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A COMPOSITE NONPARAMETRIC TEST FOR A SCALE SLIPPAGE ALTERNATIVE

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Consider the 2-sample problem where the null cdf $F(x)$ satisfies $F(0) = 0$ and the alternative is $F_\theta(x) = F(x/(1+\theta))$ with $\theta > 0$. An asymptotically optimum statistic z is obtained for a parametric model where $F(x)$ is a gamma distribution. The Mann-Whitney U and Savage T statistics are compared to z for several null densities. It is shown that the Pitman asymptotic relative efficiency, ARE (U/z), can approach zero if $\mu/\sigma \to 0$, where μ is the mean and σ^2 the variance of the null distribution. However, a lower bond on ARE (U/z) is obtained as a function of μ/σ for general $F(x)$. Using the bound a composite test is constructed which has a specified minimum ARE of any desired value between 0 and .864. Densities exist for the composite test which result in arbitrarily large values of efficiency.

1. Introduction. Consider the two-sample problem,

$$H: X_1, X_2, \cdots X_{n_1}, Y_1, Y_2 \cdots Y_{n_2} \quad \text{i.i.d.} \sim F(x)$$
$$K: X_1, X_2, \cdots X_{n_1} \quad \text{i.i.d.} \sim F_\theta(x)$$
$$Y_1\, Y_2 \cdots Y_{n_2} \quad \text{i.i.d.} \sim F(x)\,,$$

where $F(x)$ is an absolutely continuous cdf with $F(0) = 0$ and corresponding density $f(x)$ and mean μ and variance σ^2. The X and Y data are independent. The alternative cdf is $F_\theta(x) = F(x/(1+\theta))$ with $\theta > 0$. In a parametric model of interest $f(x)$ is a gamma density,

$$f(x) = (s^\lambda/\Gamma(\lambda))x^{\lambda-1}\exp(-sx) \qquad \lambda, s > 0\,, \quad x > 0\,, \tag{1.1}$$

with known shape parameter λ and unknown scale parameter s. This model arises in a target detection problem [19] where the X and Y data are obtained by spectral analysis of a stationary Gaussian time-series. The parameter λ is the time-bandwidth product used in the analyzer and s is inversely proportional to the input noise power in the analyzer band. The presence of an input sinusoid induces a noncentral gamma density which at small signal-to-noise ratio can be characterized as a scale alternative. If the form of the distribution of the input time-series data is unknown then the form of the distribution of the spectral data is unknown and a nonparametric formulation is appropriate.

In the parametric case (1.1) an asymptotically optimum statistic z defined in (2.2) is used. This statistic depends on the ratio of sample means. The restriction $F(0) = 0$ makes the scale alternative a one-sided slippage alternative, i.e.

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$F_\theta(x) \leqq F(x)$, and the Mann-Whitney-Wilcoxon U and Savage T tests are suitable for use in the nonparametric model. T is the locally most powerful rank test [4] for (1.1) when $\lambda = 1$. Pitman asymptotic relative efficiency (ARE) is used to make comparisons. ARE results are obtained for the gamma and other densities. For (1.1) and $\lambda = 1$ it follows from [2], [4] that ARE $(U/z) = \frac{3}{4}$ and ARE $(T/z) = 1$. It is shown that for $\lambda > 1$, ARE $(U/z) > \frac{3}{4}$ and ARE $(T/z) >$ .816. For other densities such as a mixture of gamma densities, large values of ARE can be obtained.

Of particular interest is the result that for general $f(x)$ with $f(x) = 0$ for $x < 0$ and finite second moment,

$$\begin{aligned} \text{ARE}\,(U/z) &\geqq .864\left(1 - .458\frac{\sigma^2}{\mu^2}\right) && \text{if } \frac{\mu}{\sigma} \geqq 2^{\frac{1}{2}}\,; \\ &\geqq \frac{27}{4}\frac{(\mu/\sigma)^6}{(1 + \mu^2/\sigma^2)^4} && \text{if } \frac{\mu}{\sigma} < 2^{\frac{1}{2}}\,. \end{aligned}$$

Using this result a composite test can be designed which has a specified minimum ARE of any desired value between 0 and .864. It is shown that densities exist for the composite test which result in an arbitrarily large ARE. The composite test is constructed by forming an estimate of μ/σ; if the estimate is smaller than a specified value, z is used otherwise U is used as the test statistic.

