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A NOTE ON METRIC TRANSITIVITY FOR STATIONARY GAUSSIAN PROCESSES ON GROUPS[1]

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Maruyama (1949) and Grenander (1950) derive necessary and sufficient conditions for stationary Gaussian processes on the real line or the integers to be metrically transitive. Their work is based on ergodic theorems for such processes. This paper studies conditions for metric transitivity for stationary Gaussian processes for which there are no ergodic theorems. Instead the work is based on results on the absolute continuity of measures corresponding to random processes.

0. Introduction. The ergodic theorem implies that if X is a stationary Gaussian process on the integers, then $\lim (1/N) \sum_1^N x_n$ exists a.e. $d\mu_x$. It follows that the limit is invariant under the shift transformation $T: x_n \rightarrow x_{n+1}$. For processes parameterized by an arbitrary group it is more difficult to generalize the sum $(1/N) \sum x_g$ and the corresponding ergodic theorems are much weaker. Nevertheless, the notion of metric transitivity, that $T_g f = f$ for all $g \in G$ and f in $L^1(d\mu_x)$ implies $f =$ constant, does generalize easily and is a useful concept. It says that all shift invariant events have probability zero or one.

In this paper ideas of absolute continuity are used in place of ergodic theorems to find necessary conditions for metric transitivity. A corollary of the Feldman-Segal [3, 8] dichotomy theorem for Gaussian measures suggests that these conditions are also sufficient. Sufficiency is proved here only for the group R^2. In a paper to appear, written with J. R. Blum, useful ergodic theorems will be proved for such processes, and sufficiency of the above conditions will be proved for a wide class of groups.

1. A necessary condition for metric transitivity. Assume X is a stationary Gaussian process on a locally compact Abelian group G with mean zero and continuous covariance $R(g) = E(x_h x_{hg})$. Most concepts for processes on the real line extend trivially to such processes (cf. Blanc-Lapierre and Fortet [1]). For example.

$$R(g) = \int_\Gamma g(\alpha)\, dF(\alpha) ,$$

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where F is the nonnegative spectral measure on the dual group $\Gamma = \hat{G}$. Also

$$x(g) = \int_\Gamma g(\alpha)\, dZ(\alpha)$$

where Z is a Gaussian random measure on Γ with $E(dZ) = 0$ and $E(|dZ(\alpha)|^2) = dF(\alpha)$. But ergodic notions seem to depend on ideas about order and generators for the group. In this section it is shown that if X is metrically transitive, then its spectral measure F has no atoms. This corresponds to the Maruyama-Grenander result where $G = R$ or Z. The proof is different.

Blum and Hanson [2] have shown that if μ and ν are probability measures invariant under a one-one bimeasurable point transformation T, where $TA = A$ implies $\mu(A) = 0$ or 1; then if ν is absolutely continuous with respect to μ, $\nu = \mu$.

The first proposition is a modification of this result useful for our purposes.

Proposition 1. *Let μ and ν be probability measures invariant under a group of transformations T_g, where μ is metrically transitive with respect to T_g. Then if ν is absolutely continuous with respect to μ, $\nu = \mu$.*

Proof. Assume $\nu \ll \mu$. Then $d\nu/d\mu \in L^1(d\mu)$ and

$$\int_A T_g \frac{d\nu}{d\mu}\, d\mu = \int_{T_g^* A} \frac{d\nu}{d\mu}\, d\mu = \int_{T_g^* A} d\nu = \int_A d\nu = \int_A \frac{d\nu}{d\mu}\, d\mu .$$

Hence

$$T_g \frac{d\nu}{d\mu} = \frac{d\nu}{d\mu} \quad \text{a.e.} \quad d\mu .$$

But μ is metrically transitive. Therefore $d\nu/d\mu = \text{constant} = 1$. □

Proposition 2 is a consequence of a theorem stated below due to Feldman [4]. To keep the paper self-contained a simple proof of this particular result is given.

