received July 1, 1971.)

71T-75. On probabilities of large deviation of sums of random variables which are attracted to the normal law. V. K. ROHATGI, The Catholic University of America.

For a sequence $\{X_n\}$ of independent random variables identically distributed with common law $\mathscr{L}(X)$ and satisfying $P(X > x) \approx L(x)x^{-\rho}$ and $P(-X > x) \sim$

$L_1(x)x^{-\rho}$ where $\rho \geqq 2$ and L, L_1 are functions of slow variation such that $L_1(x)/L(x) \to p$ as $x \to \infty$, it is shown that $P(S_n > t_n) \sim nP(X > t_n)$ for certain sequences t_n. Here $S_n = \sum_1^n X_k$. (Received July 2, 1971.)

71T-76. Classes of optimal incomplete designs. V. G. KUROTSCHKA, University of Michigan.

Consider a three way layout, $Y_{ijkl} = a_i + b_j + c_k + Z_{ijkl}$, $i = 1, \cdots, I$; $j = 1, \cdots, J$; $k = 1, \cdots, K$; $1 - 0, 1, \cdots, n_{ijk}$. A specific choice of the system $\{n_{ijk}\}$ can be looked upon as a particular design of the three way layout. Under different assumptions on the availability of observations, classes of optimal designs have been recently investigated by P. S. Dwyer and V. Kurotschka. These results can be used to obtain optimality statements concerning incomplete block designs. These optimality statements include, modify, and extend the results of A. Wald (1943), M. Masuyama and T. Okuno (1957), A. M. Khirsagar (1958), V. L. Mote (1958) and others, as well as the more general results of J. Kiefer (1958) regarding models for two way elimination of heterogeneities. In particular, uniform optimality of some classical incomplete block designs and some replicated designs, as for example replicated Latin squares, within certain classes of admitted designs, can be established. (Received July 9, 1971.)

71T-77. Optimum design of three way layouts with interactions. V. G. KUROTSCHKA, The University of Michigan.

Optimum design of general three way layouts without interactions has been recently [abstract *Ann. Math. Statist.* **42** (1971) 1483] discussed by P. S. Dwyer and V. G. Kurotschka. Applications of the obtained results to incomplete settings have been given at the Symposium on Symmetric Functions in Statistics (Windsor, Canada, 1971) by V. G. Kurotschka [Abstract *Ann. Math. Statist.* **42** (1971)]. Optimum design of three way layouts with special types of interactions can be investigated by using the above results. In particular, if interactions of only one pair of factors at the given levels are present and of interest, the design problem can be reduced to the design problem of a general two way layout without interaction. For this case results of J. Kiefer [*Ann. Math. Statist.* **29** (1958) 675–699] and of V. G. Kurotschka [*Metrika* **16** (1970) 30–47] where uniformly optimum design were characterized provide optimality conditions for the designs. If interaction between all factors is present and of interest, the design problems can be reduced to that of a one way classification, and, again, known results can be applied. Intermediate cases are investigated separately and conditions for uniformly optimal designs—as far as these exist—are given. For cases where they do not exist optimal designs obeying weaker optimality conditions (D-optimality and A-optimality) are derived by studying the corresponding functionals of the relevant information matrices. (Received July 9, 1971.)

71T-78. Six central limit theorems for linear combinations of random variables.
MARTIN G. WEINRICH, University of Michigan.

Let $X_n = \sum_{i=1}^{k_n} a_{n_i} Y_{n_i}$ be a weighted sum of the components of the vector Y_n. Assume that Y_n has finite second moments, and that Var $(X_n) = 1$. Necessary and sufficient conditions that $\mathfrak{L}(X_n - EX_n) \to \mathfrak{N}(0, 1)$ are determined when all the vectors Y_n have distributions from a single one of the following six families: Y_n is a vector of independent gamma, Poisson, or compound Poisson distributed random variables, or Y_n is a vector of random variables with a joint Dirichlet, multinomial, or Dirichlet-multinomial distribution. The condition resembles the Lindeberg–Feller condition in that it states that $\forall\, \delta > 0$, $\lim_{n\to\infty} T_n(\delta) = 0$, but here $T_n(\delta)$ is expressed in terms of the coefficients $\{a_{n_i}\}_{i=1}^{k_n}$ and the parameters of the distribution of Y_n. The theorem for the Dirichlet-multinomial case provides a direct Bayesian analogue to Erdös and Rényi's (*Publ. Math. Inst. Hungar. Acad. Sci.* **4** (1959) 49–57) and Hájek's (*Publ. Math. Inst. Hungar. Acad. Sci.* **5** (1960) 361–374) results on asymptotic normality of sample means from finite populations. (Received July 13, 1971.)