It should be noted that the literature contains several papers, for example, [4], [9], [15], concerning nonparametric tests against a scale alternative. The emphasis is usually on dispersion, i.e., $F(0) = \frac{1}{2}$. The statistics of Puri and Puri [13] and the statistic of Ansari and Bradley [1] reduce to the Mann-Whitney statistic if it is known that $F(0) = 0$. Sukhatme's S statistic [17] appears efficient for the problem considered. However, although it is not mentioned the derivation of Sukhatme's [17] efficiency equations assumes $F(0) = \frac{1}{2}$. The dispersion statistic of Mood [16] is efficient for testing for a change in variance in a Gaussian distribution [1]. However, for $F(0) = 0$ this statistic appears to be very inefficient [3].

2. Parametric statistic. For the problem considered and the gamma density of (1.1) it can be shown in a lengthy but straightforward manner that a statistic equivalent to the likelihood ratio statistic for *all* known λ is,

$$t = \bar{X}/\bar{Y}\,, \tag{2.1}$$

where $\bar{X}$ and $\bar{Y}$ are the sample means. The critical region consists of large values of t. It has been shown [7] that in the case $\lambda = 1$, t is uniformly most powerful. The ratio t, is F-distributed with $2\lambda n_1$ and $2\lambda n_2$ degrees of freedom under H if 2λ is an integer. If λ is unknown or if the density is not given by (1.1), t cannot be used since the critical region cannot be specified, not even asymptotically. Note also that a maximum likelihood estimator of λ is not available in closed form [6].

Consider the following statistic,

$$z = \hat{\phi} \log \bar{X}/\bar{Y}\,, \tag{2.2}$$

where

$$\hat{\phi} = n_1(n_1 + n_2)^{-1}\bar{X}/S_X + n_2(n_1 + n_2)^{-1}\bar{Y}/S_Y\,, \tag{2.3}$$

and $S_x{}^2$, $S_y{}^2$ are the sample variances of the X and Y sample, respectively. For the nonparametric formulation, $F(x)$ continuous and $F(0) = 0$, $\hat{\phi} \rightarrow_{\text{a.s.}} \mu/\sigma$ as $\min(n_1, n_2) \rightarrow \infty$, for all θ. Also from Lehmann ([10] page 274) and the central limit theorem it follows that

$$(r(1-r)N)^{\frac{1}{2}}(\mu/\sigma)(\log \bar{X}/\bar{Y} - \log(1+\theta)) \tag{2.4}$$

is asymptotically distributed according to $\phi(x)$ the standard normal cdf, where $N = n_1 + n_2$ and $r = n_1/N$ provided $\lim_{N\to\infty} r \neq 0, 1$. It follows from ([8] page 236) that (2.4) with μ/σ replaced by $\hat{\phi}$ is still asymptotically distributed according to $\phi(x)$ and, therefore, t and z are asymptotically equi-efficient. Then from the properties of the likelihood ratio [18], z is asymptotically optimum for the gamma density and all values of λ. Clearly this remains true if $\hat{\phi}$ in (2.2) is replaced by any consistent estimate of μ/σ. The statistic z can be used when λ is unknown or for general $F(x)$. The critical region consisting of large values of z can be specified asymptotically from (2.4).

