The Annals of Mathematical Statistics
1971, Vol. 42, No. 6, 2044–2049

A NOTE ON HARMONIC FUNCTIONS AND MARTINGALES[1]

BY CHARLES W. LAMB
Stanford University

1. A decomposition theorem. In this note we will be concerned with the problem of decomposing a positive harmonic function h into a sum of three positive harmonic functions h_1, h_2, and h_3, each of which behaves quite differently when composed with Brownian motion. This problem has been treated in a very general context by Blumenthal and Getoor (1968) and, in fact, our main result (Theorem 1) is contained in Theorem 5.14 of Chapter IV. We present here a direct treatment based on the theory of conditional Brownian motion which, in addition to giving the required decomposition, characterizes the functions h_1, h_2 and h_3 in terms of their Martin boundary representations (Corollary 1). As will be seen in the examples, this last characterization is useful in understanding the nature of the non-uniformly integrable martingale component h_2. It is assumed throughout that the reader is familiar with the relationship between harmonic functions and Brownian motion as described in Doob (1954) and (1957a), and with the theory of the Martin boundary and conditional processes as developed in Doob (1957b) and (1958). Before stating the main result, we introduce some notation and recall a few facts.

Let D be a domain in n-dimensional Euclidean space which has a Green's function g. Let ∂D denote the Martin boundary of D and let ∂D_e denote the subset of ∂D consisting of the minimal points. $K(\eta, \cdot)$ will denote the minimal harmonic function associated with $\eta \in \partial D_e$ which is normalized so that $K(\eta, \xi_0) = 1$ for a fixed $\xi_0 \in D$.

Let Ω be the function space consisting of all continuous functions $\omega:[0, \infty) \to D \cup \partial D_e$ with the property that, if $\omega(s) = \eta \in \partial D_e$, then $\omega(t) = \eta$ for all $t \geqq s$. $X(t)$ will denote the tth coordinate function on Ω. Using the notation of Blumenthal and Getoor (1968), let $(\Omega, \mathscr{F}, \mathscr{F}(t), X(t), \theta(t), P_\xi)$ denote the standard Brownian motion process on D, stopped when ∂D_e is hit. Note that we can write P_ξ (or E_ξ) for $\xi \in D \cup \partial D_e$, but that each point of ∂D_e acts as an absorbing point. Define the lifetime τ by the equation

$$\tau(\omega) = \inf\{t : X(t) \in \partial D_e\}.$$

If h is a positive harmonic function on D, then $\lim_{t\uparrow\tau} h[X(t)]$ exists P_ξ-almost everywhere ($\xi \in D$) and, in fact, h defines a Borel measurable boundary function (which we continue to denote by h) on ∂D_e such that

$$\lim_{t\uparrow\tau} h[X(t)] = h[\lim_{t\uparrow\tau} X(t)] \tag{1}$$

P_ξ-almost everywhere ($\xi \in D$). When dealing with a given measure $P_\xi(\xi \in D)$, we adopt the convention that $h[X(t)]$ equals the quantity in (1) if $t \geqq \tau$. With this

Received February 2, 1971.

[1] Research supported in part under Contract AF–F44620–67–C–0049 at Stanford University.

notation, the process $\{h[X(t)], 0 \leqq t \leqq \infty\}$ is a supermartingale with respect to each of the measures $P_\xi(\xi \in D)$.

We shall make use of conditional Brownian motion processes which are conditioned to converge to certain minimal points $\eta \in \partial D_e$. If p denotes the transition density function of killed Brownian motion on D, then p^η will denote the conditional transition density function

$$p^\eta(t, \xi, \sigma) = p(t, \xi, \sigma)K(\eta, \sigma)/K(\eta, \xi).$$

A conditional process governed by p^η will be denoted by

$$(\Omega, \mathscr{F}^\eta, \mathscr{F}^\eta(t), X(t), \theta(t), P_\xi^\eta).$$

THEOREM 1. *Let h be a positive harmonic function defined on D. The function h can be written uniquely as a sum of three positive harmonic functions $h = h_1 + h_2 + h_3$ where, for every $\xi \in D$ and corresponding measure P_ξ,*

