The Annals of Mathematical Statistics
1971, Vol. 42, No. 6, 2056–2063

REGULARITY OF EXCESSIVE FUNCTIONS II[1]

BY R. K. GETOOR

University of California, San Diego

1. Introduction. In [3] we introduced some conditions which are equivalent to the regularity of excessive functions under certain additional hypotheses. Unfortunately the additional hypotheses assumed in [3] are much too strong, and, in fact, can be eliminated completely. Thus the present paper represents a considerable extension and simplification of the results of [3]. Moreover, it may be read independently of [3].

All terminology and notation are the same as in [1] unless explicitly stated otherwise. In particular we fix once and for all a standard process $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ with state space $(E, \mathscr{E})$, and all stopping times are $\{\mathscr{F}_t\}$ stopping times unless explicitly stated otherwise. In Section 2 we characterize those stopping times which are accessible on $\{T < \zeta\}$—here accessibility is defined as in the general theory of processes, [4] or [7], and not as in [1]. See Section 2 for the precise definition. In the case of a special standard process our result reduces to a criterion of Meyer [6]. Even though all of the techniques needed for the construction in Section 2 are well known, we have given some details since the result seems to have been overlooked in the literature and is, perhaps. of some independent interest. In any case it is crucial for the discussion in Section 4. In Section 3 we introduce a topology on the state space E which for lack of a better name we call the d-topology. In Section 4 we relate this topology to the regularity of excessive functions. Roughly speaking, an excessive function is regular if and only if it is d-continuous. See Proposition 4.2 for the precise statement. Also we give a necessary and sufficient condition that all excessive functions be regular. See Proposition 4.4.

2. Accessibility of stopping times. Recall that a stopping time T is *accessible* if for each initial measure μ there exists a sequence $\{\Lambda_k\}$ of sets in $\mathscr{F}_T$ such that $\{T > 0\} = \cup \Lambda_k$ almost surely P^μ and for each k there exists an increasing sequence $\{T_n{}^k\}$ of stopping times bounded by T and such that almost surely P^μ, $\{T_n{}^k\}$ increases to T strictly from below on Λ_k; i.e., $\lim_n T_n{}^k = T$ and $T_n{}^k < T$ for all n almost surely P^μ on Λ_k. If T is a stopping time and $\Lambda \in \mathscr{F}_T$ we say that T is accessible on Λ provided T_Λ is accessible where $T_\Lambda = T$ on Λ and $T_\Lambda = \infty$ on Λ^c. It is easily seen that this is equivalent to the statement that for each μ, almost surely P^μ we have $\{T > 0\} \cap \Lambda = \cup \Lambda_k$ with each $\Lambda_k \in \mathscr{F}_T$ and with T the limit strictly from below on Λ_k of an increasing sequence of stopping times. In general we use the terms accessible, totally inaccessible, and previsible as in the general theory of processes

Received March 15, 1971.

[1] This research was partially supported by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant AF–AFOSR 67–1261B.

[7]. Note that a stopping time T is accessible if and only if it is accessible on $\{T < \infty\}$.

There is one trivial technical point in applying the results from the general theory of processes on which a comment is, perhaps, in order. As in [1], $\mathscr{F}^0 = \sigma\{X_s : s \geqq 0\}$ and $\mathscr{F}_t{}^0 = \sigma\{X_s : 0 \leqq s \leqq t\}$. For each initial measure μ, $\mathscr{F}^\mu$ denotes the completion of $\mathscr{F}^0$ with respect to P^μ and $\mathscr{F}_t{}^\mu$ is the σ-algebra generated by $\mathscr{F}_t{}^0$ and all P^μ null sets. The fundamental σ-algebras for X are then $\mathscr{F} = \cap_\mu \mathscr{F}^\mu$ and $\mathscr{F}_t = \cap_\mu \mathscr{F}_t{}^\mu$ where the intersection is over all initial probability measures μ. Now the proof of (I$-$8.12) in [1] shows that for each μ the family $(\mathscr{F}_t{}^\mu)$ is right continuous. Thus the results from the general theory of processes can be applied to $(\mathscr{F}^\mu, \mathscr{F}_t{}^\mu, P^\mu)$ for each μ. But if T is an $(\mathscr{F}_t{}^\mu)$ stopping time, then there exists an $(\mathscr{F}_t)$—actually an $(\mathscr{F}^0_{t+})$—stopping time T^μ such that $P^\mu(T \neq T^\mu) = 0$. See (I$-$7.3) of [1]. Making use of these facts one can apply the general results to $(\mathscr{F}_t)$ stopping times provided one exercises a modicum of care. We shall say no more about such trivialities.