3. Asymptotic relative efficiency. The nonparametric statistics can be defined in terms of the ranks R_i, $i = 1, 2 \cdots n_1$, where R_i is the rank of X_i in the pooled Y, X data. The linearly equivalent Mann-Whitney-Wilcoxon [11] statistic is,

$$U = (n_1 n_2)^{-1} \textstyle\sum_{i=1}^{n_1} R_i - (n_1 + 1)/2n_2 \tag{3.1}$$

and the Savage statistic [14] is,

$$T = n_1^{-1} \textstyle\sum_{i=1}^{n_1} (\sum_{l=N-R_i+1}^{N} l^{-1})\,. \tag{3.2}$$

The Savage statistic is the optimum rank statistic [14] for an exponential distribution and a scale alternative. Tables of the null distribution of U and T are available and the critical regions can be specified approximately by using the asymptotic normality of U and T.

Subject to the usual regularity conditions for Pitman efficiency [12], the ARE can be obtained from the efficacy of each test. The procedure is outlined below.

Let $\mathrm{E}_\theta(Q_i)$ and $\sigma_0{}^2(Q_i) = \sigma^2_{\theta=0}(Q_i)$ be the moments of Q_i representing z, U or T. The efficacy of Q_i is,

$$\varepsilon(Q_i) = \left[\frac{d\mathrm{E}_\theta(Q_i)}{d\theta}\bigg|_{\theta=0}\right]^2 \Big/ \sigma_0{}^2(Q_i)$$

and $$\text{ARE}(Q_1/Q_2) = \lim_{N\to\infty} \varepsilon(Q_1)/\varepsilon(Q_2) .$$

From Section 2. z is asymptotically normal under H and K and it follows that the efficacy of z is,

$$\varepsilon(z) = n_1 n_2 N^{-1}(\mu/\sigma)^2 . \tag{3.3}$$

From [11], $\sigma_0^2(U) = (N+1)/12n_1n_2$ and $\mathbf{E}_\theta(U) = \int_0^\infty [1 - F_\theta(x)]\, dF(x)$, using $F_\theta(x) = F(x/(1+\theta))$ gives

$$\varepsilon(U) = 12n_1n_2(N+1)^{-1}[\textstyle\int_0^\infty xf^2(x)\, dx]^2 . \tag{3.4}$$

From Chernoff and Savage [5],

$$\mathbf{E}_\theta(T) = \int_0^\infty J[n_1N^{-1}F_\theta(x) + n_2N^{-1}F(x)]\, dF_\theta(x) , \tag{3.5}$$

where $J(x) = -\log(1-x)$, $0 < x < 1$, and $\sigma_0^2(T) = n_2/(n_1N)$ so that

$$\varepsilon(T) = n_1n_2N^{-1}\left[\int_0^\infty \frac{xf^2(x)}{1-F(x)}\, dx\right]^2 . \tag{3.6}$$

Note that Basu and Woodworth [3] give the efficacy of T for general $f(x)$ as shown in (3.6) but with the lower limit of integration $-\infty$ and $1 - F(x)$ incorrectly replaced by e^{-x}. However, they only make a numerical calculation for an exponential $f(x)$. In that case their result and (3.6) agree.

It follows from (3.3), (3.4) and (3.6) that

$$\text{ARE}(U/z) = 12\left(\frac{\sigma}{\mu}\right)^2\left[\int_0^\infty xf^2(x)\, dx\right]^2 , \tag{3.7}$$

$$\text{ARE}(T/z) = \left(\frac{\sigma}{\mu}\right)^2\left[\int_0^\infty \frac{xf^2(x)}{1-F(x)}\, dx\right]^2 . \tag{3.8}$$

For the gamma density of (1.1),

$$\text{ARE}(U/z) = 12\Gamma^2(2\lambda)/(\lambda 2^{2\lambda}\Gamma^4(\lambda)) , \tag{3.9}$$

$$\text{ARE}(T/z) = I^2/(\lambda 2^{2\lambda}\Gamma^4(\lambda)) , \tag{3.10}$$

where

$$I = \int_0^\infty dx e^{-x}x^{2\lambda-1}\Big/\left[1 - \frac{\gamma(\lambda, x/2)}{\Gamma(\lambda)}\right]$$

and $\gamma(\lambda, x/2)$ is the incomplete gamma function.

