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ON MOMENTS OF INFINITELY DIVISIBLE DISTRIBUTION FUNCTIONS[1]

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Let $F(x)$ be an infinitely divisible distribution function with a Lévy-Khintchine function $G(u)$ and let p be any positive number. It is shown that $F(x)$ has an absolute moment of the pth order if and only if $G(u)$ has an absolute moment of the pth order, and $F(x)$ has an exponential moment of the pth order if and only if $G(u)$ has an exponential moment of the pth order. This result generalizes a theorem of J. M. Shapiro. Other related results are also obtained.

1. Introduction and summary. A distribution function $F(x)$ is said to be infinitely divisible if for every positive integer n there exists a distribution function $F_n(x)$ such that $F(x)$ is the convolution of $F_n(x)$ with itself n times. It is well known that a distribution function $F(x)$ is infinitely divisible if and only if its characteristic function $\hat{f}(t)$ has a unique representation of the form

$$\hat{f}(t) = \exp\{i\gamma t + \int_{-\infty}^{\infty}(e^{iut} - 1 - iut(1+u^2)^{-1})((1+u^2)/u^2)\,dG(u)\} \tag{1}$$

where γ is a constant and $G(u)$ is a bounded, non-decreasing function. The constant γ is called the centering constant of $F(x)$, $G(u)$ is called the Lévy-Khintchine function of $F(x)$, and formula (1) is called the Lévy-Khintchine representation of $\hat{f}(t)$.

Let p be a positive constant and let k be a positive integer. A distribution function $F(x)$ is said to have an absolute moment of the pth order if $\int_{-\infty}^{\infty}|x|^p dF(x) < \infty$ and it is said to have an exponential moment of the pth order if $\int_{-\infty}^{\infty} e^{p|x|}\, dF(x) < \infty$. It is said to have an algebraic moment of the kth order if $\int_{-\infty}^{\infty} x^k dF(x)$ exists, and it is said to have a symmetric moment of the kth order if $\lim_{T\to\infty} \int_{-T}^{T} x^k dF(x)$ exists. A distribution function has an algebraic moment of the kth order if and only if it has an absolute moment of the same order. However, a distribution function may have a symmetric kth moment and not have an absolute kth moment if k is odd.

Let $F(x)$ be an infinitely divisible distribution function with a Lévy-Khintchine function $G(u)$. J. M. Shapiro (1956) showed that if k is an even positive integer then $F(x)$ has an absolute moment of the kth order if and only if $G(u)$ has an absolute moment of the kth order. In this paper it is shown that if p is any positive number, then $F(x)$ has an absolute moment of the pth order if and only if $G(u)$ has an absolute moment of the pth order. Also $F(x)$ has an exponential moment of the pth order if and only if $G(u)$ has an exponential moment of the pth order. Other related theorems are also obtained.

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2. A theorem concerning convolutions of distribution functions. A. Wintner (1947, Section 23) has shown that if $F(x)$ and $G(x)$ are distribution functions and p is a positive number, then $F^*G(x)$ has an absolute moment of the pth order if and only if both $F(x)$ and $G(x)$ have absolute moments of the pth order. In this section a similar result will be obtained.

THEOREM 1. *Let F and G be distribution functions. Let p be any positive number. Then* $1-F^*G(x) = O(x^{-p})$ *as* $x \to \infty$ *if and only if* $1-F(x) = O(x^{-p})$ *as* $x \to \infty$ *and* $1-G(x) = O(x^{-p})$ *as* $x \to \infty$. *Also* $F^*G(-x) = O(x^{-p})$ *as* $x \to \infty$ *if and only if* $F(-x) = O(x^{-p})$ *as* $x \to \infty$ *and* $G(-x) = O(x^{-p})$ *as* $x \to \infty$. *The theorem remains true if O is replaced by o.*

PROOF. Assume that $x > 0$ and that x and $x/2$ are points of continuity of $F(y)$ and $G(y)$. Then

$$1-F^*G(x) = \iint_{u+v>x} dF(u)\,dG(v)$$
$$\leqq \iint_{u>x/2} dF(u)\,dG(v) + \iint_{v>x/2} dF(u)\,dG(v) = 1-F(x/2)+1-G(x/2).$$

