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COMPLETION OF A DOMINATED ERGODIC THEOREM

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In 1937 [1], Marcinkiewicz and Zygmund proved that for $r \geqq 1$, independent, identically distributed (i.i.d.) random variables $\{X_n, n \geqq 1\}$ satisfy

$$E \sup_{n \geqq 1} n^{-r} |\sum_1^n X_i|^r < \infty \tag{1}$$

provided

$$E|X_1|^r < \infty, r > 1 \quad \text{and} \quad E|X_1|^r \log^+ |X_1| < \infty, r = 1. \tag{2}$$

In the following year Wiener [4] demonstrated the analogous result in the more general context of measure—preserving transformations—and this as well as subsequent operator generalizations have come to be known as dominated ergodic theorems.

Reverting to the i.i.d. case, it was proved in 1967 [3] that if, in addition the rv's satisfy $EX_1 = 0$, then for $r \geqq 2$ (this restriction is necessary)

$$E \sup_{n \geqq 1} c_n |\sum_1^n X_i|^r < \infty \tag{3}$$

for c_n such as $n^{-r/2}(\log n)^{-(r/2k)-\delta}$ with $\delta > 0$ and k = greatest integer $\leqq r$ provided $r = 2$ plays the role of $r = 1$ in condition (2). A major step forward was taken by Siegmund [2] who proved the theorem below for *integral values*[2] *of* r. It is the purpose of this note to complete the analogy with the result of Marcinkiewicz and Zygmund by proving the theorem for non-integral values of r as well. This is accomplished by modification of an idea of [3] in conjunction with the approach of [2].

THEOREM. *For* $r \geqq 2$, *independent, identically distributed random variables* $\{X_n, n \geqq 1\}$ *with* $EX_1 = 0$ *satisfy*

$$E \sup_{n \geqq e^e} \frac{|\sum_1^n X_i|^r}{(n \log \log n)^{r/2}} < \infty \tag{4}$$

if and only if

$$E|X_1|^r < \infty, r > 2 \quad \text{and} \quad E \frac{X_1^2 \log |X_1|}{\log \log |X_1|} I_{[|X_1| > e^e]} < \infty, \qquad r = 2.$$

The proof of the theorem will be facilitated by the following proposition which may have independent interest.

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[2] Strictly speaking, the theorem is proved explicitly for $r = 2$ and stated for $r = 3, 4, \cdots$.

LEMMA. *If* $\{Y_n, n \geqq 1\}$ *are independent, nonnegative random variables and* $\{c_n, n \geqq 1\}$ *are positive constants, then* $E(\sum_{n=1}^{\infty} c_n Y_n)^r < \infty$ *for some* $r > 1$ *provided,*

(i) $\sum_{n=1}^{\infty} c_n^r EY_n^r < \infty$ *and* (ii) $\sum_{n=1}^{\infty} c_n^{\alpha} EY_n^{\alpha} < \infty$

where $\alpha = 1$ *if* r *is an integer and* $\alpha = r-[r]$ = *fractional part of* r, *otherwise.*

PROOF. By convexity of $\log E|Y|^r$ (Lyapounov's inequality), whenever $0 < a < b < d$

$$EY_i^b \leqq (EY_i^a)^{(d-b)/(d-a)} (EY_i^d)^{(b-a)/(d-a)}, \qquad i \geqq 1$$

implying

$$\sum_i c_i^b \, EY_i^b \leqq \sum_i (c_i^a EY_i^a)^{(d-b)/(d-a)} (c_i^d EY_i^d)^{(b-a)/(d-a)}$$
$$\leqq (\sum_i c_i^a EY_i^a)^{(d-b)/(d-a)} (\sum_i c_i^d EY_i^d)^{(b-a)/(d-a)}$$

via Hölder. Thus, (i) and (ii) imply that

(iii) $\sum_{n=1}^{\infty} c_n^h EY_n^h < \infty, \quad \alpha \leqq h \leqq r.$

Consider only the case where r is not an integer since the alternative situation is analogous but simpler. Setting $k = r-\alpha$,

