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ON THE ASYMPTOTIC DISTRIBUTION OF THE SEQUENCES OF RANDOM VARIABLES WITH RANDOM INDICES

BY SILVIU GUIASU

Mathematical Institute, Rumanian Academy of Science

1. Introduction. In the present paper we investigate the asymptotic distribution of the sequence of random variables $(Y_{Nr})_{1 \leqq r < \infty}$ where $(Y_n)_{1 \leqq n < \infty}$ is a sequence of random variables and $(N_r)_{1 \leqq r < \infty}$ is a sequence of positive integer-valued random variables, both defined on a probability space $\{\Omega, \mathscr{K}, P\}$. About the behavior of the random indices we assume the following condition:

(C0) The sequence $(N_r/n_r)_{1 \leqq r < \infty}$ converges in probability to a positive random variable λ where $(n_r)_{1 \leqq r < \infty}$ is an increasing sequence of positive integer numbers tending to infinity when r tends to infinity.

The main problem is the following: *when does the sequence of random variables with random indices* $(Y_{Nr})_{1 \leqq r < \infty}$ *have the same asymptotic distribution as the sequence* $(Y_n)_{1 \leqq n < \infty}$? To simplify let us denote this main question by Q.

The first general result in this area (but with $\lambda \equiv 1$) was obtained by F. J. Anscombe (1952). Theorem 2 gives us the exact content of this result. Here condition (C4), known as "Anscombe's condition," is very important.

After the papers of W. Richter (1965a) (1965b), the latest general assertion in this direction (with λ arbitrary positive random variable) was formulated by J. Mogyoródi (1967). According to it an affirmative answer to the question Q is possible if both the classical Anscombe's condition (C4) and a mixing condition (similar to (C5)) hold. Unfortunately the proof given to this assertion is not correct. Professor Mogyoródi has kindly pointed out to me the error in his paper (1967). Therefore the validity of Mogyoródi's assertion is still an open question.

In the present paper we shall establish some theorems answering to the problem Q mentioned above. The main result is Theorem 3, similar to Mogyoródi's assertion. In fact, in Theorem 3 one condition is stronger while the other one is weaker than the analogous Mogyoródi conditions. At the same time, taking $\lambda \equiv 1$, Theorem 3 becomes just the classical Anscombe theorem.

Theorem 1 is more complicated but it is useful from the point of view of applications in the present topic. Simplifications are obtained if the random variables $(Y_n)_{1 \leqq n < \infty}$ are asymptotically independent with respect to λ (Theorem 4) or if the random variable λ takes on only discrete values (Theorem 6). Finally, all essential conditions are satisfied by sums of independent identically distributed random variables.

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2. Generalities and a useful lemma. Let $(\mathscr{P}_m)_{1 \leqq m < \infty}$ be a sequence of partitions of the sample space Ω, i.e. $\mathscr{P}_m = (A_{k,m})_{1 \leqq k < \infty}$, $(m = 1, 2, \cdots)$ where

$$A_{k,m} = \{(k-1)2^{-m} < \lambda \leqq k2^{-m}\} = \{v_m = k2^{-m}\},$$

$(v_m)_{1 \leqq m < \infty}$ being the usual sequence of elementary random variables which approximates the random variable λ. Obviously

$$A_{k',m} \cap A_{k'',m} = \varnothing, (k' \neq k''); \qquad \bigcup_{k=1}^{\infty} A_{k,m} = \Omega, (m = 1, 2, \cdots).$$

Call $(\mathscr{P}_m)_{1 \leqq m < \infty}$ *the sequence of partitions corresponding to the random variable* λ.

Since for every m $(m = 1, 2, \cdots)$,

$$\sum_{k=1}^{\infty} P(A_{k,m}) = 1$$

then, for every $\eta > 0$ and every m there exists a natural number $l_0 = l_0(m, \eta)$ such that

$$\sum_{k=l_0+1}^{\infty} P(A_{k,m}) < \eta,$$

or equivalently

$$\sum_{k=1}^{l_0} P(A_{k,m}) \geqq 1 - \eta.$$

We shall call the set of events $\{A_{1,m}, A_{2,m}, \cdots, Al_{0,m}\}$. *the essential part of the partition* $\mathscr{P}_m$. Without any loss of generality we may suppose that all events of the essential parts of the partitions have positive probabilities.

