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A "FATOU EQUATION" FOR RANDOMLY STOPPED VARIABLES[1]

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Let X_n be a sequence of random variables adapted to an increasing sequence of σ-fields. In this note, convergence properties of EX_t are studied as $t \to \infty$ through the directed set of stopping variables. The analogue of the inequality in Fatou's Lemma turns out to be an equation, which strengthens Fatou's Lemma.

These problems arise naturally in the theory of gambling.

Let (Ω, F, P) be a probability space, $\{F_n\}_{n \geqq 1}$ an increasing sequence of σ-fields contained in F, and $\{X_n\}_{n \geqq 1}$ a sequence of random variables such that X_n is F_n-measurable for every n. A random variable t is a *stopping variable* (sv) if its range is contained in $\{1, 2, \cdots, +\infty\}$, $P[t < +\infty] = 1$, and, for every positive integer n, $[t \leqq n] \in F_n$. If t and s are stopping variables, write $t \leqq s$ if $t(\omega) \leqq s(\omega)$ for all $\omega \in \Omega$. With this natural partial ordering, the stopping variables form a directed set. Some of the convergence properties of the net EX_t are considered below. In particular, Theorem 2 is an analogue of Fatou's Lemma in which an equation replaces the usual inequality.

In what follows, the letters "s" and "t" always denote stopping variables and the letters "k" and "n" positive integers.

THEOREM 1. *The following inequalities hold whenever all the expectations occurring in them are well-defined:*

$$E(\limsup_{n \to \infty} X_n) \leqq \limsup_{t \to \infty} EX_t \tag{1}$$

$$E(\liminf_{n \to \infty} X_n) \geqq \liminf_{t \to \infty} EX_t. \tag{1'}$$

PROOF. It is enough to prove (1). For convenience, let $X^* = \limsup_{n \to \infty} X_n$. Equation (1) is obvious if $EX^* = -\infty$. Let us assume now that X^* is integrable and return later to the case when $EX^* = +\infty$.

Let $\varepsilon > 0$ and let s be a sv. Define

$$t(\omega) = \inf\{n : n \geqq s(\omega) \quad \text{and} \quad E(X^* \mid X_1, \cdots, X_n)(\omega) < X_n(\omega) + \varepsilon\}.$$

By Lévy's martingale convergence theorem (29.4, [2]), $E(X^* \mid X_1, \cdots, X_n) \to X^*$ almost surely as $n \to \infty$ and, hence, $P[t < +\infty] = 1$. Thus t is a sv, $t \geqq s$, and

$$\begin{aligned} EX_t &\geqq \textstyle\sum_{n=1}^{\infty} \int_{[t=n]} E(X^* \mid X_1, \cdots, X_n)\,dP - \varepsilon \\ &= EX^* - \varepsilon, \end{aligned}$$

which proves (1).

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Finally, suppose $EX^* = +\infty$. Let c be a real number and apply the case just considered to the random variables $\min(X_n, c)$. Then

$$\limsup_{t\to\infty} EX_t \geqq \limsup_{t\to\infty} E(\min(X_t, c)) \geqq E(\min(X^*, c)) \to EX^*$$

as $c \to +\infty$. □

Certain results of Siegmund in [3] are closely related to the ideas of this note. In particular, the previous theorem seems to follow from his Theorem 3(a) in the special case that $EX_n^- < +\infty$ for all n.

THEOREM 2. *Suppose Z and W are integrable random variables. If $X_n \leqq Z$ for all n, then*

$$E(\limsup_{n\to\infty} X_n) = \limsup_{t\to\infty} EX_t. \tag{2}$$

If $X_n \geqq W$ for all n, then

$$E(\liminf_{n\to\infty} X_n) = \liminf_{t\to\infty} EX_t. \tag{2'}$$

PROOF. The proof is a simple modification of the proof of Fatou's Lemma ([2], p. 125). Let $W_n = \sup_{k\geqq n} X_k$. Then

$W_n \downarrow X^* = \limsup_{n\to\infty} X_n$ as $n \to \infty$. Also, $X_t \leqq W_n$ for $t \geqq n$. Hence

$$\limsup_{t\to\infty} EX_t \leqq \lim_{n\to\infty} \{\sup_{t\geqq n} EX_t\} \leqq \lim_{n\to\infty} EW_n = EX^*.$$

The opposite inequality is true by Theorem 1. □

When the X_n are dominated above, it is easy to see that $\limsup_{t\to\infty} EX_t \geqq \limsup_{n\to\infty} EX_n$. Thus, Theorem 2 implies the usual Fatou Lemma.

