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ON THE CONTINUITY PROPERTIES OF L FUNCTIONS[1]

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A necessary and sufficient condition is obtained for an L function to have a continuous derivative of the kth order. Other results concerning the continuity properties of infinitely divisible distribution functions are also obtained.

1. Introduction and summary. The definition of an L function, i.e., a distribution function in class L, and a discussion of its elementary properties can be found in Gnedenko and Kolmogorov (1954, Chapter 6).

By 1963 it was shown that every non-degenerate L function is absolutely continuous (see Fisz and Varadarajan (1963, page 336)) or (Zolotarev (1963, Theorem 2, page 125)). It is quite easy to show that an L function with a normal component has continuous derivatives of all orders. Let $F(x)$ be an L function without a normal component and with Lévy spectral function $M(u)$. Let $\lambda(u) = uM'(u)$. In this paper it will be shown that a necessary and sufficient condition for $F(x)$ to have continuous derivatives (or absolutely continuous derivatives) of the first k orders is that $\lambda(+0)+|\lambda(-0)| > k$. The sufficiency of the condition follows from a slight extension of a theorem of V. M. Zolotarev (1963, Theorem 3, page 131). However, a much deeper analysis of the continuity properties of L functions than that made by Zolotarev is necessary in order to show the necessity of the condition.

2. Three lemmas. Three lemmas will be needed for the proof of the main theorem of this paper. The statements and proofs of the lemmas will now be given.

LEMMA 1. *Let $F(x)$ and $G(x)$ be two distribution functions such that $F(x)$ has absolutely continuous and integrable derivatives of the first n orders. Then $F^*G(x)$ has absolutely continuous and integrable derivatives of the first n orders that vanish at infinity and*

$$(F^*G)^{(i)}(x) = \int_{-\infty}^{\infty} F^{(i)}(x-y)\,dG(y)$$

for $1 \leqq i \leqq n$. If in addition $F^{(n+1)}(x)$ is integrable then

$$(F^*G)^{(n+1)}(x) = \int_{-\infty}^{\infty} F^{(n+1)}(x-y)\,dG(y)$$

for almost all x and $(F^*G)^{(n+1)}(x)$ is integrable.

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PROOF. Since $F'(x)$ is absolutely continuous the function

$$q(x) = \int_{-\infty}^{\infty} F'(x-y)\, dG(y)$$

is also absolutely continuous. Since $F'(x)$ is integrable it follows from Fubini's theorem that $q(x)$ is integrable. It also follows from Fubini's theorem that

$$F^*G(x) = \int_{-\infty}^{x} q(t)\, dt.$$

Thus $(F^*G)'(x) = q(x)$. Since $(F^*G)'(x)$ is absolutely continuous and integrable it vanishes at infinity. The remainder of Lemma 1 can be proved in a similar manner. □

LEMMA 2. *Let $p > 0$, $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, and $\lambda_1 + \lambda_2 > 0$. Let $H(x)$ be a distribution function with characteristic function*

$$\hat{h}(t) = \exp\{-\lambda_1 \int_{-p}^{0} (e^{iut}-1)/u\, du + \lambda_2 \int_0^p (e^{iut}-1)/u\, du\}.$$

Let k be the largest integer smaller than $\lambda_1 + \lambda_2$. Then $H(x)$ has absolutely continuous derivatives of the first k orders and integrable derivatives of the first $k+1$ orders. However, $H(x)$ does not have a continuous derivative of the $(k+1)$th order. All derivatives of $H(x)$ vanish at infinity.