Of course, for every $\eta > 0$ we have one essential part of the partition $\mathscr{P}_m$, containing a finite number $l_0 = l_0(m, \eta)$ of events. We shall denote this essential part by $\mathscr{E}(l_0(m, \eta))$. The sequence $\mathscr{E}_\lambda(\eta) = (\mathscr{E}(l_0(m, \eta)))_{1 \leqq m < \infty}$ will be called *the sequence of the essential parts corresponding to* λ. Naturally, for another sequence of elementary random variables $(v_m)_{1 \leqq m < \infty}$ which approximates the random variable λ we will have generally a different sequence of essential parts corresponding to λ and it is possible, of course, to take it into consideration.

If λ is a positive random variable which takes on discrete values, $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots$ then we shall consider as sequence of partitions corresponding to λ the constant sequence $(\mathscr{P}_m)_{1 \leqq m < \infty}$, $\mathscr{P}_m = (A_k)_{1 \leqq k < \infty}$ for every m $(m = 1, 2, \cdots)$, where

$$A_k = \{\lambda = \lambda_k\}, \qquad (k = 1, 2, \cdots).$$

Particularly, if $\lambda \equiv c$, where c is a positive constant, then for every $\eta > 0$ we shall consider $\mathscr{E}_\lambda(\eta) = \{\Omega\}$.

Let us prove now a useful lemma, a slightly different form of a result of Blum, Hanson and Rosenblatt (1963):

LEMMA. *Let*

$$(W_r)_{1 \leqq r < \infty}, (x_{m,r})_{1 \leqq m < \infty, 1 \leqq r < \infty}, (y_{m,r})_{1 \leqq m < \infty, 1 \leqq r < \infty}$$

be sequences of random variables such that for every m and r we have

$$W_r = x_{m,r} + y_{m,r}$$

Let us suppose that the following conditions are satisfied, i.e.

(A) *The distribution functions of the sequence* $(x_{m,r})_{1 \leqq r < \infty}$ *converge to the distribution function F for each fixed m;*

(B) *For every* $\varepsilon > 0$ *we have*

$$\lim_{m \to \infty} \limsup_r P(|y_{m,r}| > \varepsilon) = 0.$$

Then the distribution functions of the sequence $(W_r)_{1 \leqq r < \infty}$ *converge also to F.*

PROOF. If x and y are two random variables and $b \geqq 0$ then we have the obvious inequalities

(1) $P(x+y \leqq a-b) - P(|y| > b) \leqq P(x \leqq a) \leqq P(x+y \leqq a+b) + P(|y| > b).$

Let now α be an arbitrary point of continuity of the distribution function F and let $\varepsilon > 0$ be an arbitrary real number. We want to show that for r sufficiently large we have

(2) $$|P(W_r \leqq \alpha) - F(\alpha)| < \varepsilon.$$

We put

$$x = W_r, \qquad y = -y_{m,r}, \qquad a = \alpha, \qquad b = \delta > 0$$

in the inequalities (1) and we obtain

$$P(x_{m,r} \leqq \alpha - \delta) - F(\alpha) - P(|y_{m,r}| > \delta) \leqq P(W_r \leqq \alpha) - F(\alpha) \leqq P(x_{m,r} \leqq \alpha + \delta) - F(\alpha) + P(|y_{m,r}| > \delta).$$

Then

(3) $$\begin{aligned} |P(W_r \leqq \alpha) - F(\alpha)| &\leqq \max_{j=\pm 1} |P(x_{m,r} \leqq \alpha + j\delta) - F(\alpha)| + P(|y_{m,r}| > \delta) \\ &\leqq \max_{j=\pm 1} |F(\alpha + j\delta) - F(\alpha)| \\ &\quad + \max_{j=\pm 1} |P(x_{m,r} \leqq \alpha + j\delta) - F(\alpha + j\delta)| + P(|y_{m,r}| > \delta). \end{aligned}$$