For uniformly bounded X_n, a version of Theorem 2 has been proved in which sv's are not assumed to be measurable. This result has an interpretation for the Dubins and Savage utility of a measurable strategy and together with a gambling theorem (3.9.5, [1]) has been used to establish an optimal stopping result (Theorem 5, [5]) parallel to Siegmund's Theorem 4 in [3]. Further applications to gambling theory are in [6].

The next result is immediate from Theorem 2 and is also easy to prove directly.

COROLLARY. *Suppose $X_n \to X$* a.s. *and there is an integrable random variable Z such that $|X_n| \leqq Z$ for all n. Then X is integrable and* $\lim_{t\to\infty} EX_t = EX$.

Recall that if $X_n \to X$ a.s. and the family $\{X_n\}_{n\geqq 1}$ is uniformly integrable, then X is integrable and $EX_n \to EX$. The following example shows that, under the same hypotheses, it is not necessarily true that $EX_t \to EX$. It also provides a non-trivial example (though not the simplest) in which (2) fails to hold.

EXAMPLE. The X_n constructed here will be nonnegative, uniformly integrable, convergent to zero a.s., and such that

$$\limsup_{t\to\infty} EX_t \geqq 1. \tag{3}$$

Let (Ω, F, P) be the unit interval with its Borel sets and Lebesgue measure. Let $\{Z_n\}_{n \geqq 1}$ be independent rv's and assume Z_n is uniformly distributed on [0, 1] for all n. Now define random variables Y_i^n for $n = 1, 2, \cdots$ and $i = 1, \cdots, n$ by

$$\begin{aligned} Y_i^n &= n \qquad \text{if } (i-1)/n^2 \leqq Z_n < i/n^2, \\ &= 0 \qquad \text{if not.} \end{aligned}$$

Notice that the random vectors $(Y_1^n, \cdots, Y_n^n)$, $n = 1, 2, \cdots$ are independent.

Choose a sequence of positive integers $n_1 < n_2 < \cdots$ such that

$$\prod_{j \geqq k} \left(1 - \frac{1}{n_j}\right) \to 1 \qquad \text{as } k \to \infty.$$

Let $X_1, X_2, \cdots$ be the sequence $Y_1^{n_1}, Y_2^{n_1}, \cdots, Y_{n_1}^{n_1}, Y_1^{n_2}, Y_2^{n_2}, \cdots, Y_{n_2}^{n_2}, \cdots$ and let F_n be the σ-field generated by $X_1, \cdots, X_n$ for all n.

The X_n are uniformly integrable since

$$\sup_n \int_{[|X_n| \geqq n_k]} |X_n| \, dP = \frac{1}{n_k} \to 0 \qquad \text{as } k \to \infty.$$

Also, if $\varepsilon > 0$ and $X_j = Y_1^{n_k}$, then

$$\begin{aligned} P[|X_n| \leqq \varepsilon \quad \text{for all} \quad n \geqq j] &= \textstyle\prod_{i \geqq k} P[|Y_1^{n_i}| \leqq \varepsilon, \cdots, |Y_{n_i}^{n_i}| \leqq \varepsilon] \\ &\geqq \prod_{i > k} \left(1 - \frac{1}{n_i}\right) \to 1 \qquad \text{as } k \to \infty . \end{aligned}$$

Hence, $X_n \to 0$ a.s.

Let s be any sv. To prove (3), it suffices to exhibit a sv t such that $t \geqq s$ and $EX_t \geqq 1$.

Let $\omega \in \Omega$ and suppose $s(\omega) = n$. Let j_n be the first j such that $j > n$ and $X_j = Y_1^m$ for some m. Define

$$\begin{aligned} t(\omega) &= j_n + i \qquad \text{if} \quad Y_1^m = \cdots = Y_i^m = 0, \ Y_{i+1}^m = m, \\ &= j_n + m - 1 \qquad \text{if} \quad Y_1^m = \cdots = Y_m^m = 0. \end{aligned}$$

Thus $X_t(\omega) = \max(Y_1^m(\omega), \cdots, Y_m^m(\omega))$ if $s(\omega) = n$. Also, $(Y_1^m, \cdots, Y_m^m)$ is independent of $[s = n]$. Therefore, if $s(\omega) = n$,

$$\begin{aligned} E(X_t \mid s = n)(\omega) &= E(\max(Y_1^m, \cdots, Y_m^m)) \\ &= 1. \end{aligned}$$

Hence, $EX_t = 1$.

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