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A GENERALIZATION OF SEPARABLE STOCHASTIC PROCESSES

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Doob introduced the standard modifications or extensions of a stochastic process and proved that every stochastic process has a separable standard modification. In 1964 Elliott and Morse developed a general theory of product measures with implications in the theory of continuous parameter processes with mutually independent random variables. In particular, they gave a new method for obtaining extensions which considerably generalizes the notion of separability. For a separable process only certain events specified by restrictions of the random variables at a nondenumerable collection of time points are measurable. Under their generalization, the restriction to only certain events is virtually removed. The key to the new method for obtaining extensions is a modification by means of nilsets. The definition of nilsets has recently been adjusted to enable the application of this method to general stochastic processes.

1. Introduction. Let $\{X_t, t \in T\}$ be a real stochastic process with a linear parameter set T and let P be the associated probability measure on the space Ω.

Suppose that for each $t \in T$, A_t is a subset of the real line and that $\{\omega : X_t(\omega) \in A_t\}$ is a measurable event (i.e., $P(X_t \in A_t)$ is defined). For J any subset of T, let us define

$$A_J = \bigcap_{t \in J} \{\omega : X_t(\omega) \in A_t\} . \tag{1}$$

Thus A_J is the event specified by restricting the random variable X_t to have values in the set A_t at each of the time points t in the set J. When J is a countable set it follows that A_J is a measurable event, since it is a countable intersection of measurable events and thus its probability $P(A_J) = P(X_t \in A_t$ for all $t \in J)$ is well defined. If J is not countable, this may not be the case. However, if the process is separable [1], [2], then A_J will be measurable whenever J is the intersection of an open interval and T, and for each $t \in J$, A_t is a fixed closed interval of the real line (say $A_t = a$ for all $t \in J$). The essential feature of a separable process is that both $\sup_{t \in I \cap T} X_t(\omega)$ and $\inf_{t \in I \cap T} X_t(\omega)$ are random variables when I is any open interval. Associated with any real stochastic process is another essentially equivalent process called a standard modification [2] which is separable.

The methods for obtaining these separable modifications or extensions were given by Doob [2]. A new method for obtaining a more general extension of a process with mutually independent random variables was developed in con-

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nection with a general theory of product measures [3]. Under this method, the modified process enjoys the property that for any $J \subset T$, the set A_J is measurable when the sets A_t, $t \in J$, are arbitrary measurable sets. This is a considerable generalization of separability.

This general method is based on modifying a prabability measure P with its associated sigma-field $\mathscr{F}^*$ by means of a family $\mathscr{K}$ of "nilsets." This family needs to be closed under countable union and to satisfy the condition that $\bar{P}(A - \alpha) = \bar{P}(A)$ whenever $A \in \mathscr{F}^*$ and $\alpha \in \mathscr{K}$. Then, a probability measure P' having a sigma-field $\mathscr{F}'$ of measurable sets with $\mathscr{F}^* \subset \mathscr{F}'$ may be obtained through outer measure extensions by the defining equation

$$\bar{P}'(A) = \inf\{P(A - \alpha) : \alpha \in \mathscr{K}\} \qquad A \subset S .$$

This extension of P preserves the probabilities of members of $\mathscr{F}^*$ but enlarges the family of measurable sets. For various choices of the family $\mathscr{K}$ of "nilsets," a wide variety of events which are specified by a nondenumerable collection of random variables become measurable. Doob's separable standard extension of a stochastic process of function space type is seen to be a special case of this new type of of extension.

2. Arbitrary processes of function space type. Take $\mathscr{S}$ to be the space of all sample paths of a process so that $\mathscr{S}$ consists of all real functions defined on the linear parameter set T, and take the Ω space to be $\mathscr{S}$ itself (i.e., to be of function space type). Doing this, we then identify a point $\omega \in \Omega$ with a sample path (function) x where $x(t) = X_t(\omega)$.

The event A_J defined in (1) may now be expressed by

$$A_J = \bigcap_{t \in J} \{x : x(t) \in A_t\} . \tag{2}$$

Suppose P is now an arbitrary probability measure on $\mathscr{S}$, and for each $t \in T$, let $\mathscr{F}_t$ denote the family of measurable events of the form $\{x : x(t) \in A_t\}$. Thus, $\mathscr{F}_t$ is a sigma-field, i.e., is closed to countable unions, countable intersections, and complementation, and the probability measure P is defined on $\mathscr{F}$ and is countably additive, i.e., $P(\bigcup_k b^k) = \sum_k P(b^k)$ whenever $b^1, b^2, \cdots$ are disjoint members of $\mathscr{F}_t$.