Proposition 2. *Let X be a stationary Gaussian process on a locally compact Abelian group with continuous covariance. If the spectral function of X has an atom, there exists a stationary Gaussian process $Y \neq X$ with μ_x and μ_y mutually absolutely continuous. In fact, Y may be chosen to be non-Gaussian.*

Proof. Assume the atom is at α_0. Then by the spectral representation

$$x_g = \int_\Gamma g(\alpha)\, dZ(\alpha) = \int_{\alpha \neq \alpha_0} g(\alpha)\, dZ(\alpha) + \int_{\alpha_0} g(\alpha)\, dZ(\alpha) = u_g + v_g ,$$

where the two integrals in the sum are independent random variables. As g varies we write $X =_{\text{law}} U + V$, where U and V are independent random processes. But clearly V and cV are mutually absolutely continuous where c is any nonzero constant. It follows that μ_x and μ_y are mutually absolutely continuous where $Y =_{\text{law}} U + cV$. □

PROPOSITION 3. *Let X be a stationary Gaussian process with continuous covariance function on a locally compact Abelian group. If μ_x is metrically transitive, then the spectral function of X has no atoms.*

PROOF. By Proposition 1 any stationary process equivalent to X must equal X. By Proposition 2 there is a stationary process $Y \neq X$ with μ_x and μ_y absolutely continuous. $\square$

COROLLARY. *There are no nonzero metrically transitive stationary Gaussian processes on a compact Abelian group.*

PROOF. The dual is discrete. $\square$

Finally we mention a theorem of Feldman.

THEOREM (Feldman). *Let X and Y be stationary Gaussian processes on a locally compact Abelian group G and spectral measures F and G. Then μ_x and μ_y are mutually absolutely continuous if and only if F and G have identical non-atomic parts, their points of positive mass are the same and if the masses are F_i and G_i at α_i then $\sum (1 - F_i/G_i)^2$ is finite. Otherwise $\mu_x \perp \mu_y$.*

The proof of this result is in [5] and uses no ergodic theorems. Rather it is based only on the dichotomy theorem for Gaussian measures. The significance of the result to this paper is that it suggests that if a stationary Gaussian process is not metrically transitive, then its spectral measure is discontinuous. That is, if a process X is not metrically transitive there is an $f \geqq 0$ in $L^1(d\mu_x)$ with $T_g f = f$ and $f =$ constant. Defining $\mu_y = (f/\|f\|)\, d\mu_x$, we see there is a stationary process $Y \neq X$ with $\mu_y \ll \mu_x$. If Y were Gaussian the Feldman theorem would imply the spectral measure of X must have had an atom. This would complete the generalization of the Maruyama-Grenander theorem. It remains to be shown that if there is a process Y with $\mu_y \ll \mu_x$, there is a Gaussian Y_1 with $\mu_{y_1} \ll \mu_x$ and $Y_1 \neq X$.

2. Sufficiency. If X is a process on a group G, denote by X_K the process restricted to the subgroup K.

PROPOSITION 4. *If X is a stationary process on G which is not metrically transitive, then for every subgroup $K \subset G$, X_K is not metrically transitive.*

LEMMA. *If μ is invariant under an invertible transformation T and if H is a subspace of $L^2(d\mu)$ with $TH = H$, then $P_H T = TP_H$, where P_H is the orthogonal projection on H.*

PROOF. Exercise. $\square$

PROOF OF PROPOSITION 4. If X is not metrically transitive $\exists f \in L^2(d\mu_x)$ with $T_g f = f$, $\forall g \in G$ and $f \neq$ constant.

Let H_g be the Hilbert space of L^2 functions measurable with respect to the σ-field generated by x_{gk}, $k \in K$, $g \in G$. Then $T_k H_g = H_g \forall k \in K$. Hence by the lemma

$$P_{H_g} f = P_{H_g}(T_k f) = T_k P_{H_g} f .$$

Thus $P_{H_g} f$ is invariant in H_g. If $P_{H_g} f$ were constant for all g, then f would be constant. Hence there is a g with $P_{H_g} f \neq$ constant. The process X restricted to gK is not metrically transitive. But $x_{gk} =_{\text{law}} x_k$, so x_K is also not metrically transitive. ☐

Proposition 4, along with the following proposition, lends further support to the conjecture that if X is a stationary Gaussian process which is not metrically