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CONTINUOUS MARTINGALES WITH DISCONTINUOUS MARGINAL DISTRIBUTIONS

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We construct in this paper a continuous, nowhere constant, square integrable martingale such that $P\{M(\frac{1}{2})^k = 0\} \geqq \frac{7}{8}$ for $k \geqq 3$. This construction is used to show that in general, $\lim_{t\to 0}\int_0^t \Phi(s)dM(s, \omega)/M(t, \omega) \neq \Phi(0)$ where $\Phi(s)$ is nonrandom and right continuous, $M(t, \omega)$ is a continuous, nowhere constant, square integrable, martingale, and the limit is a limit in probability.

In a previous paper [3] we considered the question of whether or not

$$\lim_{\Delta t\to 0} \int_t^{t+\Delta t} \Phi(s)dM_s/M(t+\Delta t)-M(t) = \Phi(t)$$

where the limit is taken to be a limit in probability. In the case where $M(s, \omega)$ was a Brownian motion we obtained the desired convergence. However, for $M(s, \omega) \in \mathscr{M}$, the class of right continuous, square integrable, nowhere constant, martingales, a counterexample showed that the convergence may fail in this case. The open question at that time was whether or not the convergence held when $M(s, \omega) \in \mathscr{M}^c$, the class of continuous, square integrable, nowhere constant, martingales. In this paper we will show by another counterexample that convergence may fail in this case also.

The main idea of the counterexample in [3] was to find $M(t, \omega) \in \mathscr{M}$ such that $P\{M((\frac{1}{2})^k, \omega) = 0\} \geqq \alpha > 0$. Hence the first question to consider for $M(t, \omega) \in \mathscr{M}^c$ is whether or not the random variable one gets for a fixed time can have a discontinuous distribution function. Since there are many similarities between martingales of the above type and Brownian motion, (e.g. any such martingale can be expressed as a continuous time change on a Brownian motion [1], [4] and the sample paths of such martingales are of unbounded variation as in Brownian motion [2]) and since Brownian motion has continuous marginal distributions, one might expect the same to be true for martingales in $\mathscr{M}^c$. To resolve this question we consider the following example.

Let $(B_t, \mathscr{F}_t)$ denote a Brownian motion process where $\mathscr{F}_t = \sigma\{B_s : s \leqq t\}$. Define a random variable

$$\begin{aligned} Y(\omega) &= 1 \quad \text{if } B(1, \omega) > c, \\ &= 0 \quad \text{if } |B(1, \omega)| \leqq c, \\ &= -1 \quad \text{if } B(1, \omega) < -c. \end{aligned}$$

Then define $M(t, \omega) = E[Y \mid \mathscr{F}_t]$ for $t \in [0, 1]$. Since $M(t, \omega)$ is a martingale adapted to the σ-fields generated by a Brownian motion, we know $M(t, \omega) =$

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$\int_0^t \Phi_s dB_s$ [5]. Hence $M(t, \omega)$ is a continuous martingale. In order to show $M(t, \omega)$ is nowhere constant we write,

$$M(t, \omega) = E[Y \mid \mathscr{F}_t] = P\{B_1 > c \mid \mathscr{F}_t\} - P\{B_1 < -c \mid \mathscr{F}_t\}$$
$$= P\{B_1 > c \mid B_t\} - P\{B_1 < -c \mid B_t\}$$

since B_t is a Markov process. Recall that M_t is nowhere constant if $P[\sup_{s,t \in I} |M_t - M_s| > 0] = 1$ for all open intervals I. Now

$$P\{B_1 < -c \mid B_t = x\} = k \int_{-\infty}^{-c} \exp[-(z-x)^2/2(1-t)]dz$$

where k depends on t but not on x. So

$$P\{B_1 < -c \mid B_t\} = k \int_{-\infty}^{-c} \exp[-(z - B_t(\omega))^2/2(1-t)]dz.$$

For ω fixed, as t varies over an interval, the variation in $B_t(\omega)$ will clearly cause $P[B_1 < -c \mid B_t]$ to vary a.s. From this it can be shown that

$$E[Y \mid \mathscr{F}_t]$$
$$= k \int_c^{\infty} \exp[-(z - B_t(\omega))^2/2(1-t)]dz - k \int_{-\infty}^{-c} \exp[-(z - B_t(\omega))^2/2(1-t)]dz$$

varies a.s. over any interval. We clearly have that $M(t, \omega)$ is square integrable. We will now use the above approach to construct $M(t, \omega) \in \mathscr{M}^c$ such that $P\{M)(\frac{1}{2})^k, \omega) = 0\} \geqq \frac{7}{8}$ for $k = 3, 4, 5, \cdots$.