We choose now $\delta > 0$ such that:

(4) $\alpha + \delta$ and $\alpha - \delta$ be continuity points of the distribution function F;

(5) $$\max_{j=\pm 1} |F(\alpha) - F(\alpha + j\delta)| < \varepsilon/3.$$

Using (B) we choose m sufficiently large such that

(6) $$\limsup_r P(|y_{m,r}| > \delta) > \varepsilon/3.$$

Using now (A) and (6) we choose r_0 such that for $r \geqq r_0$ we have

(7) $$\max_{j=\pm 1} |P(x_{m,r} \leqq \alpha + j\delta) - F(\alpha + j\delta)| < \varepsilon/3,$$

(8) $$P(|y_{m,r}| > \delta) < \varepsilon/3.$$

From (5), (7), (8) and (3) we obtain for every $r \geqq r_0$ the inequality (2). □

3. A general theorem.

THEOREM 1. *We suppose that we have the convergence* (C0) *and the following two additional conditions:*

(C1) *For every $\eta > 0$ and every $A_{k,m} \in \mathscr{E}_\lambda(\eta)$,*

$$\lim_{n\to\infty} P_{A_{k,m}}(Y_n \leqq a) = F(a)$$

for all continuity points a of the function F;

(C2) *For every $\varepsilon > 0$, $\eta > 0$ and every $A_{k,m} \in \mathscr{E}_\lambda(\eta)$ there exist a small real number $s_0 = s_0(\varepsilon, \eta)$ and a natural number $n_0 = n_0(\varepsilon, \eta, k, m)$ such that for every $n > n_0$ we have*

$$P_{A_{k,m}}(\max_{i,|i-n|<s_0 n} |Y_i - Y_n| > \varepsilon) < \eta;$$

then, in every continuity point of the function F we have

$$\lim_{r\to\infty} P(Y_{N_r} \leqq a) = F(a).$$

PROOF. According to the notation of the lemma given above we put

$$x_{m,r} = Y_{[n_r v_m]},\ y_{m,r} = Y_{N_r} - Y_{[n_r v_m]},\ W_r = Y_{N_r}$$

where the sign $[v]$ denotes the integer part of the real number v. Obviously,

$$W_r = x_{m,r} + y_{m,r}$$

whichever be r, m $(r, m = 1, 2, \cdots)$. Let us show that all conditions of the lemma are satisfied. Indeed, $([n_r k 2^{-m}])_{1 \leqq r < \infty}$ is a sequence of natural numbers, for every m and k $(m, k = 1, 2, \cdots)$. The condition (C1) implies that for every $\eta > 0$, $A_{k,m} \in \mathscr{E}_\lambda(\eta)$ and every continuity point a of the function $F(a)$ there exists a natural number $r_0 = r_0(\eta, a, k, m)$ such that for every $r > r_0$ we have

$$|P_{A_{k,m}}(Y_{[n_r k 2^{-m}]} \leqq a) - F(a)| < \eta.$$

We put now

$$\bar{r} = \bar{r}_0(\eta, a, m) = \max_{1 \leqq k \leqq l_0} r_0(\eta, a, k, m), \qquad (l_0 = l_0(m, \eta)).$$