It follows from this that for all $x > 0$,

$$x^p(1-F^*G(x)) \leqq 2^p(x/2)^p(1-F(x/2))+2^p(x/2)^p(1-G(x/2)). \tag{2}$$

In a similar manner it can be shown that for all $x > 0$,

$$x^p(F^*G(-x)) \leqq 2^p(x/2)^p(F(-x/2))+2^p(x/2)^p(G(-x/2)). \tag{3}$$

Let a be a point of continuity of $G(y)$ such that $a < 0$ and $G(a) < 1$. Let b be a point of continuity of $G(y)$ such that $b > 0$ and $G(b) > 0$. Let x be chosen so that x and $2x$ are points of continuity of $F(y)$ and $x > -a$. Then

$$1-F^*G(x) = \iint_{u+v>x} dF(u)\,dG(v)$$
$$\geqq \iint_{u>x-a,v>a} dF(u)\,dG(v) \geqq \iint_{u>2x,v>a} dF(u)\,dG(v) = [1-F(2x)][1-G(a)].$$

Thus if $x > -2a$,

$$x^p(1-F(x)) \leqq (2^p/(1-G(a)))(x/2)^p(1-F^*G(x/2)). \tag{4}$$

Similarly, if $x > 2b$, then

$$x^p(F(-x)) \leqq (2^p/G(b))(x/2)^p(F^*G(-x/2)). \tag{5}$$

The theorem follows immediately from inequalities (2) to (5). □

It should be pointed out that if $F, F_1, \cdots, F_n$ are distribution functions such that $F(x) = F_1^* \cdots {}^*F_n(x)$ and if $x > 0$, then $1-F(x) \leqq n-F_1(x/n)-\cdots-F_n(x/n)$ and $F(-x) \leqq F_1(-x/n)+\cdots+F_n(-x/n)$. These inequalities are obtained in the same way as (2) and (3) are obtained.

An interesting application of Theorem 1 can be made to the study of the behavior of characteristic functions at the origin. Let $F(x)$, $F_1(x)$, and $F_2(x)$ be distribution functions such that $F(x) = F_1^*F_2(x)$. If $\hat{f}(t)$, $\hat{f}_1(t)$, and $\hat{f}_2(t)$ are the characteristic functions of $F(x)$, $F_1(x)$, and $F_2(x)$ respectively, then $\hat{f}(t) = \hat{f}_1(t)\hat{f}_2(t)$. It is well known that if k is a positive even integer, then $\hat{f}^{(k)}(0)$ exists if and only if $F(x)$ has an algebraic moment of the kth order. E. J. G. Pitman (1956) has shown that if k is a positive odd integer, then $\hat{f}^{(k)}(0)$ exists if and only if $F(x)$ has a symmetric moment of the kth order and $1-F(x)+F(-x) = o(x^{-k})$ as $x \to \infty$.

If k is a positive even integer and $\hat{f}^{(k)}(0)$ exists then $\hat{f}_1^{(k)}(0)$ and $\hat{f}_2^{(k)}(0)$ also exist. This follows from A. Wintner's theorem. However, if k is a positive odd integer and $\hat{f}^{(k)}(0)$ exists, then it does not follow that $\hat{f}_1^{(k)}(0)$ and $\hat{f}_2^{(k)}(0)$ necessarily exist. To see this, let $f_1(x) = 0$ if $x < 2$ and let $f_1(x) = c/x^2 lnx$ if $x \geqq 2$ where $c = [\int_2^\infty (x^2 lnx)^{-1}dx]^{-1}$. Let $F_1(x) = \int_{-\infty}^x f_1(y)dy$ and let $F_2(x) = 1-F_1(-x)$. It is easy to see that neither $F_1(x)$ nor $F_2(x)$ have symmetric first moments. Thus it follows that neither $\hat{f}_1'(0)$ nor $\hat{f}_2'(0)$ exist. However, $1-F_1(x)+F_1(-x) = o(x^{-1})$ as $x \to \infty$ and $1-F_2(x)+F_2(-x) = o(x^{-1})$ as $x \to \infty$. It follows from Theorem 1 that $1-F(x)+F(-x) = o(x^{-1})$ as $x \to \infty$. Since $F(x)$ is symmetric it follows that it has a symmetric moment of the first order. Thus it follows from Pitman's theorem that $\hat{f}'(0)$ exists.

3. Three lemmas.

LEMMA 1. *Let $a_1, \cdots, a_n$ be real numbers and let $p > 0$. Then* $|a_1+\cdots+a_n|^p \leqq n^p(|a_1|^p+\cdots+|a_n|)^p$.