Let

$$\begin{aligned} Y_1 &= 1 && \text{if } B_1 - B_{\frac{1}{2}} > a_1 \\ &= 0 && \text{if } |B_1 - B_{\frac{1}{2}}| \leqq a_1 \\ &= -1 && \text{if } B_1 - B_{\frac{1}{2}} < -a_1 \end{aligned}$$

where a_1 is chosen so that $P\{|B_1 - B_{\frac{1}{2}}| > a_1\} < \frac{1}{3}$. Now define $M_t^{(1)} = E[Y_1 \mid \mathscr{F}_t]$. This martingale is zero for $0 \leqq t \leqq \frac{1}{2}$, and continuous and nowhere constant for $\frac{1}{2} \leqq t \leqq 1$. Let

$$\begin{aligned} Y_2 &= \tfrac{1}{2} && \text{if } B_{\frac{1}{2}} - B_{\frac{1}{4}} > a_2 \\ &= 0 && \text{if } |B_{\frac{1}{2}} - B_{\frac{1}{4}}| \leqq a_2 \\ &= -\tfrac{1}{2} && \text{if } B_{\frac{1}{2}} - B_{\frac{1}{4}} < -a_2 \end{aligned}$$

where a_2 is chosen so that $P\{|B_{\frac{1}{2}} - B_{\frac{1}{4}}| > a_2\} < (\frac{1}{3})^2$. Define $M_t^{(2)} = E[Y_2 \mid \mathscr{F}_t]$. This martingale is zero for $0 \leqq t \leqq \frac{1}{4}$, continuous and nowhere constant for $\frac{1}{4} \leqq t \leqq \frac{1}{2}$, and Y_2 for $\frac{1}{2} \leqq t \leqq 1$. Let

$$\begin{aligned} Y_k &= (\tfrac{1}{2})^{k-1} && \text{if } B_{(\frac{1}{2})^{k-1}} - B_{(\frac{1}{2})^k} > a_k \\ &= 0 && \text{if } |B_{(\frac{1}{2})^{k-1}} - B_{(\frac{1}{2})^k}| \leqq a_k \\ &= -(\tfrac{1}{2})^{k-1} && \text{if } B_{(\frac{1}{2})^{k-1}} - B_{(\frac{1}{2})^k} < -a_k \end{aligned}$$

where a_k is chosen so that $P\{|B_{(\frac{1}{2})^{k-1}} - B_{(\frac{1}{2})^k}| > a_k\} < (\frac{1}{3})^k$. Define $M_t^{(k)} = E[Y_k \mid \mathscr{F}_t]$.

Now define $M_t = \sum_{j=1}^{\infty} M_t^{(j)}$. Since $|M_t^{(j)}| \leqq (\frac{1}{2})^{j-1}$ we have the series convergent a.s. and

$$E[M_t \mid \mathscr{F}_s] = \sum_{j=1}^{\infty} E[M_t^{(j)} \mid \mathscr{F}_s] = \sum_{j=1}^{\infty} E[E[Y_j \mid \mathscr{F}_t] \mid \mathscr{F}_s]$$
$$= \sum_{j=1}^{\infty} E[Y_j \mid \mathscr{F}_s] = M_s$$

a.s. for $s \leqq t$. Hence M_t is a martingale which is clearly continuous and square integrable. In order to show M_t is nowhere constant it is sufficient to show $M_t^{(k)}$ is nowhere constant on $((\frac{1}{2})^k, (\frac{1}{2})^{k-1})$ for each k since each $M_t^{(k)}$ has constant paths except on $((\frac{1}{2})^k, (\frac{1}{2})^{k-1})$. For simplicity let $k = 1$. Then we have

$$M_t^{(1)} = P\{B_1 - B_{\frac{1}{2}} > a_1 \mid \mathscr{F}_t\} - P\{B_1 - B_{\frac{1}{2}} < -a_1 \mid \mathscr{F}_t\}$$
$$= P\{B_1 - B_{\frac{1}{2}} > a_1 \mid B_t - B_{\frac{1}{2}}\} - P\{B_1 - B_{\frac{1}{2}} < -a_1 \mid B_t - B_{\frac{1}{2}}\}$$

and since $B_t - B_{\frac{1}{2}}$ is again a Brownian motion for $\frac{1}{2} \leqq t \leqq 1$ we can use the previous argument to show $M_t^{(1)}$ is nowhere constant on $(\frac{1}{2}, 1)$. Let $B_t^* = B_t - B_{\frac{1}{2}}$.

We now note that $P\{M_{\frac{3}{4}}^{(1)} = 0\} = 0$ since

$$P\{M_{\frac{3}{4}}^{(1)} = 0\} = P\{\int_{a_1}^{\infty} \exp[-2(z - B_{\frac{3}{4}}^*(\omega))^2]dz$$
$$= \int_{-\infty}^{-a_1} \exp[-2(z - B_{\frac{3}{4}}^*(\omega))^2]dz\}$$

and the above integrals are equal if and only if $B_{\frac{3}{4}}^* = 0$. Similarly we can show $P\{M_{(\frac{3}{2})^{k+2}}^{(k)} = 0\} = 0$ for all k. We also have

$$P\{M_{(\frac{1}{2})^k} = 0\} = 1 - P\{M_{(\frac{1}{2})^k} \neq 0\} \geqq 1 - \sum_{n=k}^{\infty} P\{M_{(\frac{1}{2})^k}^{(n)} \neq 0\}$$
$$= 1 - \tfrac{1}{2} \cdot (\tfrac{1}{3})^{k-1}.$$

We now construct a right continuous, nonrandom integrand such that $\int_0^t \Phi_s dM_s / M_t \nrightarrow \Phi_0$ as $t \to 0$. We note that for Brownian motion we were able to show convergence at every point t when the integrand was of this type (Isaacson (1969)). Let

$$\Phi(s) = 1/n \quad \text{if} \quad (\tfrac{1}{2})^n \leqq s < (\tfrac{3}{2})^{n+1}$$
$$\Phi(s) = -1/n \quad \text{if} \quad (\tfrac{3}{2})^{n+1} \leqq s < (\tfrac{1}{2})^{n-1}$$
$$\Phi(0) = 0$$

for $n = 1, 2, 3, \cdots$.

Since Φ is nonrandom and bounded it is certainly integrable on [0, 1]. Now

$$\int_{(\frac{1}{2})^k}^{(\frac{1}{2})^{k-1}} \Phi_s dM_s = [M_{(\frac{1}{2})^{k-1}} - 2M_{(\frac{3}{2})^{k+1}} + M_{(\frac{1}{2})^k}]/k$$

and with probability at least $1 - (\frac{1}{3})^{k-2}$ this integral is equal to $-2m_{(\frac{3}{2})^{k+1}}/k$. Hence

$$P\{\int_{(\frac{1}{2})^k}^{(\frac{1}{2})^{k-1}} \Phi_s dM_s \neq 0\} \geqq 1 - (\tfrac{1}{3})^{k-2} > \tfrac{3}{4} \qquad \text{for } k \geqq 4.$$

From this it follows that $P\{\int_0^{(\frac{1}{2})^k} \Phi_s dM_s \neq 0\} \geqq \frac{3}{8}$ for infinitely many k, since if $P\{\int_0^{(\frac{1}{2})^{k+1}} \Phi_s dM_s \neq 0\} < \frac{3}{8}$ then we must have $P\{\int_0^{(\frac{1}{2})^k} \Phi_s dM_s \neq 0\} \geqq \frac{3}{8}$. Therefore

$$P\{|\int_0^{(\frac{1}{2})^k} \Phi_s dM_s| \geqq |M_{(\frac{1}{2})^k}|\varepsilon\} \geqq P\{\int_0^{(\frac{1}{2})^k} \Phi_s dM_s \neq 0, M_{(\frac{1}{2})^k} = 0\} \geqq \tfrac{3}{8} - \tfrac{1}{4} = \tfrac{1}{8}$$

for infinitely many k. We conclude that when the integrator is in $\mathscr{M}^c$ one may not get the convergence to the integrand that one gets when the integrator is Brownian motion.

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