Then for every m $(m = 1, 2, \cdots)$ if $r > \bar{r}_0$ we have

$$\begin{aligned}
|P(x_{m,r} \leqq a) - F(a)| &= |P(Y_{[n_r v_m]} \leqq a) - F(a)| \\
&= |\textstyle\sum_{k=1}^{\infty} P((Y_{[n_r v_m]} \leqq a) \cap A_{k,m}) - F(a)| \\
&\leqq |\textstyle\sum_{k=1}^{l_0} P((Y_{[n_r v_m]} \leqq a) \cap A_{k,m}) - F(a) \sum_{k=1}^{l_0} P(A_{k,m})| \\
&\quad + \textstyle\sum_{k=l_0+1}^{\infty} P((Y_{[n_r v_m]} \leqq a) \cap A_{k,m}) + F(a) \sum_{k=l_0+1}^{\infty} P(A_{k,m}) \\
&\leqq \textstyle\sum_{k=1}^{l_0} |P_{A_{k,m}}(Y_{[n_r k 2^{-m}]} \leqq a) - F(a)| P(A_{k,m}) \\
&\quad + 2 \textstyle\sum_{k=l_0+1}^{\infty} P(A_{k,m}) < \eta \sum_{k=1}^{l_0} P(A_{k,m}) + 2\eta < 3\eta
\end{aligned}$$

i.e. $\lim_{r\to\infty} P(x_{m,r} \leqq a) = F(a)$, in every continuity point of the function $F(a)$, whichever be m ($m = 1, 2, \cdots$). Therefore condition (A) of the lemma is satisfied. Let us notice further that from conditions (C0) and (C2) we have for arbitrary $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{m\to\infty} \limsup_r P(|y_{m,r}| > \varepsilon) = \lim_{m\to\infty} \limsup_r P(|Y_{N_r} - Y_{[n_r\nu_m]}| > \varepsilon) \\
&\quad \leqq \lim_{m\to\infty} \limsup_r P((|Y_{N_r} - Y_{[n_r\nu_m]}| > \varepsilon) \cap (|N_r/n_r - \lambda| < 2^{-m})) \\
&\qquad + \lim_{m\to\infty} \limsup_r P(|N_r/n_r - \lambda| \geqq 2^{-m}) \\
(9) &\quad = \lim_{m\to\infty} \limsup_r P(\textstyle\bigcup_{k=1}^{\infty} ((|Y_{N_r} - Y_{[n_r\nu_m]}| > \varepsilon) \\
&\qquad \cap (|N_r/n_r - \lambda| < 2^{-m}) \cap A_{k,m})) \\
&\quad \leqq \lim_{m\to\infty} \limsup_r P(\textstyle\bigcup_{k=1}^{\infty} ((\max_{i,|i/n_r - \lambda| < 2^{-m}} |Y_i - Y_{[n_r k 2^{-m}]}| > \varepsilon) \\
&\qquad \cap A_{k,m})) \\
&\quad \leqq \lim_{m\to\infty} \limsup_r P(\textstyle\bigcup_{k=1}^{\infty} ((\max_{i,(k-2)2^{-m}n_r < i < (k+1)2^{-m}n_r} |Y_i - Y_{[n_r k 2^{-m}]}| \\
&\qquad > \varepsilon) \cap A_{k,m}))
\end{aligned}$$

where in the last inequality we have taken into account that from the inequality $|i/n_r - \lambda| < 2^{-m}$ it results

$$(10) \qquad (k-2)2^{-m}n_r < i < (k+1)2^{-m}n_r$$

because on the set $A_{k,m}$ we have $(k-1)2^{-m} < \lambda \leqq k2^{-m}$. From condition (C2) it results that for every $\varepsilon > 0$, $\eta > 0$ and every $A_{k,m} \in \mathscr{E}_\lambda(\eta)$ there exists a small real number $s_0 = s_0(\varepsilon, \eta)$ such that

$$(11) \qquad \limsup_n P_{A_{k,m}}(\max_{i,|i-n| < s_0 n} |Y_i - Y_n| > \varepsilon) < \eta.$$