Using $\lim_{\lambda\to 0}\lambda\Gamma(\lambda) = 1$ yields $\lim_{\lambda\to 0}\text{ARE}(U/z) = 0$ and by numerical evaluation $\text{ARE}(U/z)$ is a monotonically increasing function of λ. For $\lambda = \frac{1}{2}$ (density function has infinite discontinuity at the origin) $\text{ARE}(U/z) = 6/\pi^2$ and for $\lambda = 1$ (exponential density) $\text{ARE}(U/z) = \frac{3}{4}$. Also if $f(x)$ is the gamma density, $\sigma f(\sigma x + \mu) \to \phi(x)$ the standard normal density as $\lambda \to \infty$. Then from (3.7) with $x = \sigma y + \mu$ and $\mu/\sigma = \lambda^{\frac{1}{2}}$,

$$\text{ARE}(U/z) = 12[\lambda^{-\frac{1}{2}} \textstyle\int_{-\lambda^{\frac{1}{2}}}^\infty y[\sigma f(\sigma y + \mu)]^2\, dy + \int_{-\lambda^{\frac{1}{2}}}^\infty [\sigma f(\sigma y + \mu)]^2\, dy]^2 ,$$

and

$$\lim_{\lambda\to\infty} \text{ARE}(U/z) = 12[\textstyle\int_{-\infty}^{\infty} \phi^2(y)\,dy]^2 = 3/\pi\,, \tag{3.11}$$

since $\int_{-\infty}^{\infty} |y|\phi^2(y)\,dy < \infty$. This efficiency is the same as the translation value for U and a normal density.

Similarly, by numerical integration, $\lim_{\lambda\to 0} \text{ARE}(T/z) = 0$ and $\text{ARE}(T/z)$ reaches its maximum at $\lambda = 1$. At $\lambda = \frac{1}{2}$, $\text{ARE}(T/z) = .978$ and by direct evaluation $\text{ARE}(T/z) = 1$ at $\lambda = 1$. The function falls monotonically for $\lambda > 1$. As before, with $\Phi(x)$ the standard normal cdf,

$$\lim_{\lambda\to\infty} \text{ARE}(T/z) = \left[\int_{-\infty}^{\infty} \frac{\phi^2(x)}{1-\Phi(x)}\,dx\right]^2, \tag{3.12}$$

since

$$\int_{-\infty}^{\infty} \frac{|y|\phi^2(y)}{1-\Phi(y)}\,dy < \infty\,.$$

Expression (3.12) has the value .816 by numerical integration. The result of (3.12) corresponds to the translation value for T and a normal density.

It follows that $\text{ARE}(T/z) \geqq .816$ and $\text{ARE}(U/z) \geqq \frac{3}{4}$ for $\lambda \geqq 1$ if $f(x)$ is a gamma density. Note that $\text{ARE}(U/z)$ can be near zero and that this occurs for small λ or small values of μ^2/σ^2. This will be shown to hold for general densities with a "large concentration" of mass near the origin resulting in small values of $\text{ARE}(U/z)$.

If other densities are considered, large values for ARE can be obtained. For a mixture density of $f(x) = (1-\varepsilon)f(x\colon \lambda, s_1) + \varepsilon f(x\colon \lambda, s_2)$, the value of $\text{ARE}(U/z)$ can be obtained by multiplying (3.9) by

$$M = \frac{[1+\varepsilon(R^2-1)+\lambda(1-\varepsilon)\varepsilon(R-1)^2]}{[1+\varepsilon(R-1)^2]} \times \left[1-2\varepsilon(1-\varepsilon)\left(1-\frac{2^{2\lambda}R^{\lambda}}{(R+1)^{2\lambda}}\right)\right]^2, \tag{3.13}$$

where $R = s_1/s_2 > 1$. The factor M is the relative improvement due to nonparametric processing when there is contamination of the underlying gamma density. Note that

$$\lim_{\varepsilon\to 0,\, R\to\infty,\, \varepsilon R^2\to A} M = 1 + (1+\lambda)A\,, \tag{3.14}$$

so large improvements are possible. With $\lambda = 8$ and $A = \frac{1}{3}$ the limiting value of M is 4. For the Savage statistic the limiting value of M is the same as in (3.14) and the actual value approximately the same as (3.13).

