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A LOCALLY MOST POWERFUL TIED RANK TEST IN A WILCOXON SITUATION[1]

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1. Introduction. Various ways of treating ties have been suggested and investigated, but little is known about tied rank tests which are optimal against specific alternatives. To my knowledge the only results in this field were achieved by Vorlíčková (1970). The author makes use of the concept of contiguity and generalizes the corresponding theorems of Hájek, Šidák (1967) to the case of rv's which take the values $k = 0, \pm 1, \pm 2, \cdots$ only. She proves that tests based on linear rank statistics are asymptotically most powerful against suitably chosen alternatives, if the method of averaged scores is used. Considering a discrete analogue of the logistic distribution one may derive in this way the two-sample Wilcoxon midrank test as an asymptotically most powerful test.

This paper presents a general framework for constructing optimal tied rank tests in the two-sample problem.[3] In particular, we will consider a class of discrete alternatives, analogous to some continuous alternatives against which the Wilcoxon test is locally most powerful, and construct a corresponding locally most powerful tied rank test. The test statistic is the sum of ranks obtained by ranking the distinct values in the pooled sample.

2. A general procedure for obtaining optimal tied rank tests. We assume that all occuring discrete distributions are lattice distributions on a real lattice

$$M = M(\xi_0, \delta) = \{\xi_k : \xi_k = \xi_0 + \delta k, k = 0, \pm 1, \pm 2, \cdots\}, \qquad \delta > 0,$$

or on a set M' derived from M by applying any continuous and strictly increasing transformation of the real line onto itself respectively.

Considering $X_{11}, \cdots, X_{1n_1}$ independent rv's with df F_1, and $X_{21}, \cdots, X_{2n_2}$ with F_2, we want to test $F_1 = F_2$ against $F_1 \leqq F_2$. ($F_1 \leqq F_2$ means, that $F_1(x) \leqq F_2(x)$ holds for all real x and this strictly for at least one x.) We get in the usual way $R(x) = (R_1(x), R_2(x)) = (r_{11}, \cdots, r_{1n_1}, r_{21}, \cdots, r_{2n_2})$ as a maximal invariant statistic relative to the group of the above mentioned transformations. The ranks are defined as follows: In the pooled sample $x = (x_{11}, \cdots, x_{2n_2})$ tied observations are regarded as one observation. Let T denote the size of the reduced sample x'. For x'

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[3] In Krauth (1969) one-sided permutation and rank tests for the matched pairs problem, and the problem of independence are similarly derived.

the ranks $i = 1, \cdots, T$ are defined as usual. These ranks then are also used for the original sample. In particular, equal ranks are assigned to equal observations. Besides these ranks, those ranks can be used which we get by applying a strictly increasing transformation to $i = 1, \cdots, T$. By this we cover e.g. midranks, but not ranks that are assigned at random.

Depending on circumstances it might be sensible to restrict oneself to a fixed lattice M. In this case R is a maximal invariant too, if we consider those transformations which stretch the distances between adjacent observations in a finite extent, and define invariance in an appropriate way (cf. Krauth (1969)).

To characterize a special test among the invariant tests we follow the proof of the corresponding theorem for continuous distributions (cf. e.g. Lehmann (1953)).

We get each unbiased test for H against K to be similar on the boundary J (of H and K). $J = \{(F_1, F_2): F_1 = F_2\}$ for the problem under consideration. We determine the rank test of level α which is optimal against a specific alternative $K_\Delta = (F_\Delta, F_0)$ given by a one-parameter subclass of distributions $P_\Delta (0 \leqq \Delta \leqq \Delta_1)$ dominated by P_0. Then a solution ψ_Δ^*, $0 < \Delta < \Delta_1$, of

(a) $$E_0\psi = \alpha, \quad \text{and}$$

(b) $$E_\Delta\psi = \sup\{E_\Delta\psi' : E_0\psi' = \alpha\} \qquad (0 < \Delta < \Delta_1)$$

is the best level α rank test against the alternative K_Δ.

In contrast to the case of continuous distributions condition (a) is not equivalent to $E_\vartheta\psi = \alpha$ for all $\vartheta \in J$, as the distribution of the rank statistic R depends on the underlying distributions even for $\vartheta \in J$. In other words, we have no distribution-free test. The reduction by invariance in this case does not yield a similar test.

