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LIMITING PROPERTIES OF THE MEAN RESIDUAL LIFETIME FUNCTION[1]

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Let X be a positive random variable, with finite mean and infinite essential supremum.

(1) Define, for $0 \leqq x < \infty$, $\quad g(x) = E[X - x \mid X > x]$.

g is called the mean residual lifetime function. We will show that under some general conditions, the conditional distribution of $(X - x)/g(x)$ given that $X > x$ converges as $x \to \infty$ to the standard exponential distribution. These conditions are satisfied, for instance, when $g(x)$ is non-decreasing, while $g(x)/x$ decreases to zero as $x \to \infty$. This will somewhat generalize results in [1]. The case $g(x)/x \to c \in (0, \infty)$ is treated by Feller in [2], page 272. In that case a limiting distribution exists, but is not exponential. The author, ([3]) used these convergence theorems to obtain good bargaining solutions when selling one expensive asset if sampling costs are small. As sampling costs decrease, the seller can permit himself to reject all but the very high offers. The problem is easily solved when the distribution of offers is exponential or the limit obtained by Feller, and its solution in these two cases provides a good asymptotic approximation to the optimal solution (as sampling costs decrease), when the distribution of offers has asymptotically exponential or Feller tails.

For any distribution function F, denote $F^*(x) = 1 - F(x)$.

LEMMA 1. *If an absolutely continuous distribution function F on* $[0, \infty)$ *has density f, then*

$$F^*(x) = \exp\{-\textstyle\int_0^x (f(t)/F^*(t))\,dt\}, \qquad x \in [0, \infty). \tag{2}$$

PROOF. The two sides agree at $x = 0$, and their logarithms are easily seen to have the same derivatives.

LEMMA 2. *Let F be any distribution function on* $[0, \infty)$, *with finite first moment* μ. *Then*

$$F^*(x) = (\mu/g(x)) \exp\{-\textstyle\int_0^x dt/g(t)\}, \tag{3}$$

(g was defined in (1)).

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PROOF. Apply Lemma 1 to the distribution function $F_1(x) = \mu^{-1} \int_0^x F^*(t)\,dt$.

LEMMA 3. *Let h be a positive, non-decreasing function on* $[0, \infty)$ *such that $h(x)/x$ decreases to zero as $x \to \infty$. Let g be a positive function on* $[0, \infty)$ *such that $g(x)/h(x) \to 1$ as $x \to \infty$. Then for any $A > 0$,*

$$g(x + ag(x))/g(x) \to 1 \quad \textit{as} \quad x \to \infty\,,$$

uniformly in $a \in [0, A]$.

PROOF. Let us first prove it for $g = h$. Since h is non-decreasing and $h(x)/x$ is non-increasing,

$$1 \leqq \frac{h(x + ah(x))}{h(x)} = \frac{h(x + ah(x))}{x + ah(x)} \cdot \frac{x + ah(x)}{h(x)} \leqq \frac{h(x)}{x} \cdot \frac{x + ah(x)}{h(x)} \leqq 1 + A\frac{h(x)}{x}\,. \tag{4}$$

Now assume $g(x)/h(x) \to 1$ as $x \to \infty$.

Denote
$$M^+(x) = \sup_{y \geqq x} g(y)/h(y)$$
$$M^-(x) = \inf_{y \geqq x} g(y)/h(y)\,.$$

($M^+(x) \downarrow 1$ and $M^-(x) \uparrow 1$ as $x \to \infty$).

$$\frac{g(x + ag(x))}{g(x)} = \frac{g(x + ag(x))}{h(x + ag(x))} \cdot \frac{h(x)}{g(x)} \cdot \frac{h(x + ag(x))}{h(x + ah(x))} \cdot \frac{h(x + ah(x))}{h(x)}\,.$$

The result to be proved follows from

$$\text{(i)} \qquad M^-(x) \leqq \frac{g(x + ag(x))}{h(x + ag(x))} \leqq M^+(x)\,,$$

$$\text{(ii)} \qquad \frac{1}{M^+(x)} \leqq \frac{h(x)}{g(x)} \leqq \frac{1}{M^-(x)}$$

and

$$\text{(iii)} \quad 1 - A[1 - M^-(x)]\frac{h(x)}{x} \leqq \frac{h(x + ag(x))}{h(x + ah(x))} \leqq 1 + A[M^+(x) - 1]\frac{h(x)}{x}\,,$$

which can be proved as in (4).