We now state the main result of this section.

(2.1) THEOREM. *Let T be a stopping time. Then T is accessible on*

$$\{X_T = X_{T-}; T < \zeta\}.$$

We shall break up the proof of Theorem 2.1 into several steps. The first fact that we need is contained implicitly in [5] and also appears in [2]. However, it is not completely clear precisely what is being assumed about X in these references and so we shall give the proof here.

(2.2) PROPOSITION. *Let f be a finite nearly Borel measurable function such that almost surely $t \to f(X_t)$ is right continuous and has left-hand limits on* $[0, \infty)$. *Let* $f(X_t)_- = \lim_{s \uparrow t} f(X_s)$ *if* $0 < t < \infty$ *and* $f(X_0)_- = f(X_0)$. *Given* $\varepsilon > 0$ *define*

$$T = \inf\{t : |f(X_t) - f(X_t)_-| > \varepsilon; X_t = X_{t-}; t < \zeta\}. \tag{2.3}$$

Then T is accessible.

PROOF. First of all it is well known (and an easy consequence of (IV–52) of [4]) that T is a terminal time. Clearly $T > 0$ almost surely and $T = \infty$ on $\{T \geqq \zeta\}$. We shall show that there exists a sequence $\{\Lambda_k\}$ in $\mathscr{F}_T$ independent of the initial measure μ such that $\{T < \zeta\} = \cup \Lambda_k$ almost surely, and such that given μ for each k there exists $\{T_n{}^k\}$ increasing to T strictly from below on Λ_k almost surely P^μ. This and the previous observation imply that T is accessible.

In view of the regularity assumptions on $t \to f(X_t)$ the infimum in (2.3) is attained if it is finite, and so $|f(X_T) - f(X_T)_-| > \varepsilon$ on $\{T < \zeta\}$. Here and in what follows we omit the phrase "almost surely." Also (II–4.8) of [1] implies that f is finely continuous. Now define

$$A_k = \left\{x : \frac{k\varepsilon}{8} \leqq f(x) < \frac{(k+1)\varepsilon}{8}\right\}$$

$$E_k = \left\{x : \left| f(x) - \frac{k\varepsilon}{8} \right| \geqq \frac{\varepsilon}{4} \right\}$$

for $k = 0, \pm 1, \pm 2, \cdots$. Clearly A_k and E_k are nearly Borel sets, and since f is finite E is the disjoint union of the A_k. Define

$$(2.4) \qquad H(\omega) = T_{E_k}(\omega) \quad \text{if } X_0(\omega) \in A_k; \qquad = \infty \quad \text{if } X_0(\omega) = \Delta.$$

It is immediate that H is a stopping time. Since $|f(X_T) - f(X_T)_-| > \varepsilon$ on $\{T < \zeta\}$, either $|f(X_0) - f(X_T)_-| > \varepsilon/2$ in which case $H < T$ or $|f(X_T) - f(X_0)| > \varepsilon/2$ in which case $H \leqq T$. Thus in all cases $H \leqq T$. Next define $H_0 = 0, H_1 = H, \cdots, H_{n+1} = \min(H_n + H \circ \theta_{H_n}, T)$. If $H_n < T$, then $H \circ \theta_{H_n} \leqq T \circ \theta_{H_n} = T - H_n$, and so $H_{n+1} = H_n + H \circ \theta_{H_n} \leqq T$ on $\{H_n < T\}$. Consequently $|f(X_{H_n}) - f(X_{H_{n+1}})| \geqq \varepsilon/4$ if $H_n < T < \zeta$. Since $t \to f(X_t)$ has left limits it follows that $H_n = T$ for all sufficiently large n on $\{T < \zeta\}$. For $k \geqq 0$, let

