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MOVING AVERAGES OF HOMOGENEOUS RANDOM FIELDS[1]

BY L. A. BRUCKNER

College of Santa Fe

Let $X(g)$ be a homogeneous random field on a discrete locally compact Abelian group G. Let $H(X)$ be the linear completion of $\{X(g) : g \in G\}$ in L_2 space. The following result is obtained: there exists a fundamental random field $Y(g)$ on G with values in $H(X)$ such that $X(g)$ is obtained as a moving average of $Y(g)$ if, and only if, $X(g)$ has a spectral density which is positive almost everywhere with respect to the Haar measure on the dual group of G.

0. Introduction and summary. The conditions under which a stationary stochastic process can be represented as a moving average have been investigated by Kolmogorov [2] and Rozanov [4]. In this paper moving averages are studied in the more general setting of homogeneous random fields on discrete locally compact Abelian (LCA) groups. It is shown that a necessary and sufficient condition for a homogeneous random field to have a representation as a moving average of a fundamental field is that it have a spectral density which is positive almost everywhere with respect to the Haar measure on the character group. This is a generalization of Kolmogorov's representation theorem [2].

1. Background. Let G be a LCA group and G^* its dual group. Then G^* is also a LCA group ([3], [5]). Let $\mathscr{B}$, $\mathscr{B}^*$ be the Borel fields of G, G^* respectively. Let (Ω, Σ, P) be a probability space. For all $g \in G$, let $X(g) \in L_2(\Omega, \Sigma, P)$. We will assume that the first moment vanishes for all g. $L_2(\Omega)$ is a Hilbert space when the inner product is defined by

$$(f_1, f_2) = Ef_1\bar{f}_2 \qquad f_1, f_2, \in L_2,$$

where E is the mathematical expectation. Let $H(X)$ be the linear completion of $\{X(g): g \in G\}$ in $L_2(\Omega)$. If the correlation function

$$B(g, g') = (X(g), X(g') = B(g-g')$$

depends only on $g-g'$, $X(g)$ is a homogeneous random field (HRF) on G. The spectral representations of $X(g)$ and $B(g)$ are

$$X(g) = \int_{G^*}(g, x)Z(dx)$$

and

$$B(g) = \int_{G^*}(g, x)F(dx)$$

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where Z is an orthogonal measure on $\mathscr{B}^*$ such that $(Z(E), Z(E')) = F(E \cap E')$ for all Borel sets E, E' of G^*.

The notation used here is that of [1] with one exception. Instead of saying that $Y(g)$ is obtained from $X(g)$ by a linear transformation, the more convenient terminology of Kolmogorov [2] will be used—$Y(g)$ will be said to be subordinate to $X(g)$.

2. Moving averages. An HRF $X(g)$ on G is said to be obtained from an HRF $Y(g)$ by a moving average if, for all $g \in G$, there exist complex-valued coefficients $a(g')$, independent of g, such that

$$X(g) = \sum_{g' \in G} a(g') Y(g-g').$$

An uncorrelated random field $Y(g)$ such that $(Y(g), Y(g)) = 1$ is called a fundamental field.

The following theorem gives a necessary and sufficient condition for representing $X(g)$ as a moving average in terms of a fundamental field.

THEOREM. *Let $X(g)$ be a homogeneous random field on a discrete* LCA *group* G. *Then there exists a fundamental random field $Y(g)$ on G, with values in $H(X)$, such that $X(g)$ is obtained as a moving average of $Y(g)$ if, and only if, $X(g)$ has a spectral density which is positive almost everywhere with respect to the Haar measure on G^*.*

PROOF. Assume that the spectral density $f(x)$ is positive almost everywhere. Then, since G^* is compact, the random field

$$Y(g) = \int_{G^*} (g, x) r(x) Z(dx)$$

where $r(x) = 1/(f(x))^{\frac{1}{2}}$, is a fundamental random field with values in $H(X)$. Let $a(g) = (Y(g), X(0)) = \int_{G^*} (g, x)(f(x))^{\frac{1}{2}} dx$. Then by Plancherel's Theorem for Abelian groups ([5], page 26),

$$(f(x))^{\frac{1}{2}} = \sum_{g \in G} a(g)(-g, x). \tag{1}$$

Write

$$X(g) = \int_{G^*} (g, x) r(x) (f(x))^{\frac{1}{2}} Z(dx)$$

and using (1) replace $(f(x))^{\frac{1}{2}}$. Since $(f(x))^{\frac{1}{2}} \in L_2(G^*)$ and $\{(g, x) : g \in G\}$ forms a complete orthonormal system in $L_2(G^*)$, the interchange of the summation and integration procedures is valid. Hence

$$X(g) = \sum_{g' \in G} a(g') Y(g-g').$$

To prove the converse note that if $Y(g)$ is fundamental then

$$\begin{aligned} B_{YY}(g) &= 0 \qquad & g \neq 0 \\ &= 1 \qquad & g = 0 \\ &= \int_{G^*} (g, x)\, dx \end{aligned}$$

so that for all $E \in \mathscr{B}^*$

$$F_{YY}(E) = x(E) \tag{2}$$

where x is the Haar measure on G^*.

By assumption $Y(g) \in H(X)$ for all g. Hence $H(Y) \subset H(X)$. Since $X(g)$ is obtained from $Y(g)$ by a moving average $X(g)$ and $Y(g)$ are mutually homogeneously correlated. By Theorems 2.2 and 2.1 ([1]) there exists a function $p(x) \in L_2(F)$ such that for all $E \in \mathscr{B}^*$.

$$F_{YY}(E) = \int_E |p(x)|^2 F(dx).$$

By (2) this becomes

$$x(E) = \int_E |p(x)|^2 F(dx)$$

and the Haar measure is seen to be absolutely continuous with respect to F. Hence, by the Radon–Nikodym Theorem, the derivative $dx/F(dx) = 1/f(x)$ exists and is finite a.e. $[F]$, and hence a.e. $[x]$. Thus the spectral density is positive a.e. $[x]$.

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