THEOREM 1. *Let F be the distribution function of a non-negative* rv *X with finite mean and infinite essential supremum. Let $g(x) = E[X - x \mid X > x]$, $x \in [0, \infty)$. If there exists a positive, non-decreasing function h on $[0, \infty)$ such that $h(x)/x$ decreases to zero and $g(x)/h(x) \to 1$ as $x \to \infty$, then*

$$\text{(i)} \qquad \frac{F^*(x + ag(x))}{F^*(x)} \to e^{-a} \quad \textit{as} \quad x \to \infty\,.$$

$$\text{(ii)} \quad \textit{For any } k > 0, \qquad a^k \left| \frac{F^*(x + ag(x))}{F^*(x)} - e^{-a} \right| \to 0 \quad \textit{as} \quad x \to \infty$$

uniformly in a.

REMARK. (i) says that the conditional distribution of $(X - x)/g(x)$ given $X > x$ converges to the standard exponential distribution as $x \to \infty$.

(ii) implies convergence of moments of all orders to the corresponding moments of the standard exponential distribution.

PROOF OF THEOREM 1.

(i) By Lemma 2,

$$\frac{F^*(x + ag(x))}{F^*(x)} = \frac{g(x)}{g(x + ag(x))} \cdot \exp\{-\textstyle\int_0^a (g(x)/g(x + ug(x)))\, du\} .$$

By Lemma 3, this expression converges to e^{-a} as $x \to \infty$.

(ii) It is enough to prove that for every $k > 0$ there exists a $x_0 > 0$ such that $(y/g(x))^k \exp\{-\int_x^{x+y} dz/g(z)\}$ is bounded on the set $\{x > x_0, y > 0\}$. Using the notations of Lemma 3, let x_1 be such that $M^+(x_1) \leqq 2$ and $M^-(x_1) \geqq \frac{1}{2}$. Then if $x \geqq x_1$ and $y > 0$,

$$\left(\frac{y}{g(x)}\right)^k \exp\left\{-\int_x^{x+y} \frac{dz}{g(z)}\right\} \leqq 2^k \left[\left(\frac{y}{h(x)}\right)^{2k} \exp\left\{-\int_x^{x+y} \frac{dz}{h(z)}\right\}\right]^{\frac{1}{2}} .$$

Hence it is enough to prove that for every $k > 0$, $(y/h(x))^k \exp\{-\int_x^{x+y} dz/h(z)\}$ is bounded on $\{x > x_0, y > 0\}$ for some $x_0 > 0$. As in the proof of Lemma 3, $h(z) \leqq h(x) + (z - x)(h(x)/x)$ for $z > x$, so

$$\exp\{-\textstyle\int_x^{x+y} dz/h(z)\} \leqq \exp\{-\int_0^y du/(h(x) + uh(x)/x)\} = (1 + y/x)^{-x/h(x)} .$$

Hence

$$\left(\frac{y}{h(x)}\right)^k \exp\left\{-\int_x^{x+y} \frac{dz}{h(z)}\right\} \leqq \left(\frac{y}{h(x)}\right)^k \left(1 + \frac{y}{x}\right)^{-x/h(x)} \leqq k^k$$

provided $h(x)/x \leqq 1/k$, as can be easily checked by differentiation with respect to y.

REMARK. There are functions other than g that normalize $(X - x)$ so that its conditional distribution (given $X > x$) converges to exponential. Obviously, any function $l(x)$ with $l(x)/g(x) \to 1$ as $X \to \infty$ will do. One important function which belongs to this class under some conditions is the inverse of the failure rate function. Assume F to be absolutely continuous, and denote its density by f. The ratio $f(x)/F^*(x)$ is called the *failure rate* of F. Absolute continuity of F implies differentiability of $g(x)$, and the equality $F^*(x)/(f(x)g(x)) = 1 + (dg/dx)(x)$ holds. So, if $(dg/dx)(x) \to 0$ as $x \to \infty$, and the conditions of Theorem 1 are satisfied, $F^*(x)/f(x)$ is a possible normalizing function. Lemma 1 provides a method for working directly with $F^*(x)/f(x)$. However, this will not remove the condition of finiteness of the first moment, since the convergence of $F^*(x)/(xf(x))$ to zero (without which the author has been unable to obtain any results leading to an exponential distribution) implies $\int_0^\infty F^*(y)\,dy < \infty$.

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