Based on the examples, for the alternative $F_\theta(x) = F(x/(1+\theta))$, the Savage statistic in general appears to perform better than the Mann-Whitney statistic. When the density has a very heavy upper tail or is concentrated far from the origin there is a slight preference for the Mann-Whitney statistic. The Savage statistic does relatively well for densities with both heavy and sharp upper tails. It does particulary well when there is a sharp cut-off on this tail. For instance if $f(x)$ is triangular (decreasing linearly from $x = 0$), $\mathrm{ARE}(U/z) = \frac{2}{3}$ while $\mathrm{ARE}(T/z) = 2$.

4. Lower bound on $\mathrm{ARE}(U/z)$. It is clear from the previous section that $\mathrm{ARE}(U/z)$ can approach zero. However it is possible to obtain a lower bound as a function of μ/σ.

Since all factors are positive, minimizing $\mathrm{ARE}(U/z)$ of (3.7) is equivalent to minimizing

$$L = \int_0^\infty x f^2(x)\, dx , \tag{4.1}$$

subject to $1 = \int_0^\infty f(x)\, dx$, $\mu = \int_0^\infty x f(x)\, dx$, $\mu_2 = \int_0^\infty x^2 f(x)\, dx$ and $f(x) \geqq 0$. Let, $V = xf^2(x) - 2(\lambda_1 + \lambda_2 x + \lambda_3 x^2) f(x)$ where the λ's are numbers determined by the integral constraints. The necessary Euler equations are $\partial V/\partial f = 0$ for $f(x) > 0$ and $\partial V/\partial f \geqq 0$ for $f(x) = 0$. The first equation yields

$$f(x) = \lambda_1/x + \lambda_2 + \lambda_3 x . \tag{4.2}$$

Assume $\lambda_1 \leqq 0$ so that the integral constraints can be satisfied with $\lambda_2 > 0$ and $\lambda_3 < 0$. The resulting $f(x)$ intersects the x axis at r_1 and r_2, $0 \leqq r_1 < r_2$ where r_1 and r_2 are solutions of

$$\lambda_1 + \lambda_2 x + \lambda_3 x^2 = 0 . \tag{4.3}$$

Taking $f(x) = 0$ outside of $[r_1, r_2]$ allows $f(x)$ of (4.2) to satisfy both Euler equations. From (4.3), $\lambda_2/\lambda_3 = -(r_1 + r_2)$, $\lambda_1/\lambda_3 = r_1 r_2$ and if $y = r_1/r_2$ it is clear that $0 \leqq y < 1$. Using the integral constraints and $\sigma^2 = \mu_2 - \mu^2$ gives after much algebra,

$$\frac{\sigma^2}{\mu^2} = \frac{3}{2} \frac{(1 - 2y + 2y^3 - y^4)(1 - y^2 + 2y \log y)}{(1 - 3y + 3y^2 - y^3)^2} - 1 \tag{4.4}$$

and

$$\left[\int_0^\infty x f^2(x)\, dx\right]^2 = \frac{(1 - 8y - 12y^2 \log y + 8y^3 - y^4)^2}{9(1 - y^2 + 2y \log y)^4} . \tag{4.5}$$