71T-79. Multivariate Bayesian stratified sampling design. MARTIN G. WEINRICH, University of Michigan.

A general Bayesian model for sampling from a stratified population is developed. It incorporates only three assumptions: that the elements within each stratum are indistinguishable, that all variances are finite, and that the regression of the variate values associated with the next element drawn from each stratum on the set of variate values in a sample from the strata is linear in the set of vectors of stratum sample means. This model includes many standard Bayesian parametric models. The dimension of the vectors may vary among the strata, the strata may be dependent a priori, and each stratum may contain finitely or infinitely many elements. Specific algorithms are found for choosing a vector of stratum sample sizes so as to minimize the preposterior expectation of a generalized quadratic loss under a cost constraint of the form $\sum c_j n_j \leqq c$ in two special cases: (i) Independent strata a priori; (ii) Scalar observations in all strata. These models generalize those considered by Ericson (*J. Amer. Statist. Assoc.* **60** (1965) 750–771, *J. Roy. Statist. Soc. Ser. B* **31** (1969) 195–233, and in *New Developments in Survey Sampling*, ed. N. L. Johnson and H. Smith, Wiley (1969) 326–357). A proof is given in the general case of the convexity of the risk function, which may therefore be minimized using conventional convex programming methods. (Received July 13, 1971.)

71T-80. A test for weak bandwidth stationarity. EDWARD L. MELNICK, New York University.

A procedure is presented for testing the null hypothesis that a harmonizable process $[x(t), t \in T]$ is weakly stationary. This approach is based upon the fact that a real harmonizable process can be represented as $x(t) = \int_0^\infty \cos \omega t \, du(\omega) + \int_0^\infty \sin \omega t \, dv(\omega)$. The technique of complex demodulation is used for estimating the random variables $du(\omega)$ and $dv(\omega)$. To investigate the assumption of weak stationarity, the estimated variables from predetermined frequency bands are subjected to statistical tests. If the null hypothesis is not rejected the process is said to be weakly bandwidth stationary. If all bands satisfy this property and if the variables from one band are statistically independent of the variables from any other band, the process is weakly stationary. The tests are illustrated by application to examples constructed by Priestley and Rao (*J. Roy. Statist. Soc. Ser. B* (1969)). (Received July 15, 1971.)

71T-81. Pseudo-Bayes estimation of multinomial probabilities: asymptotic results (preliminary report). S. E. FIENBERG, P. W. HOLLAND AND M. SUTHERLAND, University of Chicago and Harvard University.