$$\begin{aligned} E(\textstyle\sum_{n=1}^{\infty} c_n Y_n)^r &= E(\textstyle\sum c_n Y_n)^{\alpha} (\textstyle\sum c_n Y_n)^k \\ &\leqq E(\textstyle\sum c_n^{\alpha} Y_n^{\alpha})[\textstyle\sum c_n^k Y_n^k + \cdots + k! \textstyle\sum_{1 \leqq i_1 < \cdots < i_k} c_{i_1} Y_{i_1} \cdots c_{i_k} Y_{i_k}] \\ &= \textstyle\sum c_n^r EY_n^r + \textstyle\sum_{i \neq j} c_i^{\alpha} EY_i^{\alpha} c_j^k EY_j^k + \cdots \\ &\qquad + k! \textstyle\sum_{1 \leqq i_1 < \cdots < i_k,\, i \neq i_j,\, 1 \leqq j \leqq k} c_{i_1} EY_{i_1} \cdots c_{i_k} EY_{i_k} c_i^{\alpha} EY_i^{\alpha} \\ &\qquad + k! \textstyle\sum_{1 \leqq i_1 < \cdots < i_{k-1},\, i \neq i_j,\, 1 \leqq j < k} c_{i_1} EY_{i_1} \\ &\qquad\quad \cdot \cdots c_{i_{k-1}} EY_{i_{k-1}} c_i^{1+\alpha} EY_i^{1+\alpha} \end{aligned}$$

recalling independence. But every term on the right is dominated by a product of terms of the form (iii). For example, the final term is majorized by

$$k!(\sum c_i EY_i)^{k-1} (\sum c_i^{\alpha+1} EY_i^{\alpha+1}).$$

The lemma follows.

PROOF OF THEOREM. Since the case $r = 2$ is proved explicitly in Siegmund (1969), suppose that $r > 2$ and moreover that $EX_1^2 = 1$. Set $S_n = \sum_{i=1}^n X_j$, $b_n = n^{1/r}$ and $c_n = (n \log_2 n)^{-\frac{1}{2}}$ or one according as $n > e^e$ or not. Assume initially that $\{X_n, n \geqq 1\}$ are symmetric random variables and define

$$X_n' = X_n I_{[|X_n| \leqq b_n]}, \qquad X_n'' = X_n - X_n', \qquad S_n' = \sum_{j=1}^n X_j', \qquad S_n'' = \sum_{j=1}^n X_j''.$$

If α is as defined in the lemma, then for $h = \alpha$ or r and $n_0 > e^e$,

$$\sum_{n=n_0}^{\infty} c_n^h E|X_n''|^h = \sum_{n=n_0}^{\infty} \sum_{j=n}^{\infty} \frac{1}{(n \log_2 n)^{h/2}} \int_{b_j < |X| \leqq b_{j+1}} |X|^h$$

$$\leqq K_1 \sum_{j=n_0}^{\infty} \frac{j^{1-h/2}}{(\log_2 j)^{h/2}} \int_{b_j < |X| \leqq b_{j+1}} |X|^h$$

$$\leqq K_1 \sum_{j=n_0}^{\infty} \frac{j^{h[(1/r)-\frac{1}{2}]}}{(\log_2 j)^{h/2}} \int_{b_j < |X| \leqq b_{j+1}} |X|^r$$

$$\leqq K_2 E|X_1|^r < \infty.$$

Thus, invoking the lemma

$$E \sup_{n > e^e} \frac{|S_n''|^r}{(n \log_2 n)^{r/2}} \leqq E\left(\sup_{n \geqq 1} c_n \sum_{j=1}^{n} |X_j''|\right)^r \leqq E\left(\sum_{n=1}^{\infty} c_n |X_n''|\right)^r < \infty.$$

It remains to prove that $E \sup n > e^e (|S_n'|^r/(n \log_2 n)^{r/2}) < \infty$ or equivalently to show that for sufficiently large u_0

$$\textstyle\int_{u_0}^{\infty} u^{r-1} P\{\sup c_n |S_n'| > u\} \, du < \infty;$$

this is accomplished as in [2], the only point of departure being in the choice of (in the notation of [2]) $t = [(\log_2 n_{k+1})/n_{k+1}]^{\frac{1}{2}}$ rather than $t = b_{n_{k+1}}^{-1}$. The removal of the symmetry assumption is standard as in [2].

In contradistinction to the case $r = 2$, the necessity of (4) when $r > 2$, is trivial.

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