Let B denote in the following both a configuration of ties and the set of all points of the sample space which satisfy B. Two points $x, x' \in M^n$ hence belong to the same B, if in $x_{[\,]}$ (i.e. the ordered pooled sample) and in $x'_{[\,]}$ equal components of the same number stand at the same place. For example, the samples (3, 5, 7, 2, 5, 5, 7) and (30, 52, 30, 52, 1, 30, 9) belong to the same B of the type (1, 2, 3, 3, 3, 4, 4).

The conditional distribution of R given B is distribution-free under the hypothesis, and can therefore be used to construct a distribution-free conditional test. In fact for all $\vartheta \in J$ the distribution of the statistic R given B is the discrete uniform distribution over the $n!/(t_1! \cdots t_T!)$ possible rankings r. By $t_1, \cdots, t_T$ we denote the lengths of the ties in $x_{[\,]}$ including ties of lengths 1.

Then ψ_Δ^* is determined from

(a′) $$E_0(\psi \mid B) = \alpha$$

(b′) $$E_\Delta(\psi \mid B) = \sup\{E_\Delta(\psi' \mid B) : E_0(\psi' \mid B) = \alpha\} \qquad (0 < \Delta < \Delta_1).$$

For simplicity in the following a conditional expectation $E(Y \mid A)$ equals 0 when $P(A) = 0$. Without loss of generality we may exclude the case $P_\Delta(B) = 0$. From

$P_\Delta(B) > 0$ we get $P_0(B) > 0$, as P_Δ is dominated by P_0. Denote by $t_{i1}, \cdots, t_{iT_i}$ the lengths of the ties in $r_{i[\,]}$, $i = 1, 2$, and let

$$m(r, B) = \frac{n_1!}{t_{11}! \cdots t_{1T_1}!} \frac{n_2!}{t_{21}! \cdots t_{2T_2}!}.$$

Obviously we have

$$\sum_{r'} m(r', B) = \frac{n!}{t_1! \cdots t_T!}.$$

Now let $x = (x_{1[\,]}, x_{2[\,]})$, $R^*(x) = (R_1(x_{1[\,]}), R_2(x_{2[\,]}))$, f_Δ or f_0 the (discrete) density of F_1 or F_2 respectively. It suffices to compute the probabilities $P_\Delta(R^* = r^* \mid B)$ instead of $P_\Delta(R = r|B)$. In the following we omit the asterisk for simplicity.

Since $P_0(R = r|B) = m(r, B)/\sum_{r'} m(r', B)$, and

$$P_\Delta(R = r) = \frac{m(r, B)}{P_\Delta(B)} \sum_{x \in B; R(x) = r} \prod_{i=1}^{n_1} f_\Delta(x_{1[i]}) \prod_{j=1}^{n_2} f_0(x_{2[j]}),$$

we have

$$P_\Delta(R = r \mid B) = \frac{P_0(B)}{P_\Delta(B)} P_0(R = r \mid B) \pi(r, \Delta),$$

with

$$\pi(r, \Delta) =_{\mathrm{df}} E_0\left(\prod_{i=1}^{n_1} \frac{f_\Delta(X_{1[i]})}{f_0(X_{1[i]})} \middle| r \right).$$

We may interpret the above relation as a discrete analogue to Hoeffding's Formula.

According to the Neyman–Pearson Lemma the critical region of $\psi_\Delta{}^*$ is given by $\pi(r, \Delta) > \text{crit}\,(\Delta, B)$, i.e., consists of those $[\alpha\,(n!)/(t_1! \cdots t_T!)]$ points to which belong the largest values of $\pi(r, \Delta)$. If those points are not uniquely determined, we have to randomize in the usual way.

On condition that $\pi(r, \Delta)$ can be expanded into a power series about $\Delta = 0$ for each r (for the cases considered this is fulfilled), just as in the continuous case there exists (cf. Witting, Nölle (1970), Theorem 3.16) for each B a $\Delta^*(B) > 0$ such that the test $\psi_\Delta{}^*$ under B is a uniformly most powerful rank test of size α for testing $\{P_\vartheta : \vartheta \in J\}$ against $\{P_\Delta : 0 < \Delta < \Delta^*(B)\}$. That is, we have a locally uniformly most powerful rank test under the condition B.

Finally, conditional local optimality given B implies unconditional local optimality because (for given sample size n) the number of different B's is finite.