The $\min\{\mathrm{ARE}(U/z)\}$ is obtained by using (4.4) and (4.5) in (3.7). This calculation was performed on a computer for $y \in [0, 1)$. As y goes from zero to one, μ/σ monotonically increases from $2^{\frac{1}{2}}$ to infinity. The $\min\{\mathrm{ARE}(U/z)\}$ is a linearly decreasing function of σ^2/μ^2 (to the accuracy of the plotting) with a value of .864 as $\mu/\sigma \to \infty$ and $\frac{2}{3}$ at $\mu/\sigma = 2^{\frac{1}{2}}$. It then follows that

$$\mathrm{ARE}(U/z) \geqq .864(1 - .458\sigma^2/\mu^2) \qquad \text{if} \quad 2^{\frac{1}{2}} \leqq \mu/\sigma < \infty \tag{4.6}$$

except for small computational error in the lower bound.

The result is a global minimum. This is easily verified by substituting an arbitrary density into (4.1), consisting of the minimizing density plus a term $\varepsilon(x)$ with $\int_0^\infty \varepsilon(x) = \int_0^\infty x\varepsilon(x)\,dx = \int_0^\infty x^2(x)\,dx = 0$, and $\varepsilon(x) \geqq 0$ for $x \notin [r_1, r_2]$.

To obtain a solution for $0 \leqq \mu/\sigma \leqq 2^{\frac{1}{2}}$, assume that $\lambda_1 = \varepsilon_1 > 0$ with $\lambda_2 > 0$, $\lambda_3 < 0$. Taking $f(x) = 0$ outside of $(0, r_2]$ allows $f(x)$ of (4.2) to satisfy the Euler conditions if r_2 is the positive root of (4.3). In order to satisfy the constraint that the density integrates to one, let $f(x) = 0$ outside of $(\varepsilon_2, r_2]$. By letting ε_1 and ε_2 approach zero at an appropriate rate it is possible to satisfy both this constraint and in the limit the minimizing Euler equations with $0 \leqq \mu/\sigma \leqq 2^{\frac{1}{2}}$.

Using (4.2), $xf^2(x) = x(\varepsilon_1/x + \lambda_2 + \lambda_3 x)f(x)$ so that from (4.1), $L = \varepsilon_1 + \lambda_2\mu + \lambda_3\mu_2$. In the limit as $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, it follows from (4.3) and the constraints on the first and second moments that $r_2 = -\lambda_2/\lambda_3$, $\mu = \lambda_2 r_2^2/2 + \lambda_3 r_2^3/3$, and $\mu_2 = \lambda_2 r_2^3/3 + \lambda_3 r_2^4/4$. Then $r_2 \to 2\mu_2/\mu$, $\lambda_2 \to 6\mu/r_2^2 = 3\mu^3/(2\mu_2^2)$, $\lambda_3 \to -3\mu^4/(4\mu_2^3)$, $L \to 3\mu^4/(4\mu_2^2) = 3\mu^4/(4[\sigma^2 + \mu^2])$ and from (3.7),

$$\text{(4.7)} \qquad \min\{\text{ARE}(U/z)\} = 12(\sigma^2/\mu^2)L^2$$

$$\min\{\text{ARE}(U/z)\} = \frac{27}{4}\frac{(\mu/\sigma)^6}{(1 + \mu^2/\sigma^2)^4}\,.$$

For this procedure to be valid and consistent with (4.6) it is necessary to show that the constraint $1 = \int_0^\infty f(x)\,dx$ can be satisfied if and only if $0 \leqq \mu/\sigma \leqq 2^{\frac{1}{2}}$. Using the constraint yields,

$$\text{(4.8)} \qquad 1 = B + \lim_{\varepsilon_1\to 0,\, \varepsilon_2\to 0} (\lambda_2 r_2 + \lambda_3 r_2^2/2)\,,$$

where $B = \lim_{\varepsilon_1\to 0,\, \varepsilon_2\to 0} -(\varepsilon_1 \log \varepsilon_2)$. Equation (4.8) is equivalent to $1 = B + (3\mu^2/\sigma^2)/(2[1 + \mu^2/\sigma^2])$ so that $\mu^2/\sigma^2 = 2(1 - B)/(1 + 2B)$. Since $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, it follows that $B \geqq 0$. It is then possible to let ε_1 and ε_2 approach zero such that (4.7) is valid only for any given $\mu/\sigma \in [0, 2^{\frac{1}{2}}]$. Note that for $\mu/\sigma = 2^{\frac{1}{2}}$ both bounds give $\text{ARE}(U/z) = \frac{2}{3}$ and that the density resulting in this value is triangular, decreasing lineary from a peak at $x = 0$. The bounds of (4.6) and (4.7) are monotonically increasing functions of μ/σ.