$X = (X_1, \cdots, X_t)$ has a multinomial distribution: $\sum_1^t X_i = n$, $E(X_i) = np_i$, $\mathbf{p} = (p_1, \cdots, p_t)$. Let $\boldsymbol{\lambda} = (\lambda_1, \cdots \lambda_t)$ be another probability vector, and set $\hat{K} = (n^2 - \sum_1^t X_i^2)/\sum_1^t (X_i - n\lambda_i)^2$, $\hat{W} = n/(n+\hat{K})$ and $\mathbf{T} = \hat{W}\mathbf{X} + (1 - \hat{W})n\boldsymbol{\lambda}$. Elsewhere we have proposed $\hat{\mathbf{T}}$ as an estimator of $n\mathbf{p}$. Now, let $t \to \infty$ and $n \to \infty$ with $\delta = n/t$ fixed. Set $p_i = p(x_i)1/t$ where $p(x)$ is a smooth density on [0, 1] and x_i is such that $p(x_i)1/t = \int_{(i-1)/t}^{i/t} p(u)\,du$. Similarly set $\lambda_i = \lambda(x_i)1/t$ for another smooth density $\lambda(x)$ on [0, 1], and let the risk function of $\hat{\mathbf{T}}$ be $R(\hat{\mathbf{T}}, \mathbf{p}) = E_{\mathbf{p}}||\mathbf{T} - n\mathbf{p}||^2$. As $t \to \infty$ the ratio of the risk of $\hat{\mathbf{T}}$ to that of the usual estimator, $\mathbf{X}$, is given by $H(D) + O(1/t)$ where $H(D) = (D^2 + 3D + 1)/(D^2 + 4D + 4)$ and $D = \delta \int_0^1 (\lambda(u) - p(u))^2 \, du$. Since $\frac{1}{4} \leqq H < 1$, this result implies that for large t, $\hat{\mathbf{T}}$ has uniformly smaller risk than $\mathbf{X}$ over an arbitrarily large portion of the probability simplex. (Received July 19, 1971.)

71T-82. Note on the estimation of the center of symmetry by rank estimates and linear combinations of order statistics. FRIEDRICH-WILHELM SCHOLZ, University of California, Berkeley.

Given a sample from a distribution function F with strongly unimodal and symmetric density f we are interested in estimating the center of symmetry of f. We consider linear combinations of order statistics (L-estimates) and estimates derived from rank tests (R-estimates). Under regularity conditions on F Jaeckel showed (*Ann. Math. Statist.* **42** (1971) 1020–1034) that one can construct an L-estimate $L(F)$ and an R-estimate $R(F)$, which are efficient, i.e. the variance of their asymptotic distribution is $\{\int (f'/f)^2 f \, dx\}^{-1}$. Now assume that the sample comes from a symmetric distribution $H \in \mathscr{H}$, a class of distributions satisfying certain regularity conditions which guarantee asymptotic normality of $L(F)$ and $R(F)$. Let $\sigma_H^2\{L(F)\}$ and $\sigma_H^2\{R(F)\}$ denote the asymptotic variances of these two estimates under H. Then it is shown that $\sigma_H^2\{L(F)\} \geqq \sigma_H^2\{R(F)\}$ for all $H \in \mathscr{H}$. Equality holds only if $H = F$. When F is normal this result is known from the papers by Chernoff and Savage (*Ann. Math. Statist.* **29** (1958) 972–994) and Hodges and Lehmann (*Ann. Math. Statist.* (1963) 598–611). Other estimates are being investigated, among them the Huber type-two proposal where as above we denote by $M(F)$ the optimal estimate for F. If F is the logistic distribution we can show that $\sigma_H{}^2\{M(F)\} \geqq \sigma_H{}^2\{R(F)\}$ with equality only if $H = F$. (Received July 21, 1971.)

71T-83. A comparison of two estimators for polynomial regression of uncertain degree. CORWIN L. ATWOOD, Haile Sellassie I University.

We consider two estimators which have been proposed for estimating a polynomial regression function $g(x)$ for $-1 \leqq x \leqq 1$, when it is not certain whether the polynomial is of degree s or $s+1$. Let θ_2 be the coefficient of x^{s+1}. One type of estimator, denoted here $\hat{g}_C$, fits a polynomial of degree s or $s+1$ depending on whether the least squares estimator $\hat{\theta}_2$ is small or large, respectively. Another type, denoted $\hat{g}_{LC}$, estimates $f(x)$ by a linear combination of the least squares estimators of degree s and $s+1$ polynomials, where the linear combination used does not depend on the data. We prove that if the observed variables are normal with known variance σ^2, for any $\hat{g}_C$ there is a $\hat{g}_{LC}$ such that for $-1 \leqq x \leqq 1$ and θ_2 in some interval containing 0, MSE $\hat{g}_{LC}(x) \leqq$ MSE $\hat{g}_C(x)$. In two examples with $s = 1$ the intervals are substantial. (Received July 22, 1971.)

71T-84. Limit theorems for the distributions of the sums of a random number of random variables. D. SZÁSZ, Mathematical Institute of the Hungarian Academy of Science.