(a) $\{h_1[X(t)], 0 \leqq t \leqq \infty\}$ *is a martingale,*

(b) $\{h_2[X(t)], 0 \leqq t < \infty\}$ *is a martingale and* $\lim_{t\to\infty} h_2[X(t)] = 0$ *almost everywhere,*

(c) $\{h_3[X(t)], 0 \leqq t < \infty\}$ *is a supermartingale with* $\lim_{t\to\infty} h_3[X(t)] = 0$ *almost everywhere and* $\lim_{t\to\infty} E_\xi\{h_3[X(t)]\} = 0$ (*a potential*).

It will be convenient for us to state two Lemmas before proceeding to the proof.

LEMMA 1. *Let h be a positive harmonic function whose canonical measure μ_h ($h(\xi) = \int_{\partial D_e} K(\eta, \xi)\, \mu_h(d, \eta)$) is singular with respect to μ_1. Then*

$$\lim_{t\to\infty} h[X(t)] = 0$$

P_ξ-almost everywhere ($\xi \in D$). In addition

$$E_\xi\{h[X(t)]\} = \int E_\xi\{K[\eta, X(t)]\}\mu_h(d\eta). \tag{2}$$

PROOF. The statement involving almost everywhere convergence is well known and so we will prove only (2). Fubini's Theorem implies that

$$\begin{aligned}\int_{t<\tau} h[X(t)]\, dP_\xi &= \int_{t<\tau}\int K[\eta, X(t)]\mu_h(d\eta)\, dP_\xi \\ &= \int\int_{t<\tau} K[\eta, X(t)]\, dP_\xi\, \mu_h(d\eta).\end{aligned}$$

Since

$$\lim_{t\uparrow\tau} h[X(t)] = \lim_{t\to\infty} h[X(t)] = 0$$

P_ξ-almost everywhere, we have

$$\int_{t<\tau} h[X(t)]\, dP_\xi = E_\xi\{h[X(t)]\}.$$

Since μ_h is singular with respect to μ_1, it follows that the measure ε_η (unit mass concentrated at $\eta \in \partial D_e$) is singular with respect to μ_1, for μ_h-almost every η. Hence, by the same reasoning which led to the proceeding equation,

$$\int_{t<\tau} K[\eta, X(t)]\, dP_\xi = E_\xi\{K[\eta, X(t)]\}$$

for μ_h-almost every η and (2) follows.

LEMMA 2. *For a fixed $\eta \in \partial D_e$, the function v defined by*

$$v(\xi) = P_\xi^\eta[\tau = \infty\}$$

is either identically 0 *or identically* 1.

PROOF. A standard argument shows that v is a bounded $K(\eta, \cdot)$-harmonic function on D. Since $K(\eta, \cdot)$ is a minimal harmonic function, v is necessarily equal to a constant c. On the set where $\tau = \infty$

$$\begin{aligned}\lim_{t\to\infty} v[X(t)] &= \lim_{t\to\infty} P^\eta_{X(t)}\{\tau = \infty\} \\ &= \lim_{t\to\infty} P_\xi^\eta\{\tau = \infty \mid \mathscr{F}^\eta(t)\} \\ &= P_\xi^\eta\{\tau = \infty \mid \vee_{t>0} \mathscr{F}^\eta(t)\} \\ &= I_{[\tau=\infty]} \\ &= 1\end{aligned}$$

P_ξ^η-almost everywhere. It follows that, if $v(\xi) > 0$ for some ξ, then $c = 1$ and, if $v(\xi) = 0$ for all ξ, then $c = 0$.

PROOF OF THEOREM 1. Decompose the canonical measure μ_h associated with h by the formula $\mu_h = \mu_h^a + \mu_h^s$ where μ_h^a is absolutely continuous with respect to μ_1, and μ_h^s is singular with respect to μ_1. If $\mu_h^a(d\eta) = f(\eta)\mu_1(d\eta)$, then

$$h(\xi) = \int K(\eta,\xi)f(\eta)\mu_1(d\eta) + \int K(\eta,\xi)\mu_h^s(d\eta). \tag{3}$$

The first term on the right of (3) is denoted by h_1 and is simply the Perron-Wiener-Brelot solution to the Dirichlet problem corresponding to the Martin boundary function f. Hence $h_1[X(t)]$ is uniformly integrable and (a) is well known.