$$\Lambda_k = \{H_k < H_{k+1} = T < \zeta\} = \{H_k < T\} \cap \{H_{k+1} = T\} \cap \{T < \zeta\} \in \mathscr{F}_T.$$

Then $\{T < \zeta\} = \cup \Lambda_k$.

Let μ be an initial (finite) measure on E. We fix $k \geqq 0$ and we then construct an increasing sequence $\{R_n\}$ of stopping times increasing to T strictly from below almost surely P^μ on Λ_k. As remarked previously this will establish Proposition 2.2. For each integer j define

$$\nu_j(B) = P^\mu[X_{H_k} \in B \cap A_j, H_k < T \wedge \zeta]$$

and

$$\nu(B) = \textstyle\sum_j \nu_j(B) = P^\mu[X_{H_k} \in B, H_k < T \wedge \zeta].$$

The measure ν_j does not charge E_j and so there exists a decreasing sequence of open sets $\{G_{j,n}\}$ containing E_j such that $T_n{}^j = T_{G_{j,n}}$ increases to T_{E_j} almost surely P^{ν_j} on $\{T_{E_j} < \zeta\}$. From (2.4), $H = T_{E_j}$ almost surely P^{ν_j} and thus $T_n{}^j \uparrow H$ almost surely P^{ν_j} on $\{H < \zeta\}$. But E_j is finely closed and so P^{ν_j} almost surely on $\{T = H < \zeta\}$ one has $X_T \in E_j \subset G_{j,n}$. Now $X_T = X_{T-}$ on $\{T < \zeta\}$ and consequently $T_n{}^j < T$ almost surely P^{ν_j} on $\{T = H < \zeta\}$; that is $\{T_n{}^j\}$ increases to T strictly from below almost surely P^{ν_j} on $\{T = H < \zeta\}$. Define $S_n = T_n{}^j$ if $X_0 \in A_j$ and $S_n = \infty$ if $X_0 = \Delta$. Then $\{S_n\}$ is an increasing sequence of stopping times with $S_n = T_n{}^j$ almost surely P^{ν_j}. Recalling the definition of ν it follows that $\{S_n\}$ increases to T strictly from below almost surely P^ν on $\{T = H < \zeta\}$. But ν is the distribution of $X(H_k)$ on $\{H_k < T \wedge \zeta\}$ under P^μ, and so ("^{c}" denotes complement)

$$0 = E^\mu\{P^{X(H_k)}\{[S_n \uparrow T; S_n < T, \forall n]^c, H = T < \zeta\}; H_k < T \wedge \zeta\}$$
$$= P^\mu\{[H_k + S_n \circ \theta_{H_k} \uparrow T; H_k + S_n \circ \theta_{H_k} < T, \forall n]^c, H_{k+1} = T < \zeta, H_k < T \wedge \zeta\}$$

because $H_k + H \circ \theta_{H_k} = H_{k+1}$ if $H_k < T$ and T and ζ are terminal times. Finally setting $R_n = \min(H_k + S_n \circ \theta_{H_k}, T)$ we see that $\{R_n\}$ increases to T strictly from below almost surely P^μ on $\Lambda_k = \{H_k < H_{k+1} = T < \zeta\}$, completing the proof of Proposition 2.2.