A necessary condition is given for the convergence of distributions of the sums of a random number of independent random variables. This is made on the basis of a theorem which gives sufficient conditions for the convergence of distributions of randomly stopped stochastic processes. The random indices are supposed to be independent of the sequence of summands. (Received July 27, 1971.)

71T-85. Asymptotic distribution of the likelihood function in the independent not identically distributed case. A. N. PHILIPPOU AND G. G. ROUSSAS, University of Wisconsin, Madison.

Let Θ be an open subset of R^k and for each $\theta \in \Theta$, let $X_1, \cdots, X_n$ be independent rv's defined on the probability space $(\mathscr{X}, \mathscr{A}, P_\theta)$, and let $p_{j,\theta}$ be the distribution of the rv X_j. Let $f_j(\cdot\,;\theta)$ be a specified version of the Radon–Nikodym derivative of $p_{j,\theta}$ with respect to a σ-finite measure μ and set $f_j(\theta) = f_j(X_j;\theta)$. Furthermore, for θ, $\theta^* \in \Theta$, set $\phi_j(\theta, \theta^*) = [f_j(\theta^*)/f_j(\theta)]^{\frac{1}{2}}$ and suppose that $\phi_j(\theta, \theta^*)$ is differentiable in quadratic mean (q.m.) with respect to θ^* at (θ, θ), when the probability measure P_θ is employed, with q.m. derivative $\dot{\phi}_j(\theta)$. Set $\Delta_n(\theta) = 2n^{-\frac{1}{2}} \sum_{j=1}^n \dot{\phi}_j(\theta)$, $\Gamma_j(\theta) = 4\mathscr{E}_\theta[\dot{\phi}_j(\theta)\dot{\phi}_j{}'(\theta)]$, $\bar{\Gamma}_n(\theta) = n^{-1} \sum_{j=1}^n \Gamma_j(\theta)$, and suppose that $\bar{\Gamma}_n(\theta) \to \Gamma(\theta)$ and $\Gamma(\theta)$ is positive definite on Θ. Finally, for $h_n \to h \in R^k$, set $\theta_n = \theta + h_n n^{-\frac{1}{2}}$ and $\Lambda_n(\theta) = \log [dP_{n,\theta_n}/dP_{n,\theta}]$, where $P_{n,\theta}$ stands for the restriction of P_θ to $\mathscr{A}_n = \sigma(X_1, \cdots, X_n)$. Then, under suitable—and not too hard to verify—conditions, we obtain the following results. The limits are taken as $n \to \infty$. THOEREM 1. $\Lambda_n(\theta) - h'\Delta_n(\theta) \to -\frac{1}{2}h'\bar{\Gamma}(\theta)h$ *in* P_θ*-probability*, $\theta \in \Theta$. THEOREM 2. $\mathscr{L}[\Delta_n(\theta) \mid P\theta] \Rightarrow N(0, \bar{\Gamma}(\theta))$, $\theta \in \Theta$. THEOREM 3. $\mathscr{L}[\Lambda_n(\theta) \mid P_\theta] \Rightarrow N(-\frac{1}{2}h'\bar{\Gamma}(\theta)h, h'\bar{\Gamma}(\theta)h)$, $\theta \in \Theta$. THEOREM 4. $\Lambda_n(\theta) - h'\Delta_n(\theta) \to -\frac{1}{2}h'\Gamma(\theta)h$ *in* P_{θ_n}*-probability*, $\theta \in \Theta$. THEOREM 5. $\mathscr{L}[\Lambda_n(\theta) \mid P_{\theta_n}] \Rightarrow N(\frac{1}{2}h'\Gamma(\theta)h, h'\bar{\Gamma}(\theta)h)$, $\theta \in \Theta$. THEOREM 6. $\mathscr{L}[\Delta_n(\theta) \mid P_{\theta_n}] \Rightarrow N(\bar{\Gamma}(\theta)h, \bar{\Gamma}(\theta))$, $\theta \in \Theta$. A certain exponential approximation to the given family of probability measures and some statistical applications will be announced soon. (Received July 27, 1971.)