Let us choose the natural number $m_0 = m_0(\varepsilon, \eta)$ such that $m_0 s_0 > 2$ and such that for $m > m_0$

$$(12) \qquad P(\lambda < m2^{-m}) < \eta.$$

Some simple calculations show that for every $m > m_0$ and $k \geqq m$ if r is sufficiently large, the inequality (10) implies

$$(13) \qquad |i - [n_r k 2^{-m}]| < s_0[n_r k 2^{-m}].$$

Now, using (11) and (12) it results that for $m > m_0$ we have

$$\begin{aligned}
&\limsup_r P(\textstyle\bigcup_{k=1}^{\infty} ((\max_{i,(k-2)2^{-m}n_r < i < (k+1)2^{-m}n_r} |Y_i - Y_{[n_r k 2^{-m}]}| > \varepsilon) \cap A_{k,m})) \\
(14) &\quad \leqq P(\lambda < m2^{-m}) + \textstyle\sum_{k=m}^{l_0} \limsup_r P_{A_{k,m}}(\max_{i,|i-[n_r k 2^{-m}]| < s_0[n_r k 2^{-m}]} \\
&\qquad \cdot |Y_i - Y_{[n_r k 2^{-m}]}| > \varepsilon) P(A_{k,m}) + \textstyle\sum_{k=l_0+1}^{\infty} P(A_{k,m}) \\
&\quad < \eta + \eta \textstyle\sum_{k=m}^{l_0} P(A_{k,m}) + \eta < 3\eta.
\end{aligned}$$

Thus from (9) and (14) it results

$$\lim_{m\to\infty} \limsup_r P(|y_{m,r}| > \varepsilon) = 0,$$

for every $\varepsilon > 0$. Therefore the condition (B) of the lemma is satisfied too and we have

$$\lim_{r\to\infty} P(Y_{N_r} \leqq a) = \lim_{r\to\infty} P(W_r \leqq a) = \lim_{r\to\infty} P(x_{m,r} \leqq a) = F(a),$$

at every continuity point of the function $F(a)$. □

REMARK. From the proof of Theorem 1 it results that the sequences of random variables $(Y_{Nr})_{1\leqq r<\infty}$ and $(Y_{[n_r v_m]})_{1\leqq r<\infty}$ have the same asymptotic distribution.

4. The classical Anscombe theorem. Let us consider the particular case $\lambda \equiv 1$. Obviously, in this case for every $\eta > 0$ we take $\mathscr{E}_\lambda(\eta) = \{\Omega\}$, and from Theorem 1 we obtain just the classical Anscombe theorem (see Anscombe (1952)):

THEOREM 2. *If the sequence* $(N_r/n_r)_{1\leqq r<\infty}$ *converges in probability to* 1 *and if we have the following two conditions*:

(C3) *At every continuity point of the function F*,

$$\lim_{n\to\infty} P(Y_n \leqq a) = F(a);$$

(C4) *For every* $\varepsilon > 0$ *and* $\eta > 0$ *there exist a small real number* $s_0 = s_0(\varepsilon, \eta)$ *and a natural number* $n_0 = n_0(\varepsilon, \eta)$ *such that for every* $n > n_0$ *the following inequality holds*,

$$P(\max_{i,|i-n|<s_0 n} |Y_i - Y_n| > \varepsilon) < \eta.$$

Then, in every continuity point of the function F, we have

$$\lim_{r\to\infty} P(Y_{N_r} \leqq a) = F(a).$$

5. A simple general theorem. Let us denote by $\mathscr{K}_\lambda$ the σ-algebra generated by the random variable λ. Obviously, $\mathscr{K}_\lambda \subset \mathscr{K}$ and $\mathscr{P}_m \subset \mathscr{K}_\lambda$ whichever be m ($m = 1, 2, \cdots$). We are able now to formulate the following theorem:

THEOREM 3. *We suppose that together with* (C0) *the following conditions are satisfied:*

(C5) *For every* $A \in \mathscr{K}_\lambda$, $(P(A) > 0)$,

$$\lim_{n\to\infty} P_A(Y_n \leqq a) = F(a),$$

at every continuity point of the function F.