3. Locally linear alternatives. In contrast to Section 1 and Section 2 we shall now consider continuous alternatives (F_Δ, F_0) with densities $f_\Delta(x) = \kappa(x, \Delta)/(b-a)$

for $a \leqq x \leqq b$, $=0$ otherwise, a, b real numbers, $a < b$, $0 \leqq \Delta < \Delta_1$. In regard to the function $\kappa(x, \Delta)$, $0 \leqq \Delta < \Delta_1$, we require

$$\text{(1)} \qquad \int_a^b \kappa(x,\Delta)\,dx = b-a$$

$$\text{(2)} \qquad \kappa(x,\Delta) \geqq 0 \quad \text{for} \quad a \leqq x \leqq b$$

$$\text{(3)} \quad \kappa(x,0) = 1 \text{ a.e., if } a \leqq x \leqq b; \qquad \kappa(x,0) = 0, \text{ if } x < a \text{ or } x > b.$$

$$\text{(4)} \qquad \int_a^z \kappa(x,b)\,dx \leqq z-a \quad \text{for} \quad a < z \leqq b, \qquad 0 < \Delta < \Delta_1.$$

Let $\kappa(x, b)$ have an expansion into a power series about $\Delta = 0$ for $a \leqq x \leqq b$.

$$\text{(5)} \qquad \kappa'(x) \equiv \frac{\partial}{\partial\Delta}\kappa(x,\Delta)\,|_{\Delta=0} = A_1\frac{x-a}{b-a} + A_2 \quad \text{for} \quad a < x \leqq b, \qquad A_1 > 0.$$

By (1)—(4) f_Δ is a probability density, F_0 is the df of the uniform distribution over $a \leqq x \leqq b$ and P_0 dominates $\{P_\Delta : 0 < \Delta < \Delta_1\}$. For the distribution functions we have $F_\Delta(x) \leqq F_0(x)$, i.e., $(F_\Delta, F_0) \in K$ for $0 < \Delta < \Delta_1$. In view of condition (5), we use the term "locally linear alternatives."

EXAMPLES. a. Let $\kappa(x, \Delta) = \Delta(2x-a-b)+1$ for $a \leqq x \leqq b$, $=0$ otherwise, $\Delta \leqq 1/(b-a)$. Then (1)—(5) are valid.

b. If F_0 is the distribution function of the uniform distribution over $a \leqq x \leqq b$, we get for the nonparametric alternatives with

$$F_\Delta(x) = (1-\Delta)F_0(x) + \Delta F_0^{\,2}(x) \quad \text{and} \quad f_\Delta(x) = (1-\Delta)f_0(x) + 2\Delta f_0(x)F_0(x):$$

$$\kappa(x,\Delta) = \Delta\left(2\frac{x-a}{b-a} - 1\right) + 1 \quad \text{for} \quad a \leqq x \leqq b, = 0 \text{ otherwise.}$$

For $\Delta \leqq 1$ conditions (1)—(5) are valid.

For the particular case of Example b, Lehmann ((1953), page 34) proved that the Wilcoxon test is locally most powerful. Because of $F_0(x) = (x-a)/(b-a)$, Lehmann's proof may easily be extended to the general case.

A discrete analogue to the locally linear alternatives introduced above may be defined as follows.

Consider $N \geqq 2$ points $\eta_1, \cdots, \eta_N \in M(\eta_1 < \cdots < \eta_N)$ and functions $f_\Delta(\eta_i) = \kappa(\eta_i, \Delta)/N$, $i = 1, \cdots, N$, for $0 \leqq \Delta < \Delta_1$ with

$$\text{(1')} \qquad \sum_{i=1}^N \kappa(\eta_i,\Delta) = N$$

$$\text{(2')} \qquad \kappa(\eta_i,\Delta) \geqq 0, \qquad i = 1, \cdots, N,$$

$$\text{(3')} \qquad \kappa(\eta_i,0) = 1, \qquad i = 1, \cdots, N,$$

$$\text{(4')} \qquad \sum_{i=1}^k \kappa(\eta_i,\Delta) \leqq k \qquad (\Delta > 0), k = 1, \cdots, N.$$

$\kappa(\eta_i, \Delta)$ may permit an expansion into a power series about $\Delta = 0$, $i = 1, \cdots, N$.

$$\kappa'(\eta_i) \equiv \frac{\partial}{\partial \Delta} \kappa(\eta_i, \Delta) \big|_{\Delta=0} = Di + E, \qquad D > 0, i = 1, \cdots, N. \tag{5'}$$

By (1')–(4') f_Δ is a probability density, f_0 corresponds to the discrete uniform distribution over $\eta_1, \cdots, \eta_N$, and P_0 dominates $\{P_\Delta : 0 < \Delta < \Delta_1\}$. For the distribution functions F_0, F_Δ we have $F_\Delta(x) \leqq F_0(x)$, i.e., $(F_\Delta, F_0) \in K$ for $0 < \Delta < \Delta_1$.