71T-86. Characterizing measurability, distribution, and weak convergence of random variables in a Banach space by total subsets of linear functionals. MICHAEL D. PERLMAN, University of Minnesota.

Consider a generalized random variable X assuming values in a Banach space $\mathscr{X}$ with conjugate space $\mathscr{X}^*$. For separable or reflexive $\mathscr{X}$ the measurability, probability distribution, and other properties of X are characterized in terms of a collection of real random variables $\{a^*(X): a^* \in A\}$ and their linear combinations, where A is a total subset of $\mathscr{X}^*$, i.e., A distinguishes points of $\mathscr{X}$. Convergence in distribution of a sequence $\{X_n\}$ is characterized in terms of uniform convergence of finite dimensional distributions formed from $\{x^*(X_n): x^* \in \mathscr{X}^*\}$ (for $\mathscr{X}$ separable) or from $\{a^*(X_n): a^* \in A\}$ (for $\mathscr{X}$ separable and reflexive). These results extend earlier ones known for the special cases $A = \mathscr{X}^*$ or $\mathscr{X} = C[0, 1]$. The proofs are based on theorems of Banach, Krein, and Šmulian characterizing the weak* closure of a convex set in $\mathscr{X}^*$. (Received July 30, 1971.)

71T-87. Cramér-type conditions and quadratic mean differentiability. BRUCE LIND AND G. G. ROUSSAS, University of Wisconsin.

In some recent works, the assumption of differentiability in quadratic mean of a certain random function, with respect to the parameter involved, has been used (along with some other regularity conditions) by these and other authors in place of the usual Cramér-type conditions, in connection with asymptotic inference. The question then arises as to how these two sets of assumptions are related. In this paper we consider this question and we show that Cramér-type conditions are, in effect, stronger than ours. (Received July 30, 1971.)

71T-88. A remark on quadratic mean differentiability. BRUCE LIND AND G. G. ROUSSAS, University of Wisconsin.

The assumption of differentiability in quadratic mean of a certain random function, with respect to the parameter involved, has often been used by one of these authors (as well as other authors) in place of the usual pointwise differentiability. When the parameter is one-dimensional, the verification of this assumption is not excessively difficult. This is not so, however, if the parameter is multidimensional. In this note, we consider this latter problem and we show how it can be reduced, in effect, to the one-dimensional case. (Received July 30, 1971.)

71T-89. On maximal (k, t)-sets. BODH RAJ GULATI, Southern Connecticut State College.

This investigation was originally motivated by the problem of determining the maximum number of points in finite n-dimensional projective space $PG(n, s)$ based on the Galois field $GF(s)$ of order $s = p^h$, where p and h are positive integers and p is the prime characteristic of the field, such that no t of the chosen points are linearly dependent. A set of k distinct points in $PG(n, s)$, no t linearly dependent, is called a (k, t)-set; such a set is said to be maximal if there exists no other set with $k^* > k$ points. The maximal value of k is denoted by $m_t(n+1, s)$. The purpose of this paper is to find new upper bounds for some values of n, s and t. The following results have been established: (i) $m_5(5, s) = s+1$ for $s \geqq 4$ and odd, (ii) $s+1 \leqq m_5(5, s) \leqq s+2$ for $s = 2^h$, $h > 2$, (iii) $m_t(t, s) = s+1$ for $s = t+1$ and odd and (iv) $m_{t+1}(n+2, s) \leqq 1+m_t(n+1, s)$ for $n \geqq t-1$. This problem has applications in experimental design and information theory problems. (Received July 30, 1971.)

71T-90. Monotone sequential generation of D-optimal designs of experiments. CORWIN L. ATWOOD, Haile Sellassie I University.

Following Wynn (*Ann. Math. Statist.* **41** 1655–1664) to a given design ξ_n we adjoin a measure concentrated at the point where $d(x, \xi_n)$ is maximized. However we do not necessarily adjoin a single observation there, but instead use that measure which minimizes the resulting generalized variance. We allow the possibility of discarding measure where $d(x, \xi_n)$ is small, and if $d(x, \xi_n)$ is maximized or minimized at several points simultaneously we treat these points together. The sequence of generalized variances thus obtained converges monotonically to the minimum possible value. In an example convergence is fairly fast. (Received July 30, 1971.)