In order to decompose the second term on the right of (3) still further, we define

$$\Gamma_1 = \{\eta \in \partial D_e : P_\xi^\eta\{\tau = \infty\} = 1 \text{ for all } \xi \in D\},$$
$$\Gamma_2 = \{\eta \in \partial D_e : P_\xi^\eta\{\tau = \infty\} = 0 \text{ for all } \xi \in D\}.$$

According to Lemma 2, $\partial D_e = \Gamma_1 \cup \Gamma_2$. Since

$$P_\xi^\eta\{\tau > t\} = \int p(t,\xi,\sigma)K(\eta,\sigma)/K(\eta,\xi)\,d\sigma$$

is a measurable function of η, it follows that Γ_1 and Γ_2 are Borel measurable subsets of ∂D_e. Let

$$h_2(\xi) = \int_{\Gamma_1} K(\eta,\xi)\mu_h^s(d\eta),$$
$$h_3(\xi) = \int_{\Gamma_2} K(\eta,\xi)\mu_h^s(d\eta).$$

Since μ_h^s is singular with respect to μ_1, it follows from Lemma 1 that

$$\lim_{t\to\infty} h_i[X(t)] = 0$$

P_ξ-almost everywhere ($\xi \in D$) for $i = 2, 3$. Lemma 1 also implies that

$$
\begin{aligned}
E_\xi\{h_2[X(t)]\} &= \int_{\Gamma_1} E_\xi\{K[\eta, X(t)]\}\mu_h^s(d\eta) \\
&= \int_{\Gamma_1} [\int p(t,\xi,\sigma)K(\eta,\sigma)\, d\sigma]\mu_h^s(d\eta) \\
&= \int_{\Gamma_1} K(\eta,\xi)[\int p^\eta(t,\xi,\sigma)\, d\sigma]\mu_h^s(d\eta) \\
&= \int_{\Gamma_1} K(\eta,\xi)P_\xi^\eta\{\tau > t\}\mu_h^s(d\eta) \\
&= \int_{\Gamma_1} K(\eta,\xi)\mu_h^s(d\eta) \\
&= h_2(\xi).
\end{aligned} \tag{4}
$$

Equation (4) implies that $\{h_2[X(t)], 0 \leqq t < \infty\}$ is a martingale and (b) is proved. Furthermore,

$$
\begin{aligned}
E_\xi\{h_3[X(t)]\} &= \int_{\Gamma_2} K(\eta,\xi)P_\xi^\eta\{\tau > t\}\mu_h^s(d\eta) \\
&\downarrow \int_{\Gamma_2} K(\eta,\xi)P_\xi^\eta\{\tau = \infty\}\mu_h^s(d\eta) \\
&= 0
\end{aligned}
$$

as $t \to \infty$ and (c) follows easily.

The uniqueness of the decomposition follows from the following observations:

$$
\begin{aligned}
h_1(\xi) &= E_\xi\{\lim_{t\to\infty} h[X(t)]\}, \\
h_2(\xi) &= \lim_{t\to\infty} E_\xi\{h[X(t)]\} - h_1, \\
h_3 &= h - h_1 - h_2.
\end{aligned} \tag{5}
$$

The proof is now complete.

We remark that the proof of Theorem 1 could have been carried out without mentioning the Martin boundary. We could simply define h_1, h_2 and h_3 by (5) and proceed from there. However, our method has tied the decomposition to the Martin boundary representation of the functions involved and we summarize the connection in the following Corollary.

COROLLARY 1. *The canonical measures μ_{h_1}, μ_{h_2} and μ_{h_3} of the functions h_1, h_2 and h_3 of Theorem 1 can be characterized by the following conditions*:

(a) *μ_{h_1} is absolutely continuous with respect to μ_1*,

(b) *μ_{h_2} is singular with respect to μ_1 and is concentrated on Γ_1*,

(c) *μ_{h_3} is singular with respect to μ_1 and is concentrated on Γ_2*.