We assume that P has been extended from the finite dimensional sample spaces to countable dimensions so that for each countable $J \subset T$, it is defined on the smallest sigmafield $\mathscr{F}_J$ which contains all sets of the form $\bigcap_k b_k$ where $b_k \in \mathscr{F}_{t_k}$ and $J = \{t_1, t_2, \cdots\}$. Moreover P is countably additive on $\mathscr{F}_J$ (making the members of $\mathscr{F}_J$ measurable). Thus, taking $\mathscr{F}^*$ to be the union of all such families $\mathscr{F}_J$, the probability measure P is defined on $\mathscr{F}^*$ and $\mathscr{F}^*$ is the familiar sigma-field of measurable sets.

Our goal is to modify P so that events of the form (2) are measurable when J is nondenumerable. Let $\mathscr{R}$ be the family of all events of the form (2) where for each $t \in J$, $\{x: x(t) \in A_t\} \in \mathscr{F}_t$ and J is an arbitrary subset subset of T. Then our task can be stated as follows: Find a probability measure P' and a sigma-field $\mathscr{F}'$ of P' measurable sets such that

(i) $\mathscr{F}^* \subset \mathscr{F}'$, $\mathscr{R} \subset \mathscr{F}'$.
(ii) $P'(A) = P(A)$ whenever $A \in \mathscr{F}^*$.

To do this by a method resembling that used for processes with mutually independent random variables [3], [4], the definition of nilsets needs to be modified so that additional needed sets are included. Suppose that for each $t \in T$, a_t is a subset of the real line and $\alpha_t = \{x: x(t) \in a_t\} \in \mathscr{F}_t$. Then the set

$$\alpha = \bigcup_{t \in T} \alpha_t = \{x: \text{for some } t \in T,\ x(t) \in a_t\} \tag{3}$$

is called locally nil on A, $A \in \mathscr{F}^*$, provided $P(A \cap \alpha_t) = 0$ for each $t \in T$. A nilset β is then defined to be a set of the form

$$\beta = \bigcup_k \{A^k \cap \alpha^k\} \tag{4}$$

where $\{A^k\}$ and $\{\alpha^k\}$ are sequences such that for each $k\ (=1, 2, \cdots)$ α^k is locally nil on A^k.

It is convenient to first consider the outer measure extension $\bar{P}$ of P which is defined by

$$\bar{P}(D) = \inf\{P(C): D \subset C \in \mathscr{F}^*\} \tag{5}$$

where D is any arbitrary subset of $\mathscr{S}$. (Thus, if D is a measurable set then $\bar{P}(D) = P(D)$ and if D is not measurable there is then a measurable superset C of D such that $\bar{P}(D) = P(C)$.)

We then define the outer measure $\bar{P}'$ as

$$\bar{P}'(B) = \inf\{\bar{P}(B - \beta): \beta \text{ is a nilset}\} \tag{6}$$

where $B \subset \mathscr{S}$.

Since a nilset is defined as a countable union of the intersections of nil pairs it is clear that the countable union of nilsets is again a nilset. Thus our definition (6) does indeed cause $\bar{P}'$ to be an outer measure. Furthermore, it is not too difficult to check, as in [5], that $\bar{P}'$ and P assign the same values to members of $\mathscr{F}^*$ i.e., (ii) holds. Thus, $\bar{P}'$ is the outer measure extension of a measure P' on a sigma-field $\mathscr{F}'$ of measurable sets and $\mathscr{F}^* \subset \mathscr{F}'$.

To complete our task we need to establish the second part of (i), namely that $R \subset F'$. In [5] there is a generalization of Lemma 2.1 in [2] which asserts that if $B \in \mathscr{R}$ then there is a nilset D and a member A of $\mathscr{F}^*$ such that $B = A - D$. Now, since A is clearly P' measurable, $P'(D) = 0$ and D is therefore P' measurable, it follows that B is P' measurable, i.e., $B \in \mathscr{F}'$. Since B was an arbitrary member of $\mathscr{R}$ we have established the desired result that $\mathscr{R} \subset$

$\mathscr{F}'$ completing our stated goal. What we have accomplished is to simply enlarge the family of sets to which we can meaningfully assign probabilities to include all members of $\mathscr{R}$ (and also all nilsets since they are each assigned a probability of zero) without disturbing the basic probabilities assigned to the members of $\mathscr{F}^*$.

The above is a greatly simplified account of the role nilsets may play in the theory of arbitrary stochastic processes. This is not only because the detailed proofs given in [5] are not repeated here but also because there are variations in the definition of nilsets that may be made for special purposes. In brief, the two variations which we shall consider require the nilsets to satisfy the respective additional condition that:

