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# A NOTE ON COMPARISONS OF MARKOV PROCESSES

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This note contains a simple proof of the following theorem of G. I. Kalmykov. Let $\{X_n\}$ and $\{Y_n\}$ be real-valued, discrete time Markov processes. Suppose $P(X_0 \leqq z) \leqq P(Y_0 \leqq z)$ for all real $z$ and

$$P(X_n \leqq z \mid X_{n-1} = x) \leqq P(Y_n \leqq z \mid Y_{n-1} = y)$$

for $n = 1, 2, \cdots$ and all $z$, whenever $y \leqq x$. Then $P(X_n \leqq z) \leqq P(Y_n \leqq z)$ for all $n$ and $z$. Some converse results are also given.

**1. Introduction.** We consider a discrete time real-valued Markov process $X_0, X_1, \cdots$, defined on a probability space $\Omega$, with initial distribution function $F$ and *transition function* $p: R \times R \to [0, 1]$ defined by

$$p(x, y) = P(X_n \leqq y \mid X_{n-1} = x)\,.$$

Although we assume such a $p$ exists *independently of* $n$, the results of this note may easily be extended to the more general situation. We denote such a system by $(\{X_n\}, \Omega, p, F)$.

The primary purpose of this note is to give a simple proof of Theorem 1, which was first proved by Kalmykov [2] by methods of functional analysis.

THEOREM 1. *Let* $(\{X_n\}, \Omega, p, F)$ *and* $(\{Y_n\}, \Omega', q, G)$ *be Markov processes (as described above). Suppose* $F(z) \leqq G(z)$ *for all* $z \in R$ *and* $p(x, z) \leqq q(y, z)$ *for all* $(x, y, z) \in R^3$ *for which* $y \leqq x$. *Then* $P(X_n \leqq z) \leqq P(Y_n \leqq z)$ *for all* $n \in N = \{1, 2, \cdots\}$ *and all* $z \in R$.

The proof is given in Section 2. In Section 3, we discuss the connection between this proof and a class of processes studied by Daley [1]. Section 4 contains some converse results.

The analytic proof of Theorem 1 first appeared in [3]. A greatly expanded study of the comparison method, using different methods, will appear later.

**2. Proof of Kalmykov's comparison theorem.** We begin with the following lemma, whose proof we omit.

LEMMA. *Let* $f: R \to [0, 1]$ *be a decreasing function. Then there is a sequence of continuous decreasing functions* $\{f_k: R \to [0, 1], k = 1, 2, \cdots\}$ *which converge pointwise to* $f$.

PROOF OF THEOREM 1. We proceed by induction on $n$. The result holds for $n = 0$; now assume it for arbitrary $n$. Let $H$ and $J$ be the distribution functions

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of $X_n$ and $Y_n$ respectively. We define a new transition function $f$ by $f(x, z) = \inf_{y \leqq x} q(y, z)$ for all $(x, z) \in R^2$. Note that $f\colon R^2 \to [0, 1]$ is a decreasing function of $x$ for each $z$ and satisfies $p(x, z) \leqq f(x, z) \leqq q(x, z)$ for all $(x, z) \in R^2$. Fix $z \in R$. There is a sequence $\{f_k\colon R \to [0, 1], k \in N\}$ of continuous decreasing functions which converge pointwise to $f(\cdot, z)$. For any $k \in N$, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} f_k(x)\, dH(x) &= [f_k(x)H(x)]\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x)\, df_k(x) \\ &\leqq [f_k(x)J(x)]\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} J(x)\, df_k(x) \\ &= \int_{-\infty}^{\infty} f_k(x)\, dJ(x)\,.\end{aligned}$$

Applying the dominated convergence theorem, we conclude:

$$\begin{aligned}P(X_{n+1} \leqq z) &= \int_{-\infty}^{\infty} p(x, z)\, dH(x) \\ &\leqq \int_{-\infty}^{\infty} f(x, z)\, dH(x) \\ &\leqq \int_{-\infty}^{\infty} f(x, z)\, dJ(x) \\ &\leqq \int_{-\infty}^{\infty} q(x, z)\, dJ(x) \\ &= P(Y_{n+1} \leqq z)\,.\end{aligned}$$

This completes the proof.