(C6) *For every* $\varepsilon > 0$, $\eta > 0$ *and every* $A \in \mathscr{K}_\lambda$, $(P(A) > 0)$, *there exist a small real number* $s_0 = s_0(\varepsilon, \eta)$ *and a natural number* $n_0 = n_0(\varepsilon, \eta, A)$ *such that for every* $n > n_0$ *we have*

$$P_A(\max_{i,|i-n|<s_0 n} |Y_i - Y_n| > \varepsilon) < \eta.$$

Then, at every continuity point of F we have

$$\lim_{r\to\infty} P(Y_{N_r} \leqq a) = F(a).$$

Of course, condition (C5) implies (C1) and condition (C6) implies (C2). Therefore, Theorem 1 implies Theorem 3. Notice that if $\lambda \equiv 1$ we have $\mathscr{K}_\lambda = \{\phi, \Omega\}$ and Theorem 3 becomes Theorem 2.

6. The case of asymptotic independence with respect to λ. Let us suppose now that the sequence of random variables $(Y_n)_{1 \leqq n < \infty}$ is asymptotically independent with respect to λ. More precisely, we shall prove the following theorem:

THEOREM 4. *Let us suppose that together with* (C0), (C4) *and* (C5) *the following additional condition is satisfied:*

(C7) *For every sequence of events $(A_n)_{1 \leqq n < \infty}$ such that $A_n \in \mathscr{K}_n$ where $\mathscr{K}_n \subset \mathscr{K}$ is the σ-algebra generated by the sequence of random variables $(Y_k)_{n \leqq k < \infty}$ we have*

$$\limsup_n P_A(A_n) = \limsup_n P(A_n),$$

for all $A \in \mathscr{K}_\lambda$, $(P(A) > 0)$. Then we have

$$\lim_{r \to \infty} P(Y_{N_r} \leqq a) = F(a),$$

at every continuity point of F.

PROOF. It is sufficient to prove that (C4) together with (C7) imply (C6). Then Theorem 3 implies Theorem 4. Indeed, condition (C4) implies that for every $\varepsilon > 0$ and $\eta > 0$ there exists a small real number $s_0 = s_0(\varepsilon, \eta)$ such that

$$\limsup_n P(\max_{i, |i-n| < s_0 n} |Y_i - Y_n| > \varepsilon) < \eta. \tag{15}$$

We notice also that for every $\varepsilon > 0$ and $\eta > 0$ the event

$$(\max_{i, |i-n| < s_0 n} |Y_i - Y_n| > \varepsilon) \in \mathscr{K}_{[(1-s_0)n]+1},$$

and from (C7) and (15) we obtain

$$\limsup_n P_A(\max_{i, |i-n| < s_0 n} |Y_i - Y_n| > \varepsilon) = \limsup_n P(\max_{i, |i-n| < s_0 n} |Y_i - Y_n| > \varepsilon) < \eta,$$

for every $A \in \mathscr{K}_\lambda$, $(P(A) > 0)$. Thus, for every $\varepsilon > 0$, $\eta > 0$ and every $A \in \mathscr{K}_\lambda$, $(P(A) > 0)$, there exist a small real number $s_0 = s_0(\varepsilon, \eta)$ and a natural number $n_0 = n_0(\varepsilon, \eta, A)$ such that for every $n > n_0$ we have

$$P_A(\max_{i, |i-n| < s_0 n} |Y_i - Y_n| > \varepsilon) < \eta,$$

which is condition (C6). □

REMARK. If in Theorem 4 we consider the particular case $\lambda \equiv 1$ we obtain again Theorem 2 of F. J. Anscombe because condition (C5) becomes (C3) while, on the other hand, the condition (C7) is obviously satisfied, with $\mathscr{K}_\lambda$ being just $\{\phi, \Omega\}$. A more restrictive variant of Theorem 4 was proved directly by W. Richter (1965b). In his theorem, instead of $\mathscr{K}_\lambda$ appears the whole σ-algebra $\mathscr{K}$. As a consequence of this overly strong restriction, if we consider in Richter's theorem the particular case $\lambda \equiv 1$, it is not possible to obtain Anscombe's theorem.