PROOF. Let $F(x)$ and $G(x)$ be the distribution functions with characteristic functions

$$\hat{f}(t) = \exp\{\lambda_2 \int_0^p (e^{iut}-1)/u\, du\},$$
$$\hat{g}(t) = \exp\{-\lambda_1 \int_{-p}^{0} (e^{iut}-1)/u\, du\}$$

respectively. It will be assumed that $\lambda_1 > 0$ and $\lambda_2 > 0$. The proof can be modified slightly if $\lambda_1 = 0$ or $\lambda_2 = 0$. Let k_1 and k_2 be the largest integers smaller than λ_1 and λ_2 respectively. Let $h(x)$, $f(x)$, and $g(x)$ be the density functions of $H(x)$, $F(x)$, and $G(x)$ respectively. Since $\hat{h}(t) = \hat{f}(t)\hat{g}(t)$ it follows that $H(x) = F^*G(x)$. It has been shown in the proof of Lemma 1 of Wolfe (1971) that

(1) $$f(x) = 0 \quad \text{if} \quad x < 0,$$

(2) $$f(x) = c_2 x^{\lambda_2 - 1} \quad \text{if} \quad 0 < x < p, \qquad \text{where } c_2 > 0,$$

(3) $$f(x) = (\lambda_2/x)[F(x) - F(x-p)] \quad \text{if} \quad x > 0,$$

and

(4) $$xf'(x) = (\lambda_2 - 1)f(x) - \lambda_2 f(x-p) \quad \text{if} \quad x > 0.$$

In a similar manner it can be shown that

(5) $$g(x) = 0 \quad \text{if} \quad x > 0,$$

(6) $$g(x) = c_1 |x|^{\lambda_1 - 1} \quad \text{if} \quad -p < x < 0, \qquad \text{where } c_1 > 0,$$

(7) $$g(x) = (\lambda_1/x)[G(x) - G(x+p)] \quad \text{if} \quad x < 0,$$

and

(8) $$xg'(x) = (\lambda_1 - 1)g(x) - \lambda_1 g(x+p) \quad \text{if} \quad x < 0.$$

It was also shown in Wolfe (1971) that

(9) $$xh'(x) = (\lambda_1+\lambda_2-1)h(x)-\lambda_1 h(x+p)-\lambda_2 h(x-p) \qquad \text{if} \quad x \neq 0.$$

It follows from the above statements that

(10) $$f^{(n)}(x) = c_2{}^* x^{\lambda_2-n-1} \qquad \text{if} \quad 0 < x < p,$$

(11) $$f^{(n)}(x) = (1/x)[(\lambda_2-n)f^{(n-1)}(x)-\lambda_2 f^{(n-1)}(x-p)] \qquad \text{if} \quad x > 0,$$

(12) $$g^{(n)}(x) = (-1)^n c_1{}^* |x|^{\lambda_1-n-1} \qquad \text{if} \quad -p < x < 0,$$

(13) $$g^{(n)}(x) = (1/x)[(\lambda_1-n)g^{(n-1)}(x)-\lambda_1 g^{(n-1)}(x+p)] \qquad \text{if} \quad x < 0,$$

and

(14) $$h^{(n)}(x) = (1/x)[(\lambda_1+\lambda_2-n)h^{(n-1)}(x)-\lambda_1 h^{(n-1)}(x+p)-\lambda_2 h^{(n-1)}(x-p)] \qquad \text{if} \quad x \neq 0,$$

where $c_1{}^* = (\lambda_1-1)\cdots(\lambda_1-n)c_1$ and $c_2{}^* = (\lambda_2-1)\cdots(\lambda_2-n)c_2$.

The distribution functions $F(x)$ and $G(x)$ are L functions and are therefore absolutely continuous. It follows from (1), (3), (10), and (11) that if $k_2 > 0$ then $f(x)$ is absolutely continuous and has absolutely continuous derivatives of the first k_2-1 orders and integrable derivatives of the first k_2 orders. Similarly it follows from (5), (7), (12), and (13) that if $k_1 > 0$ then $g(x)$ is absolutely continuous and has absolutely continuous derivatives of the first k_1-1 orders and integrable derivatives of the first k_1 orders. Also $f^{(k_2)}(x)$ and $g^{(k_1)}(x)$ are absolutely continuous on $(-\infty, 0)$ and $(0, \infty)$.

