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TRANSFORMATIONS OF GAUSSIAN PROCESSES TO THE BROWNIAN MOTION

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Transformations of a class of Gaussian processes to the Brownian motion are obtained by reproducing kernel Hilbert space methods. These transformations are such that the value of the transformed process at any point of time is given in terms of the sample path of the original process up to that time. In certain situations the boundary-crossing behaviors of the original process and the transformed process are related.

1. Introduction. Consider a finite number of random variables $\{X(t), t = 1, \cdots, n\}$ following a nonsingular Gaussian distribution with zero mean. In dealing with many problems concerning such random variables, it is convenient to work with linear transforms $\{Y(t), t = 1, \cdots, n\}$ of the X-variables which are independent standardized Gaussian random variables. Indeed, these transforms can be carried out sequentially in t, i.e. by transforming with a lower triangular matrix. For a continuous time parameter Gaussian process $\{X(t), t \geqq 0\}$ the natural analogue of this would be to try to transform the process to a Brownian motion and the need for such transformations has been expressed by Doob (1949). In this paper, such transformations are obtained for a class of Gaussian processes by reproducing kernel Hilbert space methods. Again, these transformations are sequential in t. Furthermore, in certain situations these transforms are such that a boundary-crossing problem for the original process is related to a boundary-crossing problem for the Brownian motion. For a survey of reproducing kernel methods the reader may consult Aronszajn (1950) and Parzen (1959).

2. Preliminaries. Consider a separable real Gaussian process $\{X(t), t \geqq 0\}$ with mean value function 0 and covariance kernel R. We shall restrict our attention to a certain class of such process by imposing some conditions on R. The first condition is,

CONDITION 1. R is continuous and nonsingular on every finite subset of $[0, \infty)$.

Before stating the other conditions, let us introduce some notations. First note that the continuity of R implies that the process is continuous in mean square and therefore, continuous in probability. Consequently, any countable set T which is dense in $[0, \infty)$ satisfies the definition of separability by virtue of a theorem of Doob (1953, Chapter II, Theorem 2.2). To be specific, we are going to take the non-negative rationals for T in all our discussion. The separability condition will be needed only in Section 5. Consider now a sequence of finite sets $\{T_n\}$ satisfying $T_n \subset T_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} T_n = T$. We denote $T_n^{\tau} = T_n \cap [0, \tau]$; R^{τ} is the restriction of R to $[0, \tau] \times [0, \tau]$; $R_{n,\tau}^{-1}(s, t)$, $s, t \in T_n^{\tau}$ are the elements of the inverse

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of the matrix $R_{n,\tau} = ((R(s, t)))_{s,t \in T_n^\tau}$; $H(R)$ is the reproducing kernel Hilbert space of functions on $[0, \infty)$ with reproducing kernel R; $H(R^\tau)$ is the reproducing kernel Hilbert space of functions on $[0, \tau]$ with reproducing kernel R^τ; $R_s(\cdot) = R(s, \cdot)$ for every $s \geqq 0$; R_s^τ is the restriction of R_s to $[0, \tau]$; $\psi(s, t; \tau) = (R_s^\tau, R_t^\tau)_{H(R^\tau)}$, $s, t, \tau \geqq 0$. When there is no danger of confusion we shall sometimes use the same symbol for a function and its restriction to some subset of its domain.

It is well known that $H(R^\tau)$ consists of all functions which are restrictions to $[0, \tau]$ of functions in $H(R)$ and if f^τ, g^τ are restrictions to $[0, \tau]$ of $f, g \in H(R)$ respectively, then

$$(f^\tau, g^\tau)_{H(R^\tau)} = (E^*[f \mid R_t, t \leqq \tau], E^*[g \mid R_t, t \leqq \tau])_{H(R)} \tag{1}$$

where E^* stands for projection. We shall frequently use this fact though we may not explicitly refer to it when it is used. We shall now list two other conditions on R in terms of the function ψ. These conditions can also be interpreted in terms of conditional expectations by noting that