7. A condition imposed on λ. Condition (C4) is weaker than (C6). However, Theorem 4 contains a new restriction (i.e. condition (C7)) imposed on the sequence $(Y_n)_{1 \leqq n < \infty}$. Let us consider now Theorem 1 and let us replace the condition (C2) by the weaker Anscombe condition (C4). At the same time, let us introduce, in compensation, a new restriction imposed in this case on the random variable λ. We shall prove thus the following theorem:

THEOREM 5. *Let us suppose that together with* (C0), (C1) *and* (C2) *the following condition is satisfied:*

(C8) *For every* $\eta > 0$ *we have*

$$d(\eta) = \inf_{1 \leqq m < \infty} d(m, \eta) \neq 0,$$

where

$$d(m, \eta) = \min_{1 \leqq k \leqq l_0} \{P(A_{k,m}) \mid A_{k,m} \in \mathscr{E}_\lambda(\eta)\}.$$

Then, in every continuity point of F we have

$$\lim_{r \to \infty} P(Y_{N_r} \leqq a) = F(a).$$

PROOF. Obviously, if we are able to prove that conditions (C4) and (C8) imply (C2) then Theorem 1 will imply Theorem 5. Indeed, from the conditions (C4) and (C8) it results that for every $\varepsilon > 0$ and $\eta > 0$ there exist a small real number $s_0 = s_0(\varepsilon, \eta)$ and a natural number $n_0 = n_0(\varepsilon, \eta)$ such that for every $n > n_0$ we have

$$P(\max_{i, |i-n| < s_0 n} |Y_i - Y_n| > \varepsilon) < \eta d(\eta),$$

and thus for every $n > n_0$, using the definition of $d(\eta)$, we have

$$P_{A_{k,m}}(\max_{i, |i-n| < s_0 n} |Y_i - Y_n| > \varepsilon)$$
$$\leqq P(\max_{i, |i-n| < s_0 n} |Y_i - Y_n| > \varepsilon)/P(A_{k,m}) < \eta d(\eta)/P(A_{k,m}) < \eta,$$

for every $A_{k,m} \in \mathscr{E}_\lambda(\eta)$. □

8. The case when λ takes on only discrete values. When is the condition (C8) satisfied? An example answering this problem is given by the following theorem:

THEOREM 6. *Let λ be a positive random variable with discrete values. Then conditions* (C0). (C5) *and* (C4) *imply*

$$\lim_{r \to \infty} P(Y_{N_r} \leqq a) = F(a), \tag{16}$$

at every continuity point of F.

PROOF. Because condition (C5) implies (C1) it is sufficient to prove that condition (C8) is satisfied in this case and then Theorem 5 will imply Theorem 6 mentioned above. Indeed, let $(\lambda_k)_{1 \leqq k < \infty}$ be the values taken on by the random variable λ. The sequence of partitions corresponding to λ will be the constant sequence $(\mathscr{P}_m)_{1 \leqq m < \infty}$, $\mathscr{P}_m = (A_k)_{1 \leqq k < \infty}$ where $A_k = (\lambda = \lambda_k)$, $(k = 1, 2, \cdots)$. Obviously, in this case for every $\eta > 0$ we have $l_0(m, \eta) = l_0(\eta)$ and $\mathscr{E}(l_0(m, \eta)) = \mathscr{E}(l_0(\eta))$ whichever be m $(m = 1, 2, \cdots)$ and, of course,

$$d(\eta) = \min_{1 \leqq k \leqq l_0} \{P(A_k) \mid A_k \in \mathscr{E}(l_0(\eta))\} \neq 0,$$

which is condition (C8). □

REMARK. If we put $\lambda \equiv 1$, Theorem 6 becomes Theorem 2. Theorem 6, with condition (C6) stated in a stronger form (i.e. $\mathscr{K}_\lambda$ is replaced there by the whole σ-algebra $\mathscr{K}$) was proved directly by W. Richter (1965a). The restriction mentioned in the brackets is too strong and as consequence, in the particular case $\lambda \equiv 1$ it is not possible to obtain Anscombe's theorem. Also, J. Mogyoródi (1967) mentioned that conditions (C0), (C4) and (C5) (but with $\mathscr{K}$ in the place of $\mathscr{K}_\lambda$) imply (16) even if λ is an arbitrary positive random variable. (The proof given to this assertion is not valid. If λ is an arbitrary positive random variable it is not possible to apply the Anscombe condition to the inequality (7) of the Mogyoródi (1967) paper, hence the last inequality on page 467 is not true.)