EXAMPLES. a'. Let $\kappa(\eta_i, \Delta) = 1 + \Delta(2i - N - 1)\delta$, $i = 1, \cdots, N$, $\Delta \leqq 1/\delta(N-1)$. Then (1')—(5') are valid.

b'. For the nonparametric alternatives with $F_\Delta(x) = (1-\Delta)F_0(x) + \Delta F_0{}^2(x)$ we get

$$f_\Delta(\eta_k) = (1-\Delta) f_0(\eta_k) + \Delta f_0(\eta_k)(2F_0(\eta_k) - f_0(\eta_k)), \qquad k = 1, \cdots, N.$$

If F_0 corresponds to the discrete uniform distribution, i.e., $F_0(\eta_i) = i/N$, $i = 1, \cdots, N$, we get $\kappa(\eta_i, \Delta) = 1 + \Delta(2i - 1 - N)/N$, $i = 1, \cdots, N$. For $\Delta \leqq N/(N-1)$(1')—(5') are valid.

In Krauth (1969) it has been shown that discrete locally linear alternatives converge to continuous ones under appropriate conditions. Since Wilcoxon is locally most powerful against continuous locally linear alternatives, the test proposed in Theorem 1 (cf. Section 4) seems to be a reasonable extension of Wilcoxon for tied ranks.

4. Main result. This section contains the following theorem and its proof.

THEOREM 1. *Let r_{1i} $(1 \leqq r_{1i} \leqq T)$ be the rank of X_{1i} as defined by ranking the T distinct values in the pooled sample, i.e., $r_{1i} = k$ if X_{1i} is equal to the* kth *smallest distinct value. The conditional (given B) test based on $\sum_{i=1}^{n_1} r_{1i}$ is the locally most powerful test against the class of locally linear alternatives.*

PROOF. The locally most powerful test is based on the statistic

$$\pi'(r) = \frac{\partial}{\partial \Delta} \pi(r, \Delta) \big|_{\Delta=0} = \sum_{i=1}^{n_1} E_0\left(\frac{\partial}{\partial \Delta} f_\Delta(X_{1[i]}) \big|_{\Delta=0} / f_0(X_{1[i]}) \,\big|\, r \right),$$

which is difficult to compute for most discrete distributions. The calculation will be simplified if we consider the discrete locally linear alternatives, introduced in Section 3.

Under these alternatives we have

$$\pi'(r) = \sum_{i=1}^{n_1} E_0(\kappa'(X_{1[i]}) \,|\, r) = \frac{1}{m_1(N, B)} \sum_{i=1}^{n_1} \sum_{x \in B; R(x) = r} \kappa'(x_{1[i]})$$

with a positive constant $m_1(N, B)$. If $r_{1i} = k$, then $x_{1[i]} = x_{\{k\}}$, where $x_{\{k\}}$ denotes the kth smallest distinct observation in the pooled sample. $x_{\{k\}}$ can take the values η_{k+s} $(s = 0, \cdots, N-T)$, and in the sum over $R(x) = r$ for fixed i (in the preceding formula) it will take the value η_{k+s} exactly $\binom{k+s-1}{k-1}\binom{N-k-s}{T-k}$ times. (Each term of the sum corresponds to a choice of the values of $x_{\{1\}}, \cdots, x_{\{T\}}$, and there are $\binom{k+s-1}{k-1}$ ways of choosing $(x_{\{1\}}, \cdots, x_{\{k-1\}})$, and $\binom{N-k-s}{T-k}$ ways of choosing $(x_{\{k+1\}}, \cdots, x_{\{T\}})$.) Hence, the sum over $R(x) = r$ is

$$C(k) = \sum_{s=0}^{N-T} \kappa'(\eta_{k+s})\binom{N-k-s}{T-k}\binom{k+s-1}{k-1} = k \sum_{s=0}^{N-T} \frac{\kappa'(\eta_{k+s})}{k+s}\binom{N-k-s}{T-k}\binom{k+s}{k}.$$

Under condition (5′), using the identity

$$\sum_{i=0}^{a-b}\binom{a-i}{b}\binom{c+i}{c} = \binom{a+c+1}{b+c+1} \quad (a, b, c \text{ nonnegative integers, } a \geqq b)$$