DEFINITION. A Markov process $(\{X_n\}, \Omega, p, F)$ is said to *live on* a Borel set $A$ if $\int_A dF(z) = 1$ and $\int_A p(x, dz) = 1$ for all $x \in A$.

We wish to extend Theorem 1 to processes that live on some subsets of $R$, say on intervals or the integers. We have the following extension.

Let $A$ and $B$ be Borel subsets of $R$. Let $(\{X_n\}, \Omega, p, F)$ live on $A$ and let $(\{Y_n\}, \Omega', q, G)$ live on $B$. Suppose $F(z) \leqq G(z)$ for all $z \in R$ and $p(x, z) \leqq q(y, z)$ for all $(x, y, z) \in A \times B \times R$ for which $y \leqq x$. Then $P(X_n \leqq z) \leqq P(Y_n \leqq z)$ for all $z \in R$ and for $n = 0, 1, 2, \cdots$.

We extend the proof of Theorem 1 as follows: define $f(x, z)$ as the supremum over all $y \geqq x$ such that $y \in A$ of $p(y, z)$ (or 0 if there are no such $y$). Then $p(x, z) \leqq f(x, z)$ for all $(x, z) \in A \times R$ and $f(y, z) \leqq q(y, z)$ for all $(y, z) \in B \times R$. The remainder of the proof easily goes through.

## 3. Stochastically monotonic processes.

DEFINITION. A Markov transition function $p$ is said to be *stochastically monotonic* if $p(x, z)$ is a decreasing function of $x$ for each $z$. A Markov process $\{X_n\}$ is stochastically monotonic if it has such a transition function.

These processes were studied by Daley [1]. As we shall see, they are closely related to the proof of Theorem 1. The following examples are included to indicate the size of this class of processes.

EXAMPLES. Let $\{X_n\}$ be an "independent trials" process. That means its transition function $p$ satisfies $p(x, y) = F(y)$ for some distribution function $F$. It is clear that $\{X_n\}$ is stochastically monotonic.

Now let $(\{X_n\}, \Omega, p, G)$ satisfy $p(x, z) = F(z - x)$. $\{X_n\}$ is a sum of independent random variables processes. If $y \leqq x$, we have $p(x, z) = F(z - x) \leqq F(z - y) = p(y, z)$, so that $p$ is stochastically monotonic.

Let $p$ and $q$ be Markov transition functions which satisfy $p(x, z) \leqq q(y, z)$ for all $(x, y, z) \in R^3$ for which $y \leqq x$.

Define two new transition functions by

$$f(x, z) = \inf_{y \leqq x} q(y, z)$$

and

$$g(x, z) = \inf_{v > z} \sup_{y \geqq x} p(y, v) .$$

Note that $f$ and $g$ are stochastically monotonic. Since $f$ is the greatest such transition function which is majorized by $q$, and since $q$ majorizes $g$, we have $p \leqq g \leqq f \leqq q$.

The proof of Theorem 1 depended on finding a stochastically monotonic transition function $f$ such that $p \leqq f \leqq q$. This permitted us to integrate by parts.

Daley [1] proved a weaker version of the theorem by assuming that one of $p$ and $q$ was stochastically monotonic, in which case $p$ and $q$ need only be compared at the same point $(x, z)$. We give his result as a corollary.

COROLLARY. *Let* $(\{X_n\}, \Omega, p, F)$ *be a stochastically monotonic Markov process. Let* $(\{Y_n\}, \Omega', q, G)$ *be any Markov process. If* $p \leqq q$ *and* $F \leqq G$, *then* $P(X_n \leqq z) \leqq P(Y_n \leqq z)$ *for all* $z \in R$ *and* $n = 0, 1, 2, \cdots$. *If* $p \geqq q$ *and* $F \geqq G$, *then* $P(Y_n \leqq z) \leqq P(X_n \leqq z)$ *for all* $z \in R$ *and* $n = 0, 1, 2, \cdots$.