$$\begin{aligned}\psi(s, t; \tau) &= (R_s^\tau, R_t^\tau)_{H(R^\tau)} \\ &= (E^*[R_s \mid R_u, u \leqq \tau], E^*[R_t \mid R_u, u \leqq \tau])_{H(R)} \\ &= (E^*[X(s) \mid X(u), u \leqq \tau], E^*[X(t) \mid X(u), u \leqq \tau])_{L_2^*}\end{aligned}$$

where L_2^* is the closed linear manifold of the Hilbert space $L_2(\Omega, \mathscr{A}, P)$ of square-integrable functions on the basic sample space $(\Omega, \mathscr{A}, P)$ spanned by $\{X(t), t \geqq 0\}$. The process being Gaussian, projections and conditional expectations are the same and we have

$$\psi(s, t; \tau) = \operatorname{Cov}(E[X(s) \mid X(u), u \leqq \tau], E[X(t) \mid X(u), u \leqq \tau]).$$

CONDITION 2. There exists a constant K, not depending on (s, t) such that

$$|\psi(s, t; \sigma) - \psi(s, t; \tau)| \leqq K|\sigma - \tau|.$$

CONDITION 3. For every positive-valued function w on $[0, \infty)$ which satisfies $\int_0^\infty w(s)R(s, s)^{\frac{1}{2}} ds < \infty$, the integral, $\int_0^\infty \int_0^\infty w(s)w(t)\psi(s, t; \tau) ds\, dt$ is a strictly increasing function of τ.

REMARK. Since

$$\begin{aligned}&\textstyle\int_0^\infty \int_0^\infty w(s)w(t)\psi(s, t, \tau)\, ds\, dt \\ &\quad = \|E^*[\textstyle\int_0^\infty w(s)R_s\, ds \mid R_u, u \leqq \tau]\|^2_{H(R)} \\ &\quad = \operatorname{Var}(E[\textstyle\int_0^\infty w(s)X(s)\, ds \mid X(u), u \leqq \tau]),\end{aligned}$$

this integral is monotone non-decreasing in τ even without the condition.

We now choose a function w on $[0, \infty)$ satisfying (i) $w(s) > 0$ for all s, (ii) $\int_0^\infty w(s) ds < \infty$, (iii) $\int_0^\infty w(s)R(s, s)^{\frac{1}{2}} ds < \infty$, and define

$$f(t) = \int_0^\infty w(s)R(s, t)\, ds, \qquad t \geqq 0. \tag{2}$$

For instance, $w(s) = e^{-s} \min(R(s,s)^{-\frac{1}{2}}, 1)$ will do. The requirement (iii) on w tells us that $f \in H(R)$, since

$$\int_0^\infty \int_0^\infty w(s)w(t)R(s,t)\,ds\,dt \leqq \int_0^\infty \int_0^\infty w(s)w(t)\{R(s,s)R(t,t)\}^{\frac{1}{2}}\,ds\,dt$$
$$= \left(\int_0^\infty w(s)R(s,s)^{\frac{1}{2}}\,ds\right)^2 < \infty.$$

Consequently, for every $\tau \geqq 0$, the restriction of f to $[0, \tau]$ is in $H(R^\tau)$. We now define

$$(3) \qquad \phi(\tau) = \|f\|^2_{H(R^\tau)} = \int_0^\infty \int_0^\infty w(s)w(t)(R_s, R_t)_{H(R^\tau)}\,ds\,dt$$
$$= \int_0^\infty \int_0^\infty w(s)w(t)\psi(s,t;\tau)\,ds\,dt.$$

This function ϕ is going to play a central role in our subsequent development and the purpose of Conditions 2 and 3 is to ensure that ϕ behaves in a desirable manner. To this end, we now prove two lemmas.

LEMMA 1. *ϕ is Lipschitz.*

PROOF.

$$|\phi(\sigma)-\phi(\tau)| \leqq \int_0^\infty \int_0^\infty w(s)w(t)|\psi(s,t;\sigma)-\psi(s,t;\tau)|\,ds\,dt \leqq K|\sigma-\tau|\left(\int_0^\infty w(s)\,ds\right)^2$$

by Condition 2. Because of requirement (ii) on w, the lemma is now seen to hold.