(2.5) COROLLARY. *Let $T_0 = 0$, $T_{n+1} = T_n + T \circ \theta_{T_n}$ for $n \geqq 0$ be the iterates of T. Then if $n \geqq 1$, T_n is accessible on $\{T_n < \zeta\}$.*

PROOF. Since $T_1 = T$ it suffices to prove that T_{n+1} is accessible on $\{T_{n+1} < \zeta\}$ for $n \geqq 1$. Using the notation of the proof of (2.2) one notes that $\theta_{T_n}^{-1}\Lambda_k \in \mathscr{F}_{T_{n+1}}$ and that

$$[\bigcup_k \theta_{T_n}^{-1}\Lambda_k] \cap \{T_n < \zeta\} = \{T_{n+1} < \zeta\}.$$

Given an initial measure μ, let $\nu(B) = P^\mu[X(T_n) \in B; T_n < \zeta\}$ and let $\{R_j\}$ be an increasing sequence of stopping times that increases to T strictly from below on Λ_k almost surely P^ν. Then one easily checks that $\{T_n + R_j \circ \theta_{T_n}\}$ increases to T_{n+1} strictly from below on $(\theta_{T_n}^{-1}\Lambda_k) \cap \{T_n < \zeta\}$ almost surely P^μ, proving (2.5).

(2.6) PROPOSITION. *Let T be a stopping time such that $X_T = X_{T-}$ almost surely on $\{T < \zeta\}$. Then T is accessible on $\{T < \zeta\}$.*

PROOF. Let μ be a fixed initial measure. Let T_A and T_I be the accessible and totally inaccessible parts of T (relative to P^μ). See [4]. Then (2.6) will follow provided we show $P^\mu(T_I < \zeta) = 0$, since $T = T_A \wedge T_I$ on $\{0 < T < \infty\}$. Here and in the remainder of this proof all statements are understood to hold almost surely P^μ. Next let $R = T_I$ if $T_I < \zeta$ and $R = \infty$ if $T_I \geqq \zeta$. Then we must show that $P^\mu(R < \infty) = 0$. Clearly if $P^\mu(R < \infty) > 0$ then R is totally inaccessible and we shall obtain a contradiction to this last statement by show that R is accessible if $P^\mu(R < \infty) > 0$. Implicitly in [6], pages 111 to 116, Meyer showed that a stopping time S is accessible provided that $f(X_S) = f(X_S)_-$ on $\{S < \infty\}$ for all $f = U^\alpha g$ with $\alpha > 0$ and g a bounded universally measurable function. This part of Meyer's argument is valid for arbitrary standard processes. However, it ultimately rests on Theorem VII–47 of [4] which has as its hypothesis that the basic family of σ-algebras is free of times of discontinuity. But this theorem gives a necessary and sufficient condition that a stopping time be accessible and a careful reading of its proof reveals that the *stated condition is sufficient without the assumption that the basic family of σ-algebras be free of times of discontinuity.* Consequently Meyer's argument shows that the above criterion is a sufficient condition for a stopping time S to be accessible for general standard processes. We now apply this criterion to the stopping time R defined above. If $R < \infty$, then $R < \zeta$ and $R = T_I = T$, and so by hypothesis $X_R = X_{R-}$ on $\{R < \zeta\} = \{R < \infty\}$. But if $f(X_R) \neq f(X_R)_-$ for some $f = U^\alpha g$ as above, then for some $\varepsilon > 0$

$$P^\mu[|f(X_R) - f(X_R)_-| > \varepsilon; R < \zeta] > 0.$$

Let T^ε be defined as in (2.3) and let (T_n^ε) denote the iterates of T^ε. Then for some n, $P^\mu[R = T_n^\varepsilon < \zeta] > 0$, and since f obviously satisfies the hypotheses of (2.2) it follows from (2.5) that R is not totally inaccessible. Consequently $f(X_R) = f(X_R)_-$ on $\{R < \infty\}$ and hence R is accessible. This contradiction establishes (2.6).

The proof of Theorem 2.1 is now immediate. Namely let $\Lambda = \{X_T = X_{T-}; T < \zeta\}$ and let $S = T_\Lambda$. Then S satisfies the hypotheses of (2.6) and so S is accessible because $\{S < \zeta\} = \{S < \infty\}$. But this is just the statement that T is accessible on Λ.