71T-91. On unbiased estimation of the common mean of a bivariate normal distribution. SURESH C. RASTOGI AND V. K. ROHATGI, University of Maryland and The Catholic University of America.

Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be a random sample from a bivariate normal random variable (X, Y) with parameters $EX = EY = \mu$, $\operatorname{Var}(X) = \sigma_1^2$, $\operatorname{Var}(Y) = \sigma_2^2$ $\operatorname{Cov}(X, Y) = \sigma_{12} = \rho\sigma_1\sigma_2$. The problem of unbiased estimation of the common mean μ, when σ_1, σ_2 are unknown and $\sigma_{12} = 0$, has received considerable attention in the literature. In absence of a uniform minimum variance unbiased estimator much work has gone into the search for a linear unbiased estimator which has smaller variance than the conventional estimators. In this paper we have considered the problem of unbiased estimation of μ when $\sigma_{12} \neq 0$. An estimator is proposed which is more efficient than the obvious estimators when a sample of a suitable size is taken. (Received August 13, 1971.)

71T-92. On a characterization property of the multinomial distribution. D. N. SHANBHAG, AND I. V. BASAWA. University of Sheffield.

Let $\mathbf{X} = (x_1, \cdots, x_k)$ and $\mathbf{Y} = (y_1, \cdots, y_k)$ be independent multinomial variates with the same set of probabilities, $(p_1, \cdots, p_k)$, and indices n_1 and n_2 respectively, so that $x_1 + \cdots + x_k = n_1$, $y_1 + \cdots + y_k = n_2$, and $p_1 + \cdots + p_k = 1$. It is well known that the sum $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ is distributed as a multinomial with probabilities $(p_1, \cdots, p_k)$, and index $n = n_1 + n_2$. In this note the following converse of this result is proved. If $\mathbf{X}$ and $\mathbf{Y}$ are k-dimensional, nonnegative, independent random variables, and if $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ is a multinomial variate with probabilities $(p_1, \cdots, p_k)$, then $\mathbf{X}$ and $\mathbf{Y}$ are also multinomial variates with the same set of probabilities. The proof is based on simple arguments utilizing the polynomial structure of the generating functions of $\mathbf{X}$ and $\mathbf{Y}$. An analogous result for the binomial distribution follows as a simple corollary. (Received August 16, 1971.)

71T-93. A Markov chain model related to neuron firing (Preliminary report). I. V. BASAWA, University of Sheffield.

Suppose that pulses are arriving at a neuron according to the Poisson process with unit rate. These pulses are of two types: excitatory (E), and inhibitory (I), and are Markov dependent. $X > 0$ and $Y < 0$ are the random charges carried by the pulses E and I respectively. Each arriving pulse is likely to alter the charge-potential of the neuron. Any E-pulse registers if it arrives within a given time α after the previous pulse, while an I-pulse is effective if it arrives within time β. The effective pulses accummulate until the total charge reaches (or exceeds) a threshold-value $a > 0$, and at this epoch the neuron fires and expends all the charge stored. The same process is then repeated. The epochs of neuron firings (or responses) form a renewal process. Under suitable restrictions on the Markov transition matrix, and on the random variables X and Y, an analogue of Wald's identity for Markov chains is utilized to determine the probability density of the response-interval, and other interesting characteristics. This model could also be used to describe certain other types of networks. (Received August 16, 1971.)

71T-94. On efficient rank tests for comparing the effects of two or more treatments on a single group I. Treatment versus control. HANS K. URY, Stanford University.