LEMMA 2. *ϕ is strictly increasing.*

PROOF. The proof follows from Condition 3 and requirements (i) and (iii) on the function w.

We shall use the function f defined by (2) to transform the X-process to a Brownian motion in two steps. First we shall transform $\{X(t), t \geqq 0\}$ to a Gaussian process $\{\tilde{Y}(t), t \geqq 0\}$ with mean value function 0 and covariance kernel $\rho(s,t) = \phi(\min(s,t))$ in which all sample paths are continuous. The $\tilde{Y}$-process which is one with independent though not necessarily stationary increments, will then be transformed to a Brownian motion on $[0, \|f\|^2_{H(R)} - \phi(0))$. For a more restricted class of Gaussian processes satisfying some additional conditions, a different method will be given for transforming the $\tilde{Y}$-process to a Brownian motion on $[0, \infty)$.

REMARK. Suppose $\{X(t), t \geqq 0\}$ is a separable real Gaussian process and let $P_m{}^\tau$ denote the probability distribution of the process up to time τ with mean value function m and covariance kernel R. If $m \in H(R)$ then $P_m{}^\tau$ and $P_0{}^\tau$ are equivalent and an explicit formula for the probability density functional $dP_m{}^\tau/dP_0{}^\tau$ is known (see Parzen (1959)). It is easy to verify that both under P_0 and under P_m, $\{\log dP_m{}^t/dP_0{}^t, t \geqq 0\}$ is a Gaussian process with independent increments, which leads us to the process defined by (4) below.

3. Construction of a Gaussian process with independent increments. We define

$$(4) \qquad Y(\tau) = \lim_{n\to\infty} \sum_{s,t \in T_n{}^\tau} f(s)R_{n,\tau}^{-1}(s,t)X(t), \qquad \tau \geqq 0.$$

The main results of this section are given in the following theorem and its corollary.

THEOREM 1. *Let* $\{X(t), t \geqq 0\}$ *be a real Gaussian process with mean value function* 0 *and covariance kernel R satisfying Conditions* 1–3. *Then*

(a) *The process* $\{Y(t), t \geqq 0\}$ *given by* (4) *is a Gaussian process with mean value function* 0 *and covariance kernel* $\rho(s, t) = \phi(\min(s, t))$; *it is continuous in probability and almost all its sample functions are uniformly continuous on* $T \cap [0, \tau]$ *for every* τ.

(b) *The modified process* $\{\tilde{Y}(t), t \geqq 0\}$ *obtained by continuous extension of sample paths* $\{Y(t, \omega), t \in T\}$ *satisfies for every* t, $\tilde{Y}(t, \omega) = Y(t, \omega)$ a.s., *and all its sample paths are continuous.*

COROLLARY. $\{\bar{Y}(t) = \tilde{Y}(t) - \tilde{Y}(0), t \geqq 0\}$ *is a Gaussian process with mean value function* 0 *and covariance kernel* $\rho'(s, t) = \phi(\min(s, t)) - \phi(0)$. *Hence, this is a process with independent increments, starting at* 0 *with probability* 1 *and having* $\text{Var}(\bar{Y}(s) - \bar{Y}(t)) = |\phi(s) - \phi(t)|$. *Furthermore, all its sample paths are continuous.*

The proof of the theorem will be broken up into several lemmas.

LEMMA 3. $Y(\tau)$ *exists both as a limit in mean square and with probability* 1.

PROOF. See Parzen (1959), Theorem 9B.

LEMMA 4. $\{Y(t), t \geqq 0\}$ *is a Gaussian process with mean value function* 0.

PROOF. Follows from the linearity of the transformation given in (4).

We shall now obtain the covariance kernel of $\{Y(t), t \geqq 0\}$. The following lemma can be easily verified.