(a) $$P(x(t) \in a_t) > 0 \quad \text{for each} \quad t \in T$$

or

(b) $$a_t \text{ is an open set for each } t \in T\,.$$

In these two variations of the definition of a nilset the other conditions relating to (4) remain unchanged. In the following we shall refer to nilsets, a-nilsets and b-nilsets, according to which definition is used. As a result we get three measures, P', P_a', and P_b', depending on which nilsets are used to modify P, and three sigma-fields, $\mathscr{F}'$, $\mathscr{F}_a'$, and $\mathscr{F}_b'$. For these variations we do not obtain the result that $\mathscr{R} \subset \mathscr{F}_a'$ or $\mathscr{R} \subset \mathscr{F}_b'$. Instead we obtain in case (a) that if $B \in \mathscr{R}$ and $P(x(t) \notin B_t) > 0$ or $B_t = (-\infty, \infty)$ for each $t \in T$, then $B \in \mathscr{F}_a'$. In case (b) we obtain something more akin to that for conventional separable processes. It is that if $B \in \mathscr{R}$ and B_t is a closed set for each $t \in T$ then $B \in \mathscr{F}_b'$. Since $\mathscr{F}^* \subset \mathscr{F}_b'$ it follows that $A \cap B \in \mathscr{F}_b'$ whenever B is one of these "closed" members of $\mathscr{R}$. A similar result holds also in case (a).

As an example application of these modifications to a conventional stochastic process, let us consider the Wiener process. Let W be the Wiener probability measure on $\mathscr{S}$ and let P be the restriction of W to $\mathscr{F}^*$. Then we find that the only b-nilsets are sets of P measure zero and hence that $P_b' = P$ and $\mathscr{F}_b' = \mathscr{F}^*$.

Let C be the family of continuous functions. The relationship between P and W is that $P(A) = W(A \cap C)$ for each $A \in \mathscr{F}^*$ (W is defined on a sigma-field $\mathscr{M} \supset \mathscr{F}^*$ to be discussed later). Actually, W is concentrated on the continuous functions which have zero-crossings. But P_a' turns out to be concentrated on the set of *all* functions which have zero crossings (we should note that there are nontrivial a-nilsets and hence that $P_a' \neq P$). Even more bizarre is P' for it is not even concentrated on the functions with zero crossings.

Thus these extensions have the potential of drastically changing the subspace upon which the probability measure is concentrated. Some interesting questions arise concerning when the extended measure may be "reconcentrated" onto the original subspace and yet retain the features gained from the extension.

3. Separable processes. Let U be a probability measure on $\mathscr{S}$ which determines some stochastic process. Then Doob [2] calls this process separable provided there exists some countable subset I of T and a subset Λ of $\mathscr{S}$ of probability zero such that for any open interval Δ of the time axis and any closed subset a of the real line the two sets

$$\{x: x(t) \in a \quad \text{for all} \quad t \in \Delta \cap T\}$$

and

$$\{x: x(t) \in a \quad \text{for all} \quad t \in I \cap \Delta \cap T\}$$

differ by at most a subset of Λ. The second of these two sets is obviously measurable under the standard assumptions. Hence the first set is also measurable and the two have the same probability.

Suppose now that P is a probability measure with the associated sigma-field $\mathscr{F}^*$ as in Section 2. Then Doob ([2] Section 2, Chapter II) gives a method for changing the process into a separable one. Briefly, it consists of taking a suitably chosen countable subset I of T and an associated type b nilset Λ and defining a probability measure U on $\mathscr{S}$ by

$$U(A) = \bar{P}(A \cap (\mathscr{S} - \Lambda)) \tag{7}$$

where A is a set of the form $B - \alpha$ for some $B \in \mathscr{F}^*$ and $\alpha \subset \Lambda$.

Since Λ is a b-nilset we know that $U(A) = \bar{P}(A)$. This fact plus the other details in the selection of I and Λ result in the process relating to U being a separable process which maintains the basic probabilities of the original process associated with P.

The definition in (7) is of the form $U(A) = \bar{P}(A \cap \mathscr{S}')$ where $\mathscr{S}'$ is a subset of $\mathscr{S}$ which has outer measure one under P, i.e., $\bar{P}(\mathscr{S}') = 1$. The effect of this relativization to $\mathscr{S}'$ may be viewed as a special case of the type of modification given in Sections 1 and 2. If we define $\mathscr{N}$ by means of

$$\mathscr{N} = \{\mathscr{S} - \mathscr{S}'\}$$

and then treat $\mathscr{N}$ as though it were a family of nilsets (note that $\mathscr{N}$ is closed under countable union) it is easy to see that

$$U(A) = \inf\{\bar{P}(A - \alpha): \alpha \in \mathscr{N}\}$$

so that U is a modification of P of the same type as P', P_a', and P_b'. The big difference is that the latter three are not based on a single nilset like U is. This accounts for P', P_a', and P_b' having vastly more measurable sets than U has.

Relative measures are used in many other applications such as, for example, obtaining the Wiener measure from an appropriate measure P and the sigma-field $\mathscr{F}^*$. With C denoting the family of continous functions as before, W may be defined by

$$W(A) = P(A \cap C) \qquad A \in \mathscr{M}$$

where again we have that $\bar{P}(C) = 1$, a fact first proved by Wiener. This again may be viewed as a case of our general method of modification which we have seen enlarges the family of measurable sets. Thus the sigma-field $\mathscr{M}$ on which W may be defined includes $\mathscr{F}^*$, as we mentioned in Section 2.

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