Relation (4.7) and its derivation points out what was indicated in Section 3 for the gamma density. $\text{ARE}(U/z)$ can be small when μ/σ is small due to a "great concentration" of mass near the origin.

5. Composite test. The results of Section 4 can be used to construct a test that has a lower bound but not an upper bound on its relative efficiency.

Let z and $\hat{\phi}$ be as defined in (2.2) and (2.3) and let

$$\text{(5.1)} \qquad W_z = 1 \quad \hat{\phi} < k\,, \qquad = 0 \quad \hat{\phi} \geqq k\,;$$

$$\text{(5.2)} \qquad W_U = 1 \quad \hat{\phi} \geqq k\,, \qquad = 0 \quad \hat{\phi} < k\,.$$

The number k is a design parameter for the test. A proper choice for k will be made clearer in the following discussion. The composite test rejects H if

$$C = W_z z + W_U U \geqq W_z L_z + W_U L_U , \tag{5.3}$$

where $L_z = \Phi^{-1}(1-\alpha)/(r(1-r)N)^{\frac{1}{2}}$, $r = n_1/N$, $N = n_1 + n_2$ and α is the desired size of the test. L_U is determined from the null distribution of U such that $P[U \geqq L_U] = \alpha$ or using the asymptotic normality of U,

$$L_U = \Phi^{-1}(1-\alpha)/12r(1-r)N)^{\frac{1}{2}} + \tfrac{1}{2} .$$

Since as $N \to \infty$, $\hat{\phi} \to_{\text{a.s.}} \mu/\sigma$, it follows that W_Z and W_U approach 1 or 0 a.s. depending on whether μ/σ is less than or greater than the chosen k. Then it follows ([8], page 236) that for any $k \geqq 0$, the test of (5.3) is asymptotically size α and,

$$\begin{aligned} \text{ARE}\,(C/z) &= \text{ARE}\,(U/z) \qquad && \mu/\sigma \geqq k , \\ &= 1 && \mu/\sigma < k . \end{aligned} \tag{5.4}$$

From Section 4,

$$\text{ARE}\,(C/z) \geqq \min_{\mu/\sigma = k} \{\text{ARE}\,(U/z)\}$$

and

$$\begin{aligned} \text{ARE}\,(C/z) &\geqq (27/4)k^6/(1+k^2)^4 \qquad && \text{if} \quad 0 \leqq k \leqq 2^{\frac{1}{2}} , \\ &\geqq .864(1 - .458/k^2) && \text{if} \quad 2^{\frac{1}{2}} \leqq k < \infty . \end{aligned}$$

The parameter k for the test can be chosen to give any desired lower bound between 0 and .864.

It can be shown that for any $k \geqq 0$, ARE (C/z) does not have an upper bound. Let $g(x)$ be a density with mean μ_g and variance σ^2 such that $g(x) = 0$, $x < 0$. Take $f(x) = g(x-m)$ $m > 0$ and from (3.7)

$$\begin{aligned} \text{ARE}\,(U/z) &= 12 \frac{\sigma^2}{(\mu_g + m)^2} \left[\textstyle\int_m^\infty x g^2(x-m)\,dx\right]^2 \\ &= 12\sigma^2 \left[\frac{\int_0^\infty x g^2(x)\,dx}{\mu_g + m} + \frac{m}{\mu_g + m} \textstyle\int_0^\infty g^2(x)\,dx\right]^2 , \end{aligned}$$

$$\text{ARE}\,(U/z) \geqq \left[\frac{m}{\mu_g + m}\right]^2 \cdot 12\sigma^2 \left[\textstyle\int_0^\infty g^2(x)\,dx\right]^2 . \tag{5.5}$$