LEMMA 5. *Let* $A = ((a_{uv}))$ *be a nonsingular* $n \times n$ *matrix with inverse* $A^{-1} = ((a^{uv}))$ *and* $b_1, \cdots, b_n$ *any* n *real numbers. Then*

$$\textstyle\sum_{u=1}^{n} \sum_{v=1}^{n} a_{su} a^{uv} b_v = b_s, \qquad s = 1, \cdots, n.$$

LEMMA 6. *For* $0 \leqq \sigma \leqq \tau$, *and for each* n,

$$\text{Cov}\left(\textstyle\sum_{s,t \in T_n^\sigma} f(s) R_{n,\sigma}^{-1}(s, t) X(t), \sum_{s,t \in T_n^\tau} f(s) R_{n,\tau}^{-1}(s, t) X(t)\right) = \textstyle\sum_{s,t \in T_n^\sigma} f(s) R_{n,\sigma}^{-1}(s, t) f(t).$$

PROOF.

$$\begin{aligned} &\text{Cov}\left(\textstyle\sum_{s,t \in T_n^\sigma} f(s) R_{n,\sigma}^{-1}(s, t) X(t), \sum_{s,t \in T_n^\tau} f(s) R_{n,\tau}^{-1}(s, t) X(t)\right) \\ &= \textstyle\sum_{s,t \in T_n^\sigma} \sum_{u,v \in T_n^\tau} f(s) R_{n,\sigma}^{-1}(s, t) f(u) R_{n,\tau}^{-1}(u, v) R(t, v) \\ &= \textstyle\sum_{s,t \in T_n^\sigma} f(s) R_{n,\sigma}^{-1}(s, t) \sum_{u,v \in T_n^\tau} R(t, v) R_{n,\tau}^{-1}(u, v) f(u) \\ &= \textstyle\sum_{s,t \in T_n^\sigma} f(s) R_{n,\sigma}^{-1}(s, t) f(t), \end{aligned}$$

where the last step follows from Lemma 5 because, since $\sigma \leqq \tau$, $T_n^\sigma \subset T_n^\tau$ and since $t \in T_n^\sigma$, $t \in T_n^\tau$.

LEMMA 7. $\rho(\sigma, \tau) = \text{Cov}(Y(\sigma), Y(\tau)) = \phi(\min(\sigma, \tau))$ *for all* $\sigma, \tau \geqq 0$.

PROOF. Let $\sigma \leqq \tau$. Then

$$\begin{aligned}
&\mathrm{Cov}(Y(\sigma), Y(\tau))\\
&\quad = \mathrm{Cov}(\lim_{n\to\infty} \textstyle\sum_{s,t\in T_n{}^\sigma} f(s)R_{n,\sigma}^{-1}(s,t)X(t), \lim_{n\to\infty} \sum_{s,t\in T_n{}^\tau} f(s)R_{n,\tau}^{-1}(s,t)X(t))\\
&\quad = \lim_{n\to\infty} \mathrm{Cov}(\textstyle\sum_{s,t\in T_n{}^\sigma} f(s)R_{n,\sigma}^{-1}(s,t)X(t), \sum_{s,t\in T_n{}^\tau} f(s)R_{n,\tau}^{-1}(s,t)X(t))\\
&\quad = \lim_{n\to\infty} \textstyle\sum_{s,t\in T_n{}^\sigma} f(s)R_{n,\sigma}^{-1}(s,t)f(t)
\end{aligned}$$

by Lemma 6. To conclude the proof of the lemma we have only to note the well-known fact (see Parzen (1959)) that

$$\lim_{n\to\infty} \textstyle\sum_{s,t\in T_n{}^\sigma} f(s)R_{n,\sigma}^{-1}(s,t)f(t) = \|f^\sigma\|^2_{H(R^\sigma)} = \phi(\sigma).$$

LEMMA 8. *The process* $\{Y(t), t \geqq 0\}$ *is continuous in probability.*

PROOF. Follows from Lemma 7 and the continuity of ϕ.