We close this section by pointing out that the condition in Theorem 2.1 is necessary in the sense that if T is a stopping time that is accessible on a set $\Lambda \in \mathscr{F}_T$, then it follows from the quasi-left-continuity of X that $X_T = X_{T-}$ almost surely on $\Lambda \cap \{T < \zeta\}$.

3. The *d*-topology. In this section we introduce a topology on E that will be used to characterize regular excessive functions in the next section. *A subset D of E is called a d-set provided that it is nearly Borel measurable and that whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T*

$$(3.1) \qquad P^x\{X(T_n) \in D \quad \text{for all} \quad n, \qquad X(T) \notin D, T < \zeta\} = 0$$

for all x. That is, almost surely on $\{T < \zeta\}$, $X(T)$ is in D if $X(T_n)$ is in D for all n. We let $\mathscr{D}$ be the collection of all d-sets. In view of the quasi-left-continuity of X it is clear that all closed subsets of E are in $\mathscr{D}$.

(3.2) LEMMA. *A nearly Borel set D is in $\mathscr{D}$ if and only if whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T*

$$(3.3) \qquad P^x\{X(T_n) \in D \quad \textit{infinitely often}, \quad X(T) \notin D; T < \zeta\} = 0$$

for all x.

PROOF. Clearly it suffices to show that if $D \in \mathscr{D}$, then it satisfies the condition in (3.2). Let $\{T_n\}$ and T be as above and let $\Lambda = \{X(T_n) \in D$, i.o., $T < \zeta\}$. Since D is nearly Borel, $\Lambda \in \mathscr{F}_T$. Now define $R_n = T_n$ if $X(T_n) \in D$ and $R_n = \infty$ if $X(T_n) \notin D$ so that each R_n is a stopping time. Finally let $S_n = \inf_{k \geq n} R_k$. Then $\{S_n\}$ is an increasing sequence of stopping times and we let $S = \lim S_n$. But on Λ, $R_n = T_n$ infinitely often and so $S = T$ and $X(S_n) \in D$ for all n. Consequently $P^x(\Lambda; X(T) \notin D) = 0$ establishing (3.2).

It is now easy to check using (3.2) that $\mathscr{D}$ is closed under finite unions and countable intersections. It is evident that the complements of sets in $\mathscr{D}$ form a base for a topology on E. We call this the d-topology and it is clear that it is finer than the original topology on E.

The following definition will be useful in the next section.

(3.4) DEFINITION. A numerical function f on E is *strongly d-continuous* if $f^{-1}(C) \in \mathscr{D}$ for all closed subsets C of $\bar{R}$.

In view of the properties of $\mathscr{D}$ it is clear that f is strongly d-continuous if and only if $f^{-1}([a, b]) \in \mathscr{D}$ for all closed intervals $[a, b]$ in $\bar{R}$.

4. Regularity of excessive functions. We say that a numerical function f on E is *smooth* provided f is nearly Borel measurable and almost surely $t \to f(X_t)$ is right continuous and has left limits on $[0, \infty)$. In light of (II–2.12) and (II–4.8) of [1] every excessive function is smooth, and every smooth function is finely continuous. (Of course, continuity of numerical functions is defined relative to the usual topology on $\bar{R} = [-\infty, \infty]$.) A numerical function f on E is *regular* provided that it is smooth and that almost surely $t \to f(X_t)$ is continuous wherever $t \to X_t$ is continuous on $[0, \zeta)$. Finally a numerical function f on E is *quasi-left-continuous* (qlc) provided whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T, then $f(X_{T_n}) \to f(X_T)$ almost surely on $\{T < \zeta\}$.

Although the following proposition is well known, at least for special standard processes, we include the proof for completeness.

(4.1) PROPOSITION. *A smooth function f is regular if and only if it is* qlc.