Since $H(x) = F^*G(x)$ it follows that

(15) $$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

By Fubini's theorem it is possible to differentiate under the integral sign and get that

(16) $$h^{(i+j)}(x) = \int_{-\infty}^{\infty} f^{(i)}(x-y)g^{(j)}(y)\,dy$$

if $0 \leqq i \leqq k_2$, $0 \leqq j \leqq k_1$, and $0 \leqq i+j \leqq k_1+k_2-1$. Also

(17) $$\begin{aligned} h^{(k_1+k_2)}(x) &= \int_{-\infty}^{\infty} f^{(k_2)}(x-y)g^{(k_1)}(y)\,dy \qquad \text{if} \quad x > 0 \\ &= \int_{-\infty}^{\infty} f^{(k_2)}(y)g^{(k_1)}(x-y)\,dy \qquad \text{if} \quad x < 0. \end{aligned}$$

If $k_1 > 0$ or $k_2 > 0$ then $h(x)$ is absolutely continuous. If $0 \leqq i \leqq k_2$, $0 \leqq j \leqq k_1$, and $1 \leqq i+j \leqq k_1+k_2-1$ then $h^{(i+j)}(x)$ is absolutely continuous. From (16) it follows that $h(x)$ has integrable derivatives of the first k_1+k_2-1 orders. From (14) it follows that $h(x)$ and all derivatives of $h(x)$ vanish at infinity. From the definition of k, k_1, and k_2, it follows that either $k = k_1+k_2$ or $k = k_1+k_2+1$. Two cases must be considered.

Case 1. $k = k_1+k_2$. In this case it has been shown that $H(x)$ has absolutely continuous derivatives of the first k orders that vanish at infinity and integrable derivatives of the first $k+1$ orders. From (2), (6), and (17) it follows that if $0 < x < p/4$,

$$h^{(k_1+k_2)}(x) = c_1{}^*c_2{}^*\int_0^{p/4}(x+y)^{\lambda_2-k_2-1}y^{\lambda_1-k_1-1}\,dy + \int_{p/4}^{\infty} f^{(k_2)}(x+y)g^{(k_1)}(-y)\,dy. \tag{18}$$

Let $A_1(x)$ and $A_2(x)$ denote respectively the first and second expressions on the right side of (18). By (11) $f^{(k_2)}(x)$ is bounded on $(p/4, \infty)$. It follows from this that $A_2(x)$ is bounded on $(0, p/4)$. If $k = k_1+k_2$ then $0 < \lambda_1 - k_1 < 1$ and $0 < \lambda_2 - k_2 < 1$. It follows that $(x+y)^{\lambda_2-k_2-1} \geqq y^{\lambda_2-k_2-1}$ for $0 < x < p/4$ and $0 < y < p/4$ and that

$$A_1(x) \geqq c_1{}^*c_2{}^*\int_0^{p/4}(x+y)^{\lambda_1+\lambda_2-k_1-k_2-2}\,dy \tag{19}$$

for $0 < x < p/4$. It follows from (19) that $\lim_{x\to 0+} A_1(x) = \infty$. Thus $\lim_{x\to 0+} h^{(k_1+k_2)}(x) = \infty$ and $h^{(k_1+k_2)}(x)$ is not continuous.

Case 2. $k = k_1+k_2+1$. It has been shown that $h(x)$ has absolutely continuous derivatives of the first k_1+k_2-1 orders. It will now be shown that $h^{(k_1+k_2)}(x)$ is absolutely continuous. Using the notation previously developed it can easily be seen that since $f^{(k_2)}(x)$ is absolutely continuous on $(p/4, \infty)$ and $g^{(k_1)}(x)$ is integrable that $A_2(x)$ is continuous from the right at 0. If $0 < x < p/4$ then

$$A_1(x) \leqq c_1{}^*c_2{}^*\int_0^{p/4} y^{\lambda_1+\lambda_2-k_1-k_2-1}\,dy < \infty. \tag{20}$$

From the monotone convergence theorem and (20) it follows that $A_1(x)$ is continuous from the right at 0. Thus $h^{(k_1+k_2)}(x)$ is continuous from the right at 0. In a similar manner it can be shown that $h^{(k_1+k_2)}(x)$ is continuous from the left at 0. Thus $h^{(k_1+k_2)}(x)$ is continuous at 0. From (14) and the fact that $h^{(k_1+k_2-1)}$ is absolutely continuous it follows that $h^{(k_1+k_2)}(x)$ is absolutely continuous and $h^{(k_1+k_2+1)}(x)$ is integrable.