LEMMA 9. *Almost all sample paths of the process* $\{Y(t), t\in T\}$ *are uniformly continuous on* $T \cap [0, \tau]$ *for every* τ.

PROOF. The proof of this lemma proceeds along the same line as that in Doob (1953), Chapter VIII, Theorem 2.2.

$$\begin{aligned}
&P[\text{l.u.b.}_{|t-j/N|<1/N, t\in T, 1\leqq j\leqq N^2} |Y(t,\omega) - Y(j/N,\omega)| \geqq N^{-\frac14}]\\
&\quad \leqq \textstyle\sum_{j=1}^{N^2} P[\text{l.u.b.}_{|t-j/N|\leqq 1/N, t\in T} |Y(t,\omega) - Y(j/N,\omega)| \geqq N^{-\frac14}]\\
&\quad \leqq 2\textstyle\sum_{j=1}^{N^2} P[Y((j+1)/N,\omega) - Y((j-1)/N,\omega) \geqq N^{-\frac14}]\\
&\quad = 2\textstyle\sum_{j=1}^{N^2} (2\pi)^{-\frac12}\{\phi((j+1)/N) - \phi((j-1)/N)\}^{-\frac12}\\
&\qquad \times \textstyle\int_{N^{-1/4}}^\infty \exp[-\xi^2/2\{\phi((j+1)/N) - \phi((j-1)/N)\}]\, d\xi\\
&\quad \leqq (2/\pi)^{\frac12} \textstyle\sum_{j=1}^{N^2} N^{\frac14}\{\phi((j+1)/N) - \phi((j-1)/N)\}^{\frac12}\\
&\qquad \times \exp[-\tfrac12 N^{-\frac14}\{\phi((j+1)/N) - \phi((j-1)/N)\}^{-1}]\\
&\quad \leqq C_1 N^{7/4} \exp[-C_2 N^{\frac12}]
\end{aligned}$$

where the last step in which C_1 and C_2 are positive constants follows from Lemma 1. Since

$$\textstyle\sum_{n=1}^\infty N^{7/4} \exp[-C_2 N^{\frac12}] < \infty,$$

an application of the Borel-Cantelli lemma concludes the proof.

PROOF OF THEOREM 1. Part (a) is merely a restatement of Lemmas 4, 7, 8 and 9. To prove part (b) consider those ω for which the sample paths of $\{Y(t,\omega), t\in T\}$ are uniformly continuous on $T \cap [0, \tau]$ for every τ and for these ω, let $\{\tilde{Y}(t,\omega), t \geqq 0\}$ be the unique continuous function which coincides with $Y(t,\omega)$ for all $t\in T$. For all other ω, which account for at most a set of probability measure 0, let $\tilde{Y}(t,\omega)$ be the zero function. It is well known (see e.g. Breiman (1968, Theorem 12.16)) that by virtue of Lemmas 8 and 9, for every t, $\tilde{Y}(t,\omega) = Y(t,\omega)$ a.s. In this way, the process $\{\tilde{Y}(t, \omega), t \geqq 0\}$ in which all sample functions are continuous, is probabilistically indistinguishable from the process $\{Y(t, \omega), t \geqq 0\}$. This concludes the proof of Theorem 1.

The Corollary follows immediately from the Theorem and its proof is omitted.

4. Converting $\{\bar{Y}(t), t \geqq 0\}$ to the Brownian motion. In this section we shall discuss two methods for transforming the process $\{\bar{Y}(t), t \geqq 0\}$ to the Brownian motion.

The first method is a well-known device (see Doob (1953, page 420)) in which the time-axis is transformed in such a way as to make the increments stationary. The actual description of this method is given in the following theorem.

THEOREM 2. *Let* $g(s) = \phi(s) - \phi(0)$, $s \geqq 0$. *Then g is a continuous* $1-1$ *mapping from* $[0, \infty)$ *to* $[0, \|f\|^2_{H(R)} - \phi(0))$, *and* $Z_1(t) = \bar{Y}(g^{-1}(t))$, $0 \leqq t < \|f\|^2_{H(R)} - \phi(0)$ *is a Brownian motion of which all sample paths are continuous.*

PROOF. That g is a continuous $1-1$ mapping follows from Lemma 2. The rest follows immediately from the corollary to Theorem 1.