PROOF. First note that there is no loss of generality in assuming f bounded. Indeed if $q:[-\infty, -\infty] \to [-1, 1]$ is defined by $q(t) = e^t - 1$ if $t \leqq 0$ and $q(t) = 1 - e^{-t}$ if $t \geqq 0$, then f is smooth, regular, or qlc if and only if $q \circ f$ is. We also note for later use that f is α-excessive if and only if $q \circ f$ is. See the proof (II–2.12) of [1].

Now suppose f is regular and that $\{T_n\}$ is an increasing sequence of stopping times with limit T. On the set of ω's for which $t \to X_t(\omega)$ is continuous at $T(\omega) < \zeta(\omega)$ one has $f(X_{T_n}) \to f(X_T)$ by the definition of regularity. On the set of ω's for which $t \to X_t(\omega)$ is discontinuous at $T(\omega) < \zeta(\omega)$ one must have $T_n(\omega) = T(\omega)$ for all sufficiently large n because of the qlc of the process X, and so $f(X_{T_n}) \to f(X_T)$ is this case also. Thus f is qlc. Here again we have omitted the phrase "almost surely." Conversely suppose f is smooth and that f is not regular. Then for some $\varepsilon > 0$ and some x one has $P^x(T < \zeta) > 0$ where T is the stopping time defined in (2.3). Thus by (2.2) there exists an increasing sequence of stopping times $\{R_n\}$ such that with positive P^x probability $\{R_n\}$ increases to T strictly from below on $\{T < \zeta\}$. Consequently f is not qlc, establishing (4.1).

(4.2) PROPOSITION. *A smooth function f is regular if and only if it is strongly d-continuous.*

PROOF. Again there is no loss of generality in assuming that f is bounded. Suppose firstly that f is regular. Let F be a closed subset of the real line and let $D = f^{-1}(F)$. Let $\{T_n\}$ be an increasing sequence of stopping times with limit T such that $X(T_n) \in D$ for all n, that is $f(X_{T_n}) \in F$. But f is regular and so by (4.1), $f(X_{T_n}) \to f(X_T)$ if $T < \zeta$. Therefore $f(X_T) \in F$ or $X_T \in D$. Thus f is strongly d-continuous.

Conversely suppose that f is smooth, strongly d-continuous, but not regular. Then for some $\varepsilon > 0$ and some x one has $P^x(T < \zeta) > 0$ where T is defined in (2.3), and it follows from (2.2) that there are a set $\Lambda \in \mathscr{F}_T$ with $\Lambda \subset \{T < \zeta\}$ and $P^x(\Lambda) > 0$ and an increasing sequence of stopping times $\{R_n\}$ which increases to T

strictly from below on Λ almost surely P^x. But then there exists an integer k such that if

$$\Gamma = \left\{|f(X_T)-f(X_T)_-| > \varepsilon, \frac{k}{4}\varepsilon \leqq f(X_T) < \frac{k+1}{4}\varepsilon, T < \zeta\right\} \cap \Lambda$$

then $P^x(\Gamma) > 0$. Let

(4.3) $$D = f^{-1}\left[\left(-\infty, \frac{k-1}{4}\varepsilon\right] \cup \left[\frac{k+2}{4}\varepsilon, \infty\right)\right]; B = f^{-1}\left(\left[\frac{k}{4}\varepsilon, \frac{k+1}{4}\varepsilon\right]\right).$$

Clearly D and B are disjoint nearly Borel sets and that $D \in \mathscr{D}$ since f is strongly d-continuous. But almost surely P^x on Γ, $\{R_n\}$ increases to T strictly from below, and consequently $X(R_n) \in D$ for all large n while $X(T) \in B$. This contradicts the fact that $D \in \mathscr{D}$ completing the proof of (4.2).

We come now to the main result of this section. Recall that a nearly Borel set B is called *finely perfect* if $B = B^r$. Clearly such a B is finely closed. Let $\mathscr{P}$ denote the class of all nearly Borel finely perfect sets. Finally recall that $\mathscr{S}^\alpha$ denotes the collection of all α-excessive functions, $\alpha \geqq 0$.