To complete the proof it must be shown that $h^{(k_1+k_2+1)}(x)$ is not continuous. If $x > 0$ it follows from (11) that $f^{(k_2+1)}(y)$ is integrable over (x, ∞). From (10), (12), (17), and Fubini's theorem, it follows that for almost all x such that $0 < x < p/4$,

$$h^{(k_1+k_2+1)}(x) = \int_0^{\infty} f^{(k_2+1)}(x+y)g^{(k_1)}(-y)\,dy \tag{21}$$

and

$$h^{(k_1+k_2+1)}(x) = (\lambda_2-k_2-1)c_1{}^*c_2{}^*\int_0^{p/4}(x+y)^{\lambda_2-k_2-2}y^{\lambda_1-k_1-1}\,dy + \int_{p/4}^{\infty} f^{(k_2+1)}(x+y)g^{(k_1)}(-y)\,dy. \tag{22}$$

In a similar manner it follows from the same statements that for almost all x such that $-p/4 < x < 0$,

$$h^{(k_1+k_2+1)}(x) = \int_0^{\infty}\ {}^{(k_2)}(y)g^{(k_1+1)}(x-y)\,dy \tag{23}$$

and

(24) $h^{(k_1+k_2+1)}(x) = (\lambda_1 - k_1 - 1)c_1{}^*c_2{}^*\int_0^{p/4}(x+y)^{\lambda_1-k_1-2}y^{\lambda_2-k_2-1}\,dy$

$$+\int_{p/4}^{\infty} f^{(k_2)}(y)g^{(k_1+1)}(x-y)\,dy.$$

Let $A_3(x)$ and $A_4(x)$ denote the first and second expressions on the right side of (22). From (13) and the fact that $g^{(k_1-1)}(y)$ is bounded it follows that $g^{(k_1)}(x)$ is bounded on $(-\infty, -p/4)$. From (11) and the fact that $f^{(k_2)}(y)$ is integrable it follows that $f^{(k_2+1)}(x)$ is integrable over $(p/4, \infty)$. It follows from these two statements that $A_4(x)$ is bounded for $x > 0$. If $\lambda_2 - k_2 < 1$ then it follows by an argument similar to the one used in the proof of the previous case that $\lim_{x\to 0+} A_3(x) = -\infty$. Thus $\lim_{x\to 0+} h^{(k_1+k_2+1)}(x) = -\infty$. By a similar proof it can be shown that if $\lambda_1 - k_1 < 1$ then $\lim_{x\to 0-} h^{(k_1+k_2+1)}(x) = -\infty$. If $\lambda_1 - k_1 = 1$ and $\lambda_2 - k_2 = 1$ it follows from (10) and (12) that

(25) $$\int_0^{\infty} f^{(k_2+1)}(y)g^{(k_1)}(-y)\,dy = -c_1{}^*c_2{}^* + \int_0^{\infty} f^{(k_2)}(y)g^{(k_1+1)}(-y)\,dy.$$

It follows from (21), (22), (23), (24) and (25) that $h^{(k_1+k_2+1)}(x)$ is bounded and $h^{(k_1+k_2+1)}(+0) = h^{(k_1+k_2+1)}(-0) - c_1{}^*c_2{}^*$ if $\lambda_1 - k_1 = 1$ and $\lambda_2 - k_2 = 1$. If $k = k_1 + k_2 + 1$ then $c_1{}^* > 0$ and $c_2{}^* > 0$. It has been shown that $h^{(k_1+k_2+1)}(x)$ is not continuous at 0 if $k = k_1 + k_2 + 1$. □

LEMMA 3. *Let $F_0(x)$ be a non-degenerate L function with Lévy spectral function $M_0(u)$. Let $\lambda_0(u) = uM_0'(u)$ for $u \neq 0$. Assume that $\lambda_0(u) = -\lambda_1$ for $-p \leqq u < 0$ and $\lambda_0(u) = \lambda_2$ for $0 < u \leqq p$ where $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, and $p > 0$. Let k be the largest integer smaller than $\lambda_1 + \lambda_2$. The L function $F_0(x)$ does not have a continuous derivative of the* $(k+1)$th *order.*