Our second method is more direct giving rise to a Brownian motion on $[0, \infty)$. However, this method is applicable to a more restrictive class of Gaussian processes whose covariance kernel satisfies the following condition.

CONDITION 4. (a) For every $\tau \geqq 0$ and for $\min(s, t) > \tau$,

$$\psi'(s, t; \tau) = \lim_{\delta \to 0} \{\psi(s, t; \tau + \delta) - \psi(s, t; \tau)\}/\delta \tag{5}$$

and

$$\psi''(s, t; \tau) = \lim_{\delta \to 0} \{\psi'(s, t; \tau + \delta) = \psi'(s, t, \tau)\}/\delta \tag{6}$$

exist.

(b) For each τ, and for w satisfying requirements (i) and (iii),

$$\textstyle\int_\tau^\infty \int_\tau^\infty w(s) w(t) \psi'(s, t; \tau)\, ds\, dt > 0.$$

(c) The convergence in (6) is uniform in (s, t), the limiting function is a bounded function of (s, t), and for each (s, t), $\psi''(s, t; \tau)$ is continuous in τ.

The functions $\psi'(\cdot, \cdot, \tau)$ and $\psi''(\cdot, \cdot, \tau)$ on $(\tau, \infty) \times (\tau, \infty)$ extend in a natural manner to $[0, \infty) \times [0, \infty)$. This is because, for $\min(s, t) < \tau$ and for sufficiently small δ,

$$\psi(s, t; \tau + \delta) = \psi(s, t; \tau)$$

by virtue of (1), and

$$\lim_{\delta \to 0} \{\psi(s, t; \tau + \delta) - \psi(s, t; \tau)\}/\delta = 0.$$

However, for $\min(s, t) = \tau$, the above limit will not in general exist, but if we define $\psi'(s, t; \tau) = 0$ for $\min(s, t) \leqq \tau$, then

$$\psi'(s, t; \tau) = \lim_{\delta \to 0} \{\psi(s, t; \tau + \delta) - \psi(s, t; \tau)\}/\delta$$

holds except on a set of two-dimensional Lebesgue measure 0. Similarly, if we extend $\psi''(s, t; \tau) = 0$ for $\min(s, t) \leqq \tau$, then

$$\psi''(s,t;\tau) = \lim_{\delta\to 0}\{\psi'(s,t;\tau+\delta)-\psi'(s,t;\tau)\}/\delta$$

holds except on a set of two-dimensional Lebesgue measure 0. Furthermore, the convergence in the last expression is obviously still uniform in those (s, t) for which it holds.

LEMMA 10. *ϕ is twice differentiable; its first derivative ϕ' is everywhere positive and its second derivative ϕ'' is continuous.*

PROOF.

$$\{\phi(\tau+\delta)-\phi(\tau)\}/\delta = \int_0^\infty\int_0^\infty w(s)w(t)\,\delta^{-1}\{\psi(s,t;\tau+\delta)-\psi(s,t,\tau)\}\,ds\,dt.$$

Here the integrand is dominated by $w(s)w(t)\,K$ by Condition 2 and it tends to $w(s)w(t)\psi'(s,t;\tau)$ as $\delta \to 0$. It now follows from the Lebesgue dominated convergence theorem that the first derivative ϕ' of ϕ exists and indeed

$$\phi'(\tau) = \int_0^\infty\int_0^\infty w(s)w(t)\psi'(s,t;\tau)\,ds\,dt.$$

That ϕ' is positive now follows from Condition 4(b) and from requirements (i) and (iii) that w satisfies. We now turn to the differentiability of ϕ', but this again follows in the same way in view of Condition 4(a). The continuity of ϕ'' also follows routinely from Condition 4.