(4.4) PROPOSITION. *If $\alpha > 0$ all α-excessive functions are regular if and only if $\mathscr{P} \subset \mathscr{D}$. This statement is also true when $\alpha = 0$ provided there exists a strictly positive bounded Borel function h such that Uh is finite.*

PROOF. Suppose first of all that $\mathscr{P} \subset \mathscr{D}$ and let f be a bounded smooth function that is not regular. We argue exactly as in the second paragraph of the proof of (4.2) until we come to (4.3). In place of (4.3) we define

$$G = f^{-1}\left[\left(-\infty, \frac{k-1}{4}\varepsilon\right), \left(\frac{k+2}{4}\varepsilon, \infty\right)\right]; B = f^{-1}\left(\left[\frac{k}{4}\varepsilon, \frac{k+1}{4}\right]\right).$$

Then G is nearly Borel and finely open and $G \subset D$ where D is defined in (4.3). Let $\phi(x) = E^x\{e^{-T_G}\}$. Then if H is the fine closure of G we have

$$H = G \cup G^r = \{\phi = 1\} \subset D,$$

which shows that $H \in \mathscr{P}$ and that H and B are disjoint. We now obtain exactly the same contradiction as in the proof of (4.2) except that we use the set H in place of D. Thus $\mathscr{P} \subset \mathscr{D}$ implies that every smooth function is regular, and so the elements of $\mathscr{S}^\alpha$ are regular for any $\alpha \geqq 0$.

Conversely suppose $\alpha > 0$ and that the elements of $\mathscr{S}^\alpha$ are regular. If $F \in \mathscr{P}$ let $\phi_F^\alpha(x) = E^x\{e^{-\alpha T_F}\}$. Then $F = \{\phi_F^\alpha = 1\}$ is in $\mathscr{D}$ by Proposition 4.2. If $\alpha = 0$ and $F \in \mathscr{P}$ let

$$\phi_F(x) = Uh(x) - P_F\, Uh(x) = E^x \textstyle\int_0^{T_F} h(X_t)\, dt.$$

Since h is strictly positive $F = F^r = \{\phi_F = 0\}$. But Uh and $P_F Uh$ are finite and regular because they are excessive, and so ϕ_F is regular. Now once again Proposition 4.2 implies that $F = \{\phi_F = 0\} \in \mathscr{D}$, completing the proof of Proposition 4.4.

REMARKS. We have actually shown that $\mathscr{P} \subset \mathscr{D}$ implies that all smooth functions are regular. The above proof also shows that a sufficient condition that $\mathscr{P} \subset \mathscr{D}$ is that for some $\alpha > 0$, ϕ_G^α is regular whenever G is a finely open nearly Borel set. Finally it is not difficult to see that a finely perfect set is in $\mathscr{D}$ if and only if it is *projective* as defined in (V–4.2) of [1].

REFERENCES

[1] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic Press, New York.

[2] GETOOR, R. K. (1964). Additive functionals of a Markov process. Lecture Notes, Univ. of Hamburg.

[3] GETOOR, R. K. (1970). Regularity of excessive functions. *Proc. Twelfth Biennial Seminar Canadian Math. Congress,* ed. R. Pyke. 157–164.

[4] MEYER, P. A. (1966). *Probability and Potentials.* Ginn, Boston.

[5] MEYER, P. A. (1962). Fonctionnelles multiplicatives et additives de Markov. *Ann. Inst. Fourier (Grenoble)* **12** 125–230.

[6] MEYER, P. A. (1967). Processus de Markov. *Lecture Notes in Mathematics* **26**. Springer, Berlin.

[7] MEYER, P. A. (1968). Guide détaillé de la théorie "generale": des processus. Strasbourg Seminar II. *Lecture Notes in Mathematics* **51**. Springer, Berlin.