For any fixed $g(x)$, $\mu/\sigma = (m + \mu_g)/\sigma$ can be made arbitrarily large and $m/(\mu_g + m)$ arbitrarily close to one, by choosing a sufficiently large value of m. The second term in (5.5) is the ARE value for a translation alternative and null density $g(x)$. It is well known that densities $g(x)$ exist which make this term arbitrarily large. Therefore for any $k \geqq 0$, a density exists such that ARE (C/z) is arbitrarily large.

To implement the composite test a choice for k must be made. Although large values of k give a lower bound close to .864 and still allow the possibility of a large ARE value, in most cases this will result in essentially using the z-test.

A reasonable choice is $k = 2^{\frac{1}{2}}$ this gives a lower bound of $\frac{2}{3}$ and should frequently result in the use of the U-test.

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REFERENCES

[1] ANSARI, A. R. and BRADLEY, R. A. (1960) Rank-sum tests for dispersion. *Ann. Math. Statist.* **31** 1174–1189.

[2] BASU, A. P. (1967). On the large sample properties of a generalized Wilcoxon-Mann-Whitney statistic. *Ann. Math. Statist.* **38** 905–915.

[3] BASU, A. P. and WOODWORTH, G. (1967). A note on a nonparametric test for scale. *Ann. Math. Statist.* **38** 274–277.

[4] CAPON, J. (1961). Asymptotic efficiency of certain locally most powerful rank tests. *Ann. Math. Statist.* **32** 88–100.

[5] CHERNOFF, H. and SAVAGE, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math Statist.* **29** 972–994.

[6] CHOI, S. C. and Wette, R. (1969). Maximum likelihood estimation of the parameters of the Gamma distribution and their bias. *Technometrics* **11** 683–690.

[7] DOKSUM, K. (1969). Starshaped transfomation and the power of rank test. *Ann. Math. Statist.* **40** 1167–1176.

[8] FISZ, M. (1963). *Probability Theory and Mathematical Statistics*, 3rd ed. Wiley, New York.

[9] KLOTZ, J. H. (1962). Nonparametric tests for scale. *Ann. Math. Statist.* **33** 498–512.

[10] LEHMANN, E. L. (1964). *Testing Statistical Hypotheses*. Wiley, New York.

[11] MANN, H. B. and WHITNEY, D. R. (1947). On a test of whether one of two random variables is stochastically larger than the other. *Ann. Math. Statist.* **18** 50–60.

[12] NOETHER, E. G. (1955). On a theorem of Pitman. *Ann. Math. Statist.* **26** 64–68.

[13] PURI, P. S. and PURI, M. L. (1968). Selection procedures based on ranks scale parameter case. *Sankhyā. Sec. A* **30** 291–302.

[14] SAVAGE, I. R. (1956). Contributions to the theory of rank order statistics—the two-sample case. *Ann. Math. Statist.* **27** 590–615.

[15] SIEGEL, S. and TUKEY, J. W. (1960). A nonparametric sum of ranks procedure for relative spread in unpaired samples. *J. Amer. Statist. Assoc.* **55** 429–445.

[16] SUKHATME, B. V. (1957). On certain two-sample nonparametric tests for variances. *Ann. Math. Statist.* **28** 188–194.

[17] SUKHATME, B. V. (1958). A two-sample distribution free test for comparing variances. *Biometrika* **45** 544–548.

[18] WILKS, S. S. (1962). *Mathematicl Statistics*. Wiley, New York.

[19] WOINSKY, M. N. (1972). Nonparametric detection using spectral data. *IEEE Trans. Information Theor.* **18** 120–126.