The following theorem now gives a method for transforming the Gaussian process $\{\bar{Y}(t), t \geqq 0\}$ to the Brownian motion on $[0, \infty)$. This construction, by its very definition, depends on Condition 4 for its applicability.

THEOREM 3. *Let*

$$Z_2(t,\omega) = \int_0^t \tfrac{1}{2}\phi''(s)\{\phi'(s)\}^{-\frac{3}{2}}\bar{Y}(s,\omega)\,ds+\{\phi'(t)\}^{-\frac{1}{2}}\bar{Y}(t,\omega), \qquad t \geqq 0.$$

Then $\{Z_2(t), t \geqq 0\}$ is a Brownian motion of which all sample functions are continuous.

REMARK. The integral in the definition of $Z_2(t, \omega)$ exists as a Riemann integral for every ω because all functions appearing in the context are continuous. The choice of $Z_2(t, \omega)$ is elucidated by formally integrating by parts as if $\bar{Y}(\cdot, \omega)$ were differentiable. Then

$$Z_2(t,\omega) = \int_0^t \{\phi'(s)\}^{-\frac{1}{2}}\bar{Y}'(s)\,ds.$$

PROOF. We have only to verify that the covariance kernel of $\{Z_2(t), t \geqq 0\}$ is of the desired form.

Now since $\{\bar{Y}(t), t \geqq 0\}$ is a process with independent increments, it is easily seen that $\{Z_2(t), t \geqq 0\}$ also has independent increments. It is, therefore, enough to verify that $\text{Var}[Z_2(\tau)] = \tau$ for all $\tau > 0$. Now

$$\text{Var}[Z_2(\tau)] = \int_0^\tau \int_0^\tau \frac{\phi''(s)\phi''(t)}{4\{\phi'(s)\phi'(t)\}^{\frac{3}{2}}}\{\phi(\min(s,t))-\phi(0)\}\,ds\,dt$$

$$+(\phi'(\tau))^{-\frac{1}{2}}\int_0^\tau \frac{\phi''(t)\{\phi(t)-\phi(0)\}}{\{\phi'(t)\}^{\frac{3}{2}}}dt+\frac{\phi(\tau)-\phi(0)}{\phi'(\tau)}.$$

Integrating by parts, the last expression can be shown to be equal to τ. This concludes the proof.

5. The boundary crossing behaviors of the X-process and the $\tilde{Y}$-process. Let b be a function in $H(R)$. If we replace X by b in (4), we get $\beta(\tau) = (f, b)_{H(R^\tau)}$. We now ask the question: If a sample path $X(t, \omega) \leqq b(t)$ for all $0 \leqq t \leqq \tau$, will its transform $\tilde{Y}(t, \omega) \leqq \beta(t)$ for all $0 \leqq t \leqq \tau$? In this section we shall prove that the answer to this question is in the affirmative with probability 1, provided that the covariance kernel of the process satisfies the following condition.

CONDITION 5. For any positive integer n and any distinct $t_1, \cdots, t_n, s \geqq 0$, the set of linear equations

$$\textstyle\sum_{i=1}^n c_i(s \mid t_1, \cdots, t_n)R(t_i, t_j) = R(s, t_j), \qquad = 1, \cdots, n$$

have solutions $c_i(s \mid t_1, \cdots, t_n) \geqq 0$.

REMARK. Condition 5 can be equivalently expressed as,

$$E[X(s) \mid X(t_1), \cdots, X(t_n)] = \textstyle\sum_{i=1}^n c_i(s \mid t_1, \cdots, t_n)X(t_i)$$

with nonnegative coefficients $c_i(s \mid t_1, \cdots, t_n)$. In this latter form, this condition may seem to be a reasonable one for many physical processes, at least over a short range.