PROOF. Let $a = \max_{1\leqq j \leqq n} |a_j|$. Then $|a_1+\cdots+a_n|^p \leqq n^p a^p \leqq n^p(|a_1|^p+\cdots+|a_n|^p)$.

LEMMA 2. *Let $F(x)$ be a distribution function with a characteristic function $\hat{f}(t)$. Let $H(x)$ be the distribution function with the characteristic function $\hat{h}(t) = \exp\{\lambda(\hat{f}(t)-1)\}$ where $\lambda > 0$. Let p be any positive number. Then $H(x)$ has an absolute moment of the* p*th order if and only if $F(x)$ has an absolute moment of the* p*th order, and $H(x)$ has an exponential moment of the* p*th order if and only if $F(x)$ has an exponential moment of the* p*th order. Also $\hat{h}(t)$ has exactly as many derivatives at the origin as $\hat{f}(t)$.*

PROOF. Let $E(x)$ denote the distribution function degenerate at the origin and let $F^{*n}(x)$ denote the convolution of $F(x)$ with itself n times. Then it follows from Tucker (1967, Theorem 6, page 152) that $\hat{h}(t)$ is the characteristic function of a distribution function $H(x)$ and that

$$H(x) = e^{-\lambda}E(x)+e^{-\lambda}\lambda F(x)+(e^{-\lambda}\lambda^2/2)F^{*2}(x)+\cdots. \tag{6}$$

It is obvious that $F(x)$ has an absolute moment of the pth order if $H(x)$ has an absolute moment of the pth order, and $F(x)$ has an exponential moment of the pth order if $H(x)$ has an exponential moment of the pth order. The converse statements must be proved.

Assume that $u = \int_{-\infty}^\infty |x|^p dF(x) < \infty$. It follows from Lemma 1 that

$$\begin{aligned}\int_{-\infty}^\infty |x|^p\, dF^{*n}(x) &= \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty |x_1+\cdots+x_n|^p\, dF(x_1)\cdots dF(x_n) \\ &\leqq n^p \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty (|x_1|^p+\cdots+|x_n|^p)\, dF(x_1)\cdots dF(x_n) = n^{p+1}u.\end{aligned} \tag{7}$$

Thus it follows from (6) and (7) that

$$\int_{-\infty}^{\infty} |x|^p\, dH(x) \leqq e^{-\lambda} u \sum_{n=1}^{\infty} (\lambda^n n^p/(n-1)!) < \infty.$$

Similarly, assume that $v = \int_{-\infty}^{\infty} e^{p|x|} dF(x) < \infty$. Then

$$(8) \qquad \int_{-\infty}^{\infty} e^{p|x|}\, dF^{*n}(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{p|x_1+\cdots+x_n|}\, dF(x_1)\cdots dF(x_n)$$
$$\leqq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{p(|x_1|+\cdots+|x_n|)} dF(x_1)\cdots dF(x_n) = v^n.$$

It follows from (6) and (8) that

$$\int_{-\infty}^{\infty} e^{p|x|}\, dH(x) \leqq e^{-\lambda} \sum_{n=0}^{\infty} ((\lambda v)^n/n!) = e^{\lambda(v-1)} < \infty.$$

It is easy to see that $\hat{h}(t)$ has exactly as many derivatives at the origin as $\hat{f}(t)$. ∎

LEMMA 3. *Let $H(x)$ and $F(x)$ be the distribution functions of Lemma 2 and let p be any positive number. Then $1-F(x) = O(x^{-p})$ as $x \to \infty$ if and only if $1-H(x) = O(x^{-p})$ as $x \to \infty$, and $F(-x) = O(x^{-p})$ as $x \to \infty$ if and only if $H(-x) = O(x^{-p})$ as $x \to \infty$. The lemma remains true if O is replaced by o.*

PROOF. The proof of this lemma is similar to the proof of the first part of Lemma 2. It follows from the inequality given after the proof of Theorem 1 that $1-F^{*n}(x) \leqq n(1-F(x/n))$ *if* $x > 0$. Thus it follows from (6) that if $x > 0$, then

$$(9) \qquad x^p(1-H(x)) \leqq e^{-\lambda} \sum_{n=1}^{\infty} (\lambda^n n^p/(n-1)!)(x/n)^p(1-F(x/n)).$$

It is obvious that $1-F(x) = O(x^{-p})$ as $x \to \infty$ if $1-H(x) = O(x^{-p})$ as $x \to \infty$ and $1-F(x) = o(x^{-p})$ as $x \to \infty$ if $1-H(x) = o(x^{-p})$ as $x \to \infty$.