PROOF. It will be assumed that $\lambda_1 > 0$ and $\lambda_2 > 0$. The proof can be modified slightly if $\lambda_1 = 0$ or $\lambda_2 = 0$. Without loss of generality the centering constant γ of $F_0(x)$ can be chosen so that $F_0(x)$ has characteristic function

$$\hat{f}_0(t) = \exp\{\textstyle\int_{-\infty}^{-0} + \int_{+0}^{+\infty}((e^{iut}-1)/u)\lambda_0(u)\,du\}.$$

Let

$$\begin{aligned} \lambda_1(u) &= \lambda_0(u) && \text{if } -p \leqq u < 0 \text{ or } 0 < u \leqq p \\ &= 0 && \text{if } u < -p \text{ or } u > p \\ \lambda_2(u) &= \lambda_0(u) - \lambda_1(u) && \text{if } u \neq 0. \end{aligned}$$

Let $F_j(x)$ be a distribution function with characteristic function

$$\hat{f}_j(t) = \exp\{\textstyle\int_{-\infty}^{-0} + \int_{+0}^{+\infty}((e^{iut}-1)/u)\lambda_j(u)\,du\}$$

for $j = 1$ and $j = 2$. It can easily be seen that $F_1(x)$ is an L function and $F_2(x)$ is an infinitely divisible distribution function. Since $\hat{f}_0(t) = \hat{f}_1(t)\hat{f}_2(t)$ it follows that $F_0(x) = F_1{}^*F_2(x)$. It can easily be seen that

$$0 \leqq \textstyle\int_{-\infty}^{-0} + \int_{+0}^{+\infty} \lambda_2(u)/u\,du = \int_{-\infty}^{-p} + \int_p^{\infty} M_0'(u)\,du = M_0(-p) - M_0(p) < \infty.$$

Let $c = \int_{-\infty}^{-0} + \int_{+0}^{+\infty} \lambda_2(u)/u\, du$,

$$g(u) = \lambda_2(u)/cu \quad \text{if} \quad u \neq 0,$$
$$= 0 \quad \text{if} \quad u = 0,$$

and $G(x) = \int_{-\infty}^{x} g(u)\, du$.

The distribution function $G(x)$ is absolutely continuous and

$$\hat{f}_2(t) = \exp\{c \int_{-\infty}^{\infty} (e^{iut} - 1)\, dG(u)\}.$$

Let $E(x)$ denote the distribution function degenerate at 0. By a lemma of H. G. Tucker (1962, Lemma 3, page 1126) $F_2(x)$ has a jump of size e^{-c} at 0 and is absolutely continuous elsewhere. Thus

(26) $$F_2(x) = e^{-c}E(x) + (1 - e^{-c}) \int_{-\infty}^{x} h(y)\, dy$$

where $h(x)$ is the density function of an absolutely continuous distribution function $H(x)$. From Lemma 2 it follows that $F_1(x)$ has absolutely continuous derivatives of the first k orders and integrable derivatives of the first $k+1$ orders. From (26) and Lemma 1 it follows that for almost all x

(27) $$F_0^{(k+1)}(x) = e^{-c}F_1^{(k+1)}(x) + (1 - e^{-c}) \int_{-\infty}^{\infty} F_1^{(k+1)}(x-y)h(y)\, dy.$$

Let k_1 and k_2 be the largest integers smaller than λ_1 and λ_2 respectively. Three cases must be considered.

Case 1. $k = k_1 + k_2$. In this case it was shown in the proof of Lemma 2 that $\lim_{x \to 0+} F_1^{(k+1)}(x) = \infty$. Since $F_1^{(k)}(x)$ is absolutely continuous and vanishes at infinity it follows from (14) that $F_1^{(k+1)}(x)$ is bounded from below. It follows from (27) that $\lim_{x \to 0+} F_0^{(k+1)}(x) = \infty$. Thus $F_0^{(k+1)}(x)$ has a discontinuity at 0.