(cf. Netto (1927), page 15), we have $C(k) = D\binom{N+1}{T+1}k + E$, and hence

$$\pi'(r) = \frac{D}{m_1(N, B)}\binom{N+1}{T+1}\sum_{j=1}^{n_1} r_{1j} + n_1 E$$

or equivalently $\pi_1'(r) = \sum_{j=1}^{n_1} r_{1j}$. For fixed B we have to order the rankings $(r_{1[\,]}, r_{2[\,]})$ corresponding to R given B according to decreasing $\pi_1'(r)$ and have to consider that discrete distribution, which assigns to the point $(r_{1[\,]}, r_{2[\,]})$ the probability

$$m(r, B)/\sum_{r'} m(r', B) = \binom{t_1}{t_{11}} \cdots \binom{t_T}{t_{1T_1}}/\binom{n}{n_1}.$$

An equivalent test is obtained by regarding equal ranks formally as different and considering the discrete uniform distribution over all $\binom{n}{n_1}$ rankings.

Finally it should be noted that the test has been proved to be unbiased only for restricted hypotheses given by a relation connecting F_1 and F_2 involving monotone relative densities.

5. Asymptotic results. When applying the test of Theorem 1 the computing effort with increasing sample size n grows so quickly that in practice the test becomes useless. We therefore need the asymptotic distribution of the test statistic on the boundary J. We now present a conditional asymptotic distribution theorem, of a type similar to the theorem of Kruskal ((1952), page 538). We get the following result:

THEOREM 2.

$$\left(\left[\sum_{i=1}^{n_1} r_{1i} - \frac{n_1}{n}\sum_{i=1}^{T(n)} i t_i(n)\right] \Big/ \left[\frac{n_1 n_2}{n(n-1)}\sum_{j=1}^{T(n)} t_j(n)\left(j - \frac{1}{n}\sum_{k=1}^{T(n)} k t_k(n)\right)^2\right]^{\frac{1}{2}} \,\Big|\, B(n)\right) \to_L N(0, 1)$$

for $1/n_1 + 1/n_2 \to 0$, *if one of the following two conditions for* $B(n)$ *holds true*:

1. $2 \leqq T(n) \leqq T_0$ *for natural* T_0 *and all* $n > 1$, *and there are* $j, k \in \{1, \cdots, T_0\}$, $j \neq k$, *such that* $\liminf t_j(n)/n > 0$, $\liminf t_k(n)/n > 0$.

2. $\liminf T(n)/n > 0$.

The approximation is considerably improved by a continuity correction.

The proof of Theorem 2 is based on a result of Hoeffding ((1951), Theorem 4).

For either of the two conditions mentioned above we can prove the sufficient condition (13) of Hoeffding's paper to hold true, if we make $r = 3$. The details of the proof are given in Krauth (1969).

Looking at ties as random variables, i.e. considering the unconditional test, we may prove consistency for restricted hypotheses, which are defined by distributions with positive probability only on a finite set. By a comparison with uniformly most powerful tests in exponential families we get the test to be asymptotically uniformly most powerful (cf. Krauth (1969), Chapter 7). The proofs run along the lines of Putter's paper (1955), for which we consider the statistic

$$S_{n_1 n_2} = \sum_{i=1}^{n_1} R_{1i} = \sum_{k=1}^{N} \operatorname{sgn}(U_k + V_k) \sum_{j=k}^{N} U_j$$

or more precisely the asymptotically equivalent statistic

$$S_{n_1} = \sum_{k=1}^{N} kU_k.$$

In addition one can consider a corresponding asymptotically distribution-free rank test (in analogy to Putter (1955), Chanda (1963) and Bühler (1967)) based on the statistic

$$T_{n_1 n_2} = \left(\frac{n_1 n_2}{n}\right)^{\frac{1}{2}} \left(\frac{1}{n_1}\sum R_{1i} - \frac{1}{n_2}\sum R_{2j}\right) \Big/ Q_n,$$

where Q_n is a suitable random factor.

In either case we get the asymptotic optimality property by considering binomial alternatives

$$f_\Delta(i) = \binom{N-1}{i-1}(p+\Delta)^{i-1}(1-p-\Delta)^{N-i}, \qquad i = 1, \cdots, N,$$

$0 \leqq \Delta < 1-p$, $0 < p < 1$. This is interesting when compared with the result of Vorlíčková (1970) for the midrank test as mentioned in Section 1.

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