If $1-F(x) = O(x^{-p})$ as $x \to \infty$ then there exists a constant $A > 0$ such that $x^p(1-F(x)) \leqq A$ if $x > 0$. But this implies that

$$(10) \qquad x^p(1-H(x)) \leqq A\, e^{-\lambda} \sum_{n=1}^{\infty} (\lambda^n n^p/(n-1)!) < \infty,$$

and thus $1-H(x) = O(x^{-p})$ as $x \to \infty$. If $1-F(x) = o(x^{-p})$ as $x \to \infty$ then it follows that for each value of k there exists a constant b_k such that

$$(11) \qquad x^p(1-H(x)) \leqq 2A\, e^{-\lambda} \sum_{n=k}^{\infty} (\lambda^n n^p/(n-1)!)$$

if $x > b_k$. But this implies that $1-H(x) = o(x^{-p})$ as $x \to \infty$. The rest of the theorem follows in the same manner. ∎

4. Main theorems and corollaries.

THEOREM 2. *Let $F(x)$ be an infinitely divisible distribution function with Lévy-Khintchine function $G(u)$. Let p be any positive number. The distribution function $F(x)$ has an absolute moment of the* pth *order if and only if $G(u)$ has an absolute moment of the* pth *order. Also $F(x)$ has an exponential moment of the* pth *order if and only if $G(u)$ has an exponential moment of the* pth *order.*

PROOF. Let γ be the centering constant of $F(x)$. Let

$$\begin{aligned} G_1(u) &= 0 && \text{if} \quad u < -1 \\ &= G(u) - G(-1) && \text{if} \quad -1 \leqq u \leqq 1 \\ &= G(1) - G(-1) && \text{if} \quad u > 1 \\ G_2(u) &= G(u) - G_1(u). \end{aligned}$$

Let $F_1(x)$ be the infinitely divisible distribution function with centering constant γ and Lévy-Khintchine function $G_1(u)$. Let $F_2(x)$ be the infinitely divisible distribution function with centering constant 0 and Lévy-Khintchine function $G_2(u)$. It is obvious that $F(x) = F_1 * F_2(x)$.

Since $G_1(u)$ has compact support it follows from a proof of Y. Linnik (1954, page 171) that $F_1(x)$ has a characteristic function that is an entire function. Thus $F_1(x)$ has absolute moments and exponential moments of all orders (see Wintner (1947, Section 17)). A. Wintner has shown (1947, Section 23) that if $H(x)$, $H_1(x)$, and $H_2(x)$ are distribution functions and if $H(x) = H_1 * H_2(x)$, then $H(x)$ has an absolute moment of the pth order if and only if both $H_1(x)$ and $H_2(x)$ have absolute moments of the pth order, and $H(x)$ has an exponential moment of the pth order if and only if both $H_1(x)$ and $H_2(x)$ have exponential moments of the pth order. It follows that $F(x)$ has an absolute moment of the pth order if and only if $F_2(x)$ has an absolute moment of the pth order, and $F(x)$ has an exponential moment of the pth order if and only if $F_2(x)$ has an exponential moment of the pth order. Finally, it follows from Lemma 2 that $F_2(x)$ has an absolute moment of the pth order if and only if $G(u)$ has an absolute moment of the pth order and $F_2(x)$ has an exponential moment of the pth order if and only if $G(u)$ has an exponential moment of the pth order. □

COROLLARY 1. *Let $F(x)$ be an infinitely divisible distribution function with Lévy-Khintchine function $G(u)$. Let k be any positive integer. The distribution function $F(x)$ has an algebraic moment of the* kth *order if and only if $G(u)$ has an algebraic moment of the* kth *order.*

For relationships between the algebraic moments of $F(x)$ and the algebraic moments of $G(u)$ see Shapiro (1956).