2. Examples and comments. Let $D = \{(x, y): -\infty < x < \infty, 0 < y\}$ be the upper half plane in 2-dimensional Euclidean space E^2. It is known that the Martin boundary $\partial D(= \partial D_e)$ may be topologically identified with the lower Euclidean boundary together with the point at infinity (which we denote by ∞). We claim that every point of the lower boundary belongs to Γ_2 while ∞ belongs to Γ_1. We leave the routine verification of these points to the reader and remark only that the functions h_1, h_2, h_3 in the decomposition of a positive harmonic function h on the half plane take the form

$$h_1(x, y) = \int_{-\infty}^{\infty} f(s) y/[(x-s)^2 + y^2]\, ds,$$

$$h_2(x, y) = cy,$$

$$h_3(x, y) = \int_{-\infty}^{\infty} y/[(x-s)^2 + y^2] \mu(ds),$$

where c is a nonnegative constant and $\mu(ds)$ is singular with respect to μ_1 (or, equivalently, with respect to Lebesgue measure).

As a second example, let D be the unit disk in E^2. The Martin boundary of D coincides with the Euclidean boundary and by symmetry one of the sets Γ_1 or Γ_2 is empty. Since the lifetime of almost every Brownian path from a point ξ is finite, it follows that $P_\xi^\eta\{\tau < \infty\} = 1$ for $K(\eta, \xi)\ \mu_1(d\eta)$-almost every η and hence Γ_1 is empty. It follows that $h_2 = 0$ in the decomposition of any positive harmonic function on the disk.

As a final remark, we state a theorem concerning the decomposition of a positive supermartingale $\{x_n, \mathscr{F}_n, n \geqq 1\}$ which is analogous to Theorem 1.

THEOREM 2. *Let* $\{x_n, \mathscr{F}_n, n \geqq 1\}$ *be a positive supermartingale. There exists three uniquely determined (up to sets of measure zero) positive processes* $\{x_n^i, \mathscr{F}_n, n \geqq 1\}$ $(i = 1, 2, 3)$ *such that* $x_n = x_n^1 + x_n^2 + x_n^3$ *for* $n \geqq 1$ *and*

(a) $\{x_n^1, \mathscr{F}_n, n \geqq 1\}$ *is a uniformly integrable martingale,*

(b) $\{x_n^2, \mathscr{F}_n, n \geqq 1\}$ *is a martingale and* $\lim_{n\to\infty} x_n^2 = 0$ *almost everywhere,*

(c) $\{x_n^3, \mathscr{F}_n, n \geqq 1\}$ *is a supermartingale with* $\lim_{n\to\infty} x_n^3 = 0$ *almost everywhere and* $\lim_{n\to\infty} E\{x_n^3\} = 0$ *(a potential).*

PROOF. Let

$$x_n^1 = E\{\lim_{m\to\infty} x_m \mid \mathscr{F}_n\},$$

$$x_n^2 = \lim_{m\to\infty} E\{x_m \mid \mathscr{F}_n\} - x_n^1,$$

$$x_n^3 = x_n - x_n^1 - x_n^2.$$

The proof now follows by standard arguments (see the proof of the Riesz decomposition theorem given in Meyer (1966, page 89).

REFERENCES

BLUMENTHAL, R. and GETOOR, R. (1968). *Markov Processes and Potential Theory*. Academic Press, New York.

DOOB, J. L. (1954). Semimartingales and subharmonic function. *Trans. Amer. Math. Soc.* 77 86–121.

DOOB, J. L. (1957a). Brownian motion on a green space. *Teor. Verojatnosti. i Primenen.* **2** 3–33.

DOOB, J. L. (1957b). Conditional Brownian motion and the boundary limits of harmonic functions. *Bull. Soc. Math. France* **85** 431–458.

DOOB, J. L. (1958). Probability theory and the first boundary value problem. *Illinois J. Math.* **2** 19–36.

MEYER, P. A. (1966). *Probability and Potentials.* Blaisdell Publishing Co., Waltham.