9. Normed sums of independent random variables. Finally, we want to emphasize some examples satisfying various conditions of the theory presented above.

(a) A sequence of random variables $(Z_n)_{1 \leqq n < \infty}$ is mixing with density F if a every continuity point of F we have

$$\lim_{n \to \infty} P_A(Z_n \leqq a) = F(a),$$

for every $A \in \mathscr{K}$, $(P(A) > 0)$. Obviously, if the sequence of random variables $(Y_n)_{1 \leqq n < \infty}$ is mixing with density F then conditions (C5) and (C1) are satisfied. An example of mixing sequence of random variables is the following one (see A. Rényi (1960)). Let $(x_n)_{1 \leqq n < \infty}$ be a sequence of independent random variables, $(B_n)_{1 \leqq n < \infty}$ be a sequence of real numbers and $(D_n)_{1 \leqq n < \infty}$ be a sequence of positive real numbers tending to infinity. If

$$\lim_{n \to \infty} P(x_1 + \cdots + x_n - B_n \leqq aD_n) = F(a),$$

at every continuity point of F then the sequence

$$Y_n = (x_1 + \cdots + x_n - B_n)/D_n, \qquad (n = 1, 2, \cdots),$$

is mixing with density $F(a)$.

(b) Condition (C7) is satisfied by the following example given by J. R. Blum, D. L. Hanson and J. I. Rosenblatt (1963): Let $(x_n)_{1 \leqq n < \infty}$ be a sequence of independent random variables and $(k_n)_{1 \leqq n < \infty}$, $(m_n)_{1 \leqq n < \infty}$ be two sequences of natural numbers. If A_n is an event depending only on the random variables $x_{k_n}, \cdots, x_{m_n}$ then

$$\limsup_n P_A(A_n) = \limsup_n P(A_n)$$

for every $A \in \mathscr{K}$, $(P(A) > 0)$. In fact condition (C1) holds any time that $(Y_n)_{1 \leqq n < \infty}$ has a 0–1 tail. This follows from a result of D. Blackwell and D. A. Freedman (1964).[1]

[1] This fact was communicated to me by the referee of the present paper.

(c) Anscombe's condition (C4) is satisfied, for example, by the normed sums of independent random variables. Indeed, if $(x_n)_{1 \leqq n < \infty}$ is a sequence of independent and identically distributed random variables with mean value 0 and variance 1, let us define

(17) $$Y_n = (x_1 + \cdots + x_n)/n^{\frac{1}{2}}, \qquad (n = 1, 2, \cdots).$$

Taking into account the obvious inequality

$$P(\max_{i, |i-n| < s_0 n} |Y_i - Y_n| > \varepsilon)$$
$$\leqq P(\max_{i, |i-n| < s_0 n} |Y_i - Y_{[(1-s_0)n]}| > \varepsilon/2) + P(|Y_n - Y_{[(1-s_0)n]}| > \varepsilon/2),$$

and applying the well-known inequalities of Tchebychev and Kolmogorov one obtains condition (C4).

Thus for the sequence (17), conditions (C5), (C4) and (C7) of Theorem 4 are all satisfied, and the single condition (C0) about the behavior of the "random time" implies

$$\lim_{r \to \infty} P(x_1 + \cdots + x_{N_r} \leqq a N_r^{\frac{1}{2}}) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{a} e^{-\frac{1}{2}u^2} du.$$

This last result was conjectured by A. Rényi (1960) and was proved directly by J. Mogyoródi (1962) and independently by J. R. Blum, D. L. Hanson and J. I. Rosenblatt (1963).

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