PROOF. In the first case, $p(x, z) \leqq q(y, z)$ for all $(x, y, z) \in R^3$ for which $y \leqq x$. In the second case $q(x, z) \leqq p(y, z)$ for all such $(x, y, z)$. Thus we may apply the theorem.

**4. Converse results.** This section concerns the extent to which the hypotheses of Theorem 1 and its corollary are necessary.

Let us first consider the theorem. It is clearly false if we omit the hypothesis that $p(x, z) \leqq q(y, z)$ whenever $y \leqq x$. In fact, suppose $p(x, z) > q(y, z)$ for some $(x, y, z) \in R^3$ with $y \leqq x$. Suppose $X_0 = x$ a.s. and $Y_0 = y$ a.s. Then $P(X_1 \leqq z) > P(Y_1 \leqq z)$.

The problem remains interesting if we change the hypothesis $F(z) \leqq G(z)$ for all $z \in R$ to $F(z) = G(z)$ for all $z \in R$. In this case, the above example only works if $p(x, z) > q(x, z)$ for some $(x, z) \in R^2$. When $p(x, z) = q(x, z)$ for all $(x, z) \in R^2$, but $p(x, z) < q(y, z)$ for some $(x, y, z)$ with $y < x$, the answer is not so clear.

We give one result in the form of a partial converse to the corollary of the last section. It should give some indication of what may be achieved by way of a converse to the theorem when the initial distributions coincide.

THEOREM 2. *Let $p$ be a Markov transition function which has the following property: there exist points $x_0$, $x_1$, $x_2$ and $y \in R$ and a Borel set $A$ in $R$ such that*

(i) $x \in A$ *implies* $x_2 < x \leqq x_1$

(ii) $x \in A$ *implies* $p(x_2, y) < p(x, y)$

(iii) $\int_A p(x_0, dz) > 0$

(iv) $y \notin [x_2, x_1)$ *or* $x_0 \in A$ *or* $(x_0 \neq x_2$ *and* $\int_{\{x_0\}} p(x_0, dz) = 0)$.

*Then there exist a distribution function $F$ and a transition function $q \geqq p$ such that for any Markov processes $(\{X_n\}, \Omega, p, F)$ and $(\{Y_n\}, \Omega', q, F)$, we have $P(X_2 \leqq y) > P(Y_2 \leqq y)$.*

PROOF. Let $F(z) = 0$ for $z < x_0$ and let $F(z) = 1$ for $z \geqq x_0$. Let $(\{X_n\}, \Omega, p, F)$ be a Markov process. Define the transition function $q$ by:

$$
\begin{aligned}
q(x, z) &= p(x, z)\,, && \text{if } x \neq x_0 \text{ or } z < x_2 \\
&= p(x_0, z) + P(X_1 \in A \cap (z, \infty))\,, && \text{otherwise.}
\end{aligned}
$$

Let $(\{Y_n\}, \Omega', q, F)$ be a Markov process. Note that $\int_A q(x_0, dz) = 0$. We have

$$
\begin{aligned}
P(Y_2 \leqq y) &= \int_R q(z, y) q(x_0, dz) \\
&= \int_{R-A} q(z, y) q(x_0, dz) \\
&= \int_{R-A} q(z, y) p(x_0, dz) + \int_A q(x_2, y) p(x_0, dz) \\
&= \int_{R-A} p(z, y) p(x_0, dz) + \int_A p(x_2, y) p(x_0, dz) && \text{(by (iv))} \\
&< \int_R p(z, y) p(x_0, dz) && \text{(by (ii) and (iii))} \\
&= P(X_2 \leqq y)\,.
\end{aligned}
$$

Several remarks may be made. Conditions (i) and (ii) and the condition that $A$ is nonempty are together the negation of the statement that $p$ is stochastically monotonic. It is clear that this negation cannot be enough; we need some condition such as (iii). Condition (iv) is purely technical. It should be possible to relax (iii) and (iv) by comparing $P(X_n \leqq y)$ and $P(Y_n \leqq y)$ for $n$ larger than 2.

Daley [1] gave the above result for processes on the integers, assuming $A = \{x_0\}$ instead of Condition (iv).

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