COROLLARY 2. *Let $F(x)$ be an infinitely divisible distribution function with a characteristic function $\hat{f}(t)$ and a Lévy-Khintchine function $G(u)$. Let $\hat{g}(t)$ be the Fourier-Stieltjes transform of $G(u)$ and let k be any positive integer. Then $\hat{f}^{(k)}(0)$ exists if and only if $\hat{g}^{(k)}(0)$ exists.*

PROOF. Let $F_1(x)$ and $F_2(x)$ be defined as in the proof of Theorem 2. Let $\hat{f}_1(t)$ and $\hat{f}_2(t)$ be the characteristic functions of $F_1(x)$ and $F_2(x)$ respectively. Since $\hat{f}_1(t)$ is an entire function, it follows that $\hat{f}^{(k)}(0)$ exists if and only if $\hat{f}_2^{(k)}(0)$ exists. It follows from Lemma 2 that $\hat{f}_2^{(k)}(0)$ exists if and only if $\hat{g}^{(k)}(0)$ exists. □

THEOREM 3. *Let $F(x)$ be an infinitely divisible distribution function with a Lévy-Khintchine function $G(u)$. Let k be any odd positive integer. The distribution function $F(x)$ has a symmetric moment of the* kth *order and $1-F(x)+F(-x) = o(x^{-k})$ as $x \to \infty$ if and only if $G(u)$ has a symmetric moment of the* kth *order and $G(\infty)-G(u)+G(-u) = o(u^{-k})$ as $u \to \infty$.*

PROOF. Let $\hat{f}(t)$ be the characteristic function of $F(x)$ and let $\hat{g}(t)$ be the Fourier-Stieltjes transform of $G(u)$. By a theorem of E. J. G. Pitman (1956), $F(x)$ has a symmetric moment of the kth order and $1-F(x)+F(-x) = o(x^{-k})$ as $x \to \infty$ if and only if $\hat{f}^{(k)}(0)$ exists. By Corollary 2, $\hat{f}^{(k)}(0)$ exists if and only if $\hat{g}^{(k)}(0)$ exists. By Pitman's Theorem, $\hat{g}^{(k)}(0)$ exists if and only if $G(u)$ has a symmetric moment of the kth order and $G(\infty)-G(u)+G(-u) = o(u^{-k})$ as $u \to \infty$. □

THEOREM 4. *Let $F(x)$ be an infinitely divisible distribution function with Lévy-Khintchine function $G(u)$. Let p be any positive number. Then $1-F(x) = O(x^{-p})$ as $x \to \infty$ if and only if $G(\infty)-G(u) = O(u^{-p})$ as $u \to \infty$, and $F(-x) = O(x^{-p})$ as $x \to \infty$ if and only if $G(-u) = O(u^{-p})$ as $u \to \infty$. The theorem remains true if O is replaced by o.*

PROOF. The proof of this theorem is similar to the proof of Theorem 2. Theorem 1 is used in the proof instead of A. Wintner's theorem, and Lemma 3 is used instead of Lemma 2.

THEOREM 5. *Let $F(x)$ be an infinitely divisible distribution function with a characteristic function $\hat{f}(t)$ and a Lévy-Khintchine function $G(u)$. Let $\hat{g}(t)$ be the Fourier-Stieltjes transform of $G(u)$. Let $\alpha < 0$ and let $\beta > 0$. The characteristic function $\hat{f}(z)$ is analytic for $\alpha < \mathrm{Im}(z) < \beta$ if and only if $\hat{g}(z)$ is analytic for $\alpha < \mathrm{Im}(z) < \beta$.*

PROOF. Let γ be the centering constant of $F(x)$. Let

$$\begin{aligned} G_1(u) &= G(u) && \text{if } u < -1 \\ &= G(-1) && \text{if } u \geqq -1 \\ G_2(u) &= 0 && \text{if } u < -1 \\ &= G(u)-G(-1) && \text{if } -1 \leqq u \leqq 1 \\ &= G(1)-G(-1) && \text{if } u > 1 \\ G_3(u) &= G(u)-G_1(u)-G_2(u). \end{aligned}$$

For $1 \leqq j \leqq 3$ let $F_j(x)$ be the infinitely divisible distribution function with characteristic function

$$\hat{f}_j(t) = \exp\{i\gamma t/3 + \textstyle\int_{-\infty}^{\infty} (e^{iut}-1-iut(1+u^2)^{-1})((1+u^2)/u^2)\, dG_j(u)\}$$

and let $\hat{g}_j(t)$ be the Fourier-Stieltjes transform of $G_j(u)$. It is obvious that $\hat{f}(t) = \hat{f}_1(t)\hat{f}_2(t)\hat{f}_3(t)$, $F(x) = F_1 * F_2 * F_3(x)$, and $\hat{g}(t) = \hat{g}_1(t)+\hat{g}_2(t)+\hat{g}_3(t)$.