Case 2. $k = k_1 + k_2 + 1$ and either $\lambda_1 - k_1 < 1$ or $\lambda_2 - k_2 < 1$. In this case it was shown that either $\lim_{x \to 0+} F_1^{(k+1)}(x) = -\infty$, or $\lim_{x \to 0-} F_1^{(k+1)}(x) = -\infty$. Since $F_1^{(k)}(x)$ is absolutely continuous and vanishes at infinity it follows from (14) that $F_1^{(k+1)}(x)$ is bounded from above. It follows from (27) that either $\lim_{x \to 0+} F_0^{(k+1)}(x) = -\infty$ or $\lim_{x \to 0-} F_0^{(k+1)}(x) = -\infty$. Thus $F_0^{(k+1)}(x)$ has a discontinuity at 0.

Case 3. $k = k_1 + k_2 + 1$ and both $\lambda_1 - k_1 = 1$ and $\lambda_2 - k_2 = 1$. In this case it was shown that $F_1^{(k+1)}(x)$ is bounded and has a discontinuity at 0. It follows from the Lebesgue dominated convergence theorem that the second expression on the right side of the equality sign in (27) is continuous and therefore $F_0^{(k+1)}(x)$ has a discontinuity at 0. □

3. Main Theorem. The main theorem of this paper can now be proved.

THEOREM 4. *Let $F(x)$ be a non-degenerate L function with a normal component variance σ^2 and a Lévy spectral function $M(u)$. Let $\lambda(u) = uM'(u)$. If $\sigma^2 = 0$ and*

$k < \lambda(+0)+|\lambda(-0)| \leqq k+1$ for some integer k then $F(x)$ has absolutely continuous derivatives of the first k orders that vanish at infinity and integrable derivatives of the first $k+1$ orders. However, $F(x)$ does not have a continuous derivative of the $(k+1)$th order. If $\sigma^2 > 0$ or $\lambda(+0)+|\lambda(-0)| = \infty$ then $F(x)$ has absolutely continuous and integrable derivatives of all orders that vanish at infinity.

PROOF. The theorem will first be proved when $\sigma^2 = 0$ and $k < \lambda(+0)+|\lambda(-0)| \leqq k+1$ for some integer k. Without loss of generality the centering constant γ of $F(x)$ can be chosen so that $F(x)$ has characteristic function

$$\hat{f}(t) = \exp\{\textstyle\int_{-\infty}^{-0} + \int_{+0}^{+\infty} ((e^{iut}-1)/u)\lambda(u)\,du\}. \tag{28}$$

It will be assumed that $\lambda(+0) > 0$ and $\lambda(-0) < 0$. The proof can be modified slightly if $\lambda(+0) = 0$ or $\lambda(-0) = 0$. By Gnedenko and Kolmogorov (1954, Theorem 1, page 149) $\lambda(u)$ is non-increasing on $(-\infty, 0)$ and on $(0, \infty)$. Thus there exist constants $\lambda_1 > 0$, $\lambda_2 > 0$, and $p > 0$ such that $k < \lambda_1+\lambda_2$, $\lambda(u) < -\lambda_1$ for $-p \leqq u < 0$, and $\lambda(u) > \lambda_2$ for $0 < u \leqq p$. Let

$$\begin{aligned}
\lambda_1(u) &= 0 && \text{if } u < -p \text{ or } u > p\\
&= -\lambda_1 && \text{if } -p \leqq u < 0\\
&= \lambda_2 && \text{if } 0 < u \leqq p\\
\lambda_2(u) &= \lambda(u) - \lambda_1(u) && \text{if } u \neq 0\\
\lambda_3(u) &= \lambda(-0) && \text{if } -p \leqq u < 0\\
&= \lambda(+0) && \text{if } 0 < u \leqq p\\
&= \lambda(u) && \text{if } u < -p \text{ or } u > p\\
\lambda_4(u) &= \lambda_3(u) - \lambda(u) && \text{if } u \neq 0.
\end{aligned}$$

For $1 \leqq j \leqq 4$ let $F_j(x)$ be the distribution function with characteristic function

$$\hat{f}_j(t) = \exp\{\textstyle\int_{-\infty}^{-0} + \int_{+0}^{+\infty} ((e^{iut}-1)/u)\lambda_j(u)\,du\}.$$

It can easily be seen that $F_1(x)$ and $F_3(x)$ are L functions. Also $F_2(x)$ and $F_4(x)$ are infinitely divisible distribution functions. Since $\hat{f}(t) = \hat{f}_1(t)\hat{f}_2(t)$ and $\hat{f}_3(t) = \hat{f}(t)\hat{f}_4(t)$ it follows that $F(x) = F_1 * F_2(x)$ and $F_3(x) = F * F_4(x)$.