LEMMA 11. *Let A be a symmetric nonsingular $n \times n$ matrix with inverse $A^{-1} = ((a^{uv}))$. If $f = Aw$ where $w = (w_1, \cdots, w_n)'$ with $w_u \geqq 0$ $u = 1, \cdots, n$, then $x_v \geqq 0$ for $v = 1, \cdots, n$ implies*

$$\textstyle\sum_{n=1}^n \sum_{v=1}^n f_u a^{uv} x_v \geqq 0.$$

PROOF. The proof follows from the identity

$$\textstyle\sum_{u=1}^n \sum_{v=1}^n f_u a^{uv} x_v = \sum_{u=1}^n w_u x_u$$

which is a consequence of Lemma 5.

LEMMA 12. *If $\{X(t), t \geqq 0\}$ be a real separable Gaussian process with mean value function 0 and covariance kernel R satisfying conditions 1–3 and 5, $\{Y(t), t \geqq 0\}$ is a process defined by (4) and if $b \in H(R)$, then*

$$P[\omega \mid X(t,\omega) \leqq b(t), 0 \leqq t \leqq \tau, Y(\tau,\omega) > \beta(\tau)] = 0 \tag{7}$$

where $\beta(\tau) = (f, b)_{H(R^\tau)}$.

PROOF. By virtue of Condition 5, the restriction of f to T_n^τ is a linear combination of the restriction to T_n^τ of $\{R_s, s \in T_n^\tau\}$ with all coefficients nonnegative. Now let

$$Y_n(\tau, \omega) = \sum_{s,t \in T_n^\tau} f(s) R_{n,\tau}^{-1}(s, t) X(t, \omega),$$
$$\beta_n(\tau) = \sum_{s,t \in T_n^\tau} f(s) R_{n,\tau}^{-1}(s, t) b(t).$$

Then

$$\beta_n(\tau) - Y_n(\tau, \omega) = \sum_{s,t \in T_n^\tau} f(s) R_{n,\tau}^{-1}(s, t)\{b(t) - X(t, \omega)\},$$

and due to the nature of f as observed above, Lemma 11 becomes applicable here, giving

$$\{\omega \mid X(t, \omega) \leqq b(t), 0 \leqq t \leqq \tau, Y_n(\tau, \omega) > \beta_n(\tau)\} = \varnothing$$

for each n. To conclude the proof we have only to recall that $\lim_{n\to\infty} Y_n(\tau, \omega) = Y(\tau, \omega)$ with probability 1 and $\lim_{n\to\infty} \beta_n(\tau) = \beta(\tau)$.
We now state and prove the main result of this section.

THEOREM 4. *Under the conditions of Lemma* 12,

$$P[\omega \mid X(t, \omega) \leqq b(t), 0 \leqq t \leqq \tau, (\tilde{Y}t, \omega) > \beta(t), \quad some \quad 0 \leqq t \leqq \tau] = 0.$$

PROOF. The proof follows from the fact that all sample paths of the $\tilde{Y}$-process are continuous and by substituting rationals in $[0, \tau]$ for τ in (7).

COROLLARY. *Under the conditions of Lemma* 12, *the following holds for the Brownian motion* $\{Z_1(t), 0 \leqq t \leqq \|f\|^2_{H(R)} - \phi(0)\}$. *For every* $0 < \tau < \|f\|^2_{H(R)} - \phi(0)$,

$$P[\omega \mid X(t, \omega) \leqq b(t), 0 \leqq t \leqq \tau, Z_1(t) > \beta(g^{-1}(t)) - \tilde{Y}(0), \quad some \quad 0 \leqq t \leqq \tau] = 0.$$

REMARK. The behavior of $\{Z_1(t), 0 \leqq t < \|f\|^2_{H(R)} - \phi(0)\}$ given in the above corollary is not in general shared by $\{Z_2(t), 0 \leqq t < \infty\}$.

Theorem 4 and its corollary have interesting implications in boundary-crossing problems in Gaussian processes which have engaged a considerable amount of attention during the last decade. The references to some of the basic work in this field can be found in Cramér and Leadbetter (1967). Here, we merely note that by using Theorem 4 or its corollary we can obtain bounds for probabilities of a large class of boundary-crossing events in Gaussian processes in terms of the probabilities of boundary-crossing events in Gaussian processes with independent increments or in the Brownian motion.

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