It follows from Theorem 2 and Wintner (1947, Section 17) that $\hat{f}(z)$ is analytic if and only if $\hat{g}(z)$ is analytic. Assume that $\hat{f}(z)$ is analytic for $\alpha < \mathrm{Im}(z) < \beta$. Then

$\hat{f}_1(z)$, $\hat{f}_2(z)$, and $\hat{f}_3(z)$ are also analytic for $\alpha < \text{Im}(z) < \beta$. By a theorem of G. Baxter and J. M. Shapiro (1960), $F_1(x)$ has support on an interval of the form $(-\infty, x_1]$ where $x_1 < \infty$ and $F_3(x)$ has support on an interval of the form $[x_2\ \infty)$ where $x_2 > -\infty$. It follows from a theorem of E. Lukacs (1960, Theorem 7.2.2., page 139) that $\hat{f}_1(z)$ is analytic for Im $(z) < 0$ and $\hat{f}_3(z)$ is analytic for Im $(z) > 0$. By a result of Y. Linnik (1954, page 171), $\hat{f}_2(z)$ is an entire function. It also follows from the same theorem of Lukacs that $\hat{g}_1(z)$ is analytic for Im $(z) < 0$, $\hat{g}_2(z)$ is an entire function, and $\hat{g}_3(z)$ is analytic for Im $(z) > 0$.

Since $\hat{f}_1(z)$ is analytic for Im $(z) < \beta$ it follows that $F_1(x)$ has exponential moments of all orders less than β. By Theorem 2, $G_1(u)$ has exponential moments of all orders less than β and thus $\hat{g}_1(z)$ is analytic for Im $(z) < \beta$. In a similar manner it can be shown that $\hat{g}_3(z)$ is analytic for Im $(z) > \alpha$. Thus $\hat{g}(z)$ is analytic for $\alpha <$ Im $(z) < \beta$.

Assume conversely that $\hat{g}(z)$ is analytic for $\alpha <$ Im $(z) < \beta$. Let $\delta = \min(-\alpha, \beta)$. By Wintner (1967, Section 17) $G(u)$, and thus $G_1(u)$, $G_2(u)$ and $G_3(u)$, have exponential moments of all orders less than δ. Thus $\hat{g}_1(z)$, $\hat{g}_2(z)$, and $\hat{g}_3(z)$ are analytic functions.

By a theorem of Lukacs (1960, Theorem 7.2.2., page 139), $\hat{g}_1(z)$ is analytic for Im $(z) < 0$, $\hat{g}_2(z)$ is an entire function, and $\hat{g}_3(z)$ is analytic for Im $(z) > 0$. Since $\hat{g}(z)$ is analytic for $\alpha <$ Im $(z) < \beta$ it follows that $\hat{g}_1(z)$ is analytic for Im $(z) < \beta$ and $\hat{g}_3(z)$ is analytic for Im $(z) > \alpha$. It follows from this that $\hat{f}_1(z)$ is analytic for Im $(z) < \beta$ and $\hat{f}_3(z)$ is analytic for Im $(z) > \alpha$. Thus $\hat{f}(z)$ is analytic for $\alpha <$ Im $(z) < \beta$. □

COROLLARY 3. *Let $F(x)$ be an infinitely divisible distribution function with a characteristic function $\hat{f}(t)$ and a Lévy-Khintchine function $G(u)$. Let $\hat{g}(t)$ be the Fourier-Stieltjes transform of $G(u)$. The characteristic function $\hat{f}(z)$ is an entire function if and only if $\hat{g}(z)$ is an entire function.*

Note that Theorem 4 follows immediately from Theorem 2 in the case when $\beta = -\alpha$. It follows from Theorem 5 and a theorem of E. Lukacs (1960, Theorem 7.1.1., page 132) that $i\alpha$ is a singular point of $\hat{f}(z)$ if and only if it is a singular point of $\hat{g}(z)$, and $i\beta$ is a singular point of $\hat{f}(z)$ if and only if it is a singular point of $g(z)$. Thus $f(z)$ and $g(z)$ have the same strip of regularity.

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