Cronholm and Revusky (*Psychometrika* **30** (1965) 459–467) have proposed an $(n-1)$-step rank test R_n for comparing the effect of a treatment with that of a

replicable control (which has at most a transitory effect), using a single group of n homogeneous subjects; the Pitman efficiency of R_n relative to the single-step Mann–Whitney test U based on equal samples of size $n/2$ is $e(R_n, U) = 3$ under local shift alternatives. Ury (*Psychometrika*, to appear) has suggested a quicker j-step analogue R_j^* for $j \ll n$. The tabulation of the null distribution of R_j^* is extended here, and the test is shown to be unbiased under a wide class of alternatives. Its limiting distribution is found to be normal under local shift alternatives, with $e(R_j^*, R_n) = j/(j+2)$ and with higher small-sample efficiency. It is proven that R_j^* is the most efficient j-step analogue of R_n and that the corresponding estimator is asymptotically minimum variance unbiased. An optimality theorem is proven for the still more efficient "reallocation" rank test T_n proposed for strictly measurable observations by Bhattacharyya and Johnson (*J. Amer. Statist. Assoc.* **65** (1970) 1308–1319). Some modifications and several j-step analogues of T_n are proposed. Their limiting distributions under local shift alternatives are shown to be normal and their Pitman efficiencies relative to T_n are computed. (Received August 17, 1971.)

71T-95. On efficient rank tests for comparing the effects of two or more treatments on a single group. II. The k-sample case. HANS, K. URY, Stanford University.

For $k > 2$, and for comparing $(k-1)$ treatments with a replicable control, using a single group of n homogeneous subjects, an "optimal Mann–Whitney test" U_k^* is proposed. This is obtained by randomly assigning $n[(k-1)^{\frac{1}{2}}-1]/(k-2)$ subjects to the control with the rest distributed equally among the $(k-1)$ non-control treatments, and it is shown to be almost twice as efficient, for large k, as the test in which n/k subjects are assigned to each treatment. The null distribution of an $(n-1)/(k-1)$-step test suggested by Revusky (*J. Exper. Analysis of Behavior* **10** (1967) 319–330) is tabulated for a number of cases and a j-step analogue is proposed. For strictly measurable observations, a more efficient $(n-1)/(k-1)$-step test with reallocation and several j-step analogues of this test are proposed, as well. The limiting distributions of all these tests are shown to be normal under local shift alternatives and their Pitman efficiencies relative to each other and to U_k^* are computed and, in some cases, tabulated. All efficiencies relative to U_k^* decrease as k increases, with limiting values not exceeding 1. However, all efficiencies are greater than 1 for $k < 10$. (Received August 17, 1971.)

71T-96. Percentage points of the joint distribution of the extreme roots of the MANOVA matrix. F. J. SCHUURMANN, V. B. WAIKAR AND P. R. KRISHNAIAH, Aerospace Research Laboratories and Miami University, and Aerospace Research Laboratories.

Let S_i $(i = 1, 2)$ denote the sample covariance matrices based on samples of sizes n_i+1 from two independent p-variate normal populations $(p \leqq n_1, n_2)$ with covariance matrices Σ_i $(i = 1, 2)$. A procedure for testing the hypothesis $H_0: \Sigma_1 = \Sigma_2$ against $H_a: \Sigma_1 \neq \Sigma_2$ is to accept H_0 if $L \leqq l_1 < l_p \leqq U$ where $l_1 < \cdots < l_p$ are the characteristic roots of $S_1 S_2^{-1}$ and L and U are constants such that $P[L \leqq l_1 < l_p \leqq U \mid H_0] = 1-\alpha$. Since the optimal choice of the constants L and U is not known, one possible choice is to put $L = 1/U$. Then, the acceptance region can also be written as: $1-A \leqq \theta_1 \leqq \theta_p \leqq A$ where $\theta_1 < \theta_2 < \cdots < \theta_p$ are the characteristic roots of the MANOVA matrix $S_1(S_1+S_2)^{-1}$ and the constant A is such that $P[1-A \leqq \theta_1 < \theta_p \leqq A \mid H_0] = 1-\alpha$. In this paper, the authors have computed the values of A for $\alpha = 0.10$, 0.05, 0.025, 0.01, $p = 2\ (1)\ 10$ and various values of n_1, n_2. The exact expression for the joint probability integral for the extreme roots of θ_1 and θ_p given by Krishnaiah and Chang (*J. Multivariate Analysis* **1** (1971) 108–117) was used to compute these values. (Received August 20, 1971.)