By Lemma 2, $F_1(x)$ has absolutely continuous derivatives of the first k orders and integrable derivatives of the first $k+1$ orders. Thus, by Lemma 1, $F(x)$ has absolutely continuous derivatives of the first k orders that vanish at infinity and integrable derivatives of the first $k+1$ orders.

It also follows from Lemma 1 that

$$F_3^{(k)}(x) = \textstyle\int_{-\infty}^{\infty} F^{(k)}(x-y)\,dF_4(y). \tag{29}$$

If $F^{(k+1)}(x)$ is bounded and continuous it could be shown using Fubini's theorem and the Lebesgue dominated convergence theorem that $F_3^{(k+1)}(x)$ is continuous.

By Lemma 3, ${F_3}^{(k+1)}(x)$ is not continuous. Therefore $F^{(k+1)}(x)$ cannot be bounded and continuous. V. M. Zolotarev has shown (1963, page 132) that $xF^{(k+1)}(x)$ is bounded. It follows from this fact that $F^{(k+1)}(x)$ vanishes at infinity and thus cannot be continuous. [It should be pointed out that it follows easily from statements of Zolotarev (1963, pages 131–132) that $xF^{(k+1)}(x)$ is continuous. Thus $F^{(k+1)}(x)$ has a discontinuity at the origin.]

If $\lambda(+0)+|\lambda(-0)| = \infty$ the proof can be modified to show that $F(x)$ has absolutely continuous and integrable derivatives of all orders that vanish at infinity. Finally, if $\sigma^2 > 0$ it can easily be shown using Lemma 1 that $F(x)$ has absolutely continuous and integrable derivatives of all orders that vanish at infinity. [In fact, if $\sigma^2 > 0$ then $F(x)$ is an analytic function (see Zolotarev (1963, Theorem 1, page 124)).]

4. Example. As an example of a distribution function that satisfies the hypothesis of Theorem 4, let $F_\lambda(x)$ be a gamma distribution function with density function

$$\begin{aligned} f_\lambda(x) &= 0 && \text{if} \quad x < 0. \\ &= (1/\Gamma(\lambda))x^{\lambda-1}e^{-x} && \text{if} \quad x \geqq 0. \end{aligned}$$

This distribution function is known to have a Lévy spectral function

$$\begin{aligned} M(u) &= 0 && \text{if} \quad u < 0 \\ &= -\lambda \textstyle\int_u^\infty e^{-x}/x\,dx && \text{if} \quad u > 0. \end{aligned}$$

(See Lukacs (1960, pages 91–93)). It follows from Theorem 4 that if $k < \lambda \leqq k+1$ for some integer k then $F_\lambda(x)$ has absolutely continuous derivatives of the first k orders that vanish at infinity and integrable derivatives of the first $k+1$ orders, but that $F_\lambda(x)$ does not have a continuous derivative of the $(k+1)$th order. This previous statement can easily be verified.

5. Generalization of the main theorem. In the proof of Theorem 4 it is necessary to assume that $F(x)$ is an L function in order to show that $F^{(k+1)}(x)$ vanishes at infinity and is thus not continuous. The rest of the proof of Theorem 4 does not depend upon the fact that F is an L function but only on the fact that its Lévy spectral function has a "nice" behavior in a neighborhood of the origin. Most of the proof of Theorem 4 can be generalized to a larger class of infinitely divisible distribution functions than the L functions.

LEMMA 5. *Let $F_0(x)$ be an infinitely divisible distribution function with a Lévy spectral function $M_0(u)$ that is absolutely continuous on $(-\infty, 0)$ and on $(0, \infty)$. Let $\lambda_0(u) = uM_0'(u)$. Assume that $\lambda(u) = -\lambda_1$ for $-p < u < 0$ and $\lambda(u) = \lambda_2$ for $0 < u < p$ where $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, $\lambda_1+\lambda_2 > 0$, and $p > 0$. Let k be the largest integer smaller than $\lambda_1+\lambda_2$. The distribution function $F_0(x)$ does not have a continuous derivative of the* $(k+1)$th *order.*

PROOF. The proof of this lemma is identical to the proof of Lemma 3, and is omitted here.

THEOREM 6. *Let $F(x)$ be an infinitely divisible distribution function without a normal component and with a Lévy spectral function $M(u)$ that is absolutely continuous on* $(-\infty, 0)$ *and on* $(0, \infty)$. *Let $\lambda(u) = uM'(u)$. Let $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, $\lambda_3 \geqq 0$, $\lambda_4 \geqq 0$, and $p > 0$ be constants. Let $k \geqq 0$ be an integer. Assume that $-\lambda_1 \leqq \lambda(u) \leqq -\lambda_2$ for almost all u such that $-p < u < 0$ and $\lambda_3 \leqq \lambda(u) \leqq \lambda_4$ for almost all u such that $0 < u < p$. If $k < \lambda_2+\lambda_3 \leqq \lambda_1+\lambda_4 \leqq k+1$ then $F(x)$ is absolutely continuous and has absolutely continuous derivatives of the first k orders that vanish at infinity and integrable derivatives of the first $k+1$ orders. However, $F(x)$ does not have a bounded and continuous derivative of the* $(k+1)$th *order.*

PROOF. The proof of this theorem is similar to the proof of Theorem 4. Lemma 5 is used in the proof in place of Lemma 3.

6. Continuity properties of infinitely divisible distribution functions. An infinitely divisible distribution function with a normal component has continuous derivatives of all orders. In this paper it has been shown that for a class of infinitely divisible distribution functions containing the L functions, it is possible to give necessary and sufficient conditions for the distribution functions to have a bounded and continuous derivative (or an absolutely continuous derivative) of the kth order. It would be of interest to extend these results.

The continuity properties of infinitely divisible distribution functions have been studied by many people. P. Hartman and A. Wintner showed (1942) that a necessary and sufficient condition for an infinitely divisible distribution function $F(x)$ to be continuous is that $\sigma^2 > 0$ or $\int_{-\infty}^{\infty} dM(u) = \infty$ where σ^2 is the normal component variance of $F(x)$ and $M(u)$ is the Lévy spectral function of $F(x)$. J. R. Blum and M. Rosenblatt (1959) obtained the same necessary and sufficient condition by an entirely different proof. H. G. Tucker (1962), M. Fisz and V. S. Varadarajan (1963) showed independently that a sufficient condition for $F(x)$ to be absolutely continuous is that $\int_{-\infty}^{\infty} dM_{ac}(x) = \infty$ where $M_{ac}(x)$ is the absolutely continuous component of $M(x)$. H. G. Tucker (1965) gave a necessary and sufficient condition for $F(x)$ to be absolutely continuous. Although his condition is not very satisfactory, it is perhaps the best that can be given. S. Orey (1968) gave a sufficient condition on $M(u)$ for $F(x)$ to have continuous derivatives of all orders. This result replaces an erroneous condition of P. Hartman and A. Wintner (1942).

Let $F(x)$ be an infinitely divisible distribution function without a normal component. In view of the work by H. G. Tucker, it is probably not possible to give a necessary and sufficient condition on $M(u)$ for $F(x)$ to have an absolutely continuous derivative of the kth order. However, it might be possible to generalize the proof of Theorem 6 and give a necessary and sufficient condition on $M(u)$ for $F(x)$ to have an absolutely continuous derivative of the kth order in the case when $M(u)$ is absolutely continuous on $(-\infty,0)$ and on $(0, \infty)$. It may also be possible to give a necessary and sufficient condition on $M(u)$ for $F(x)$ to have a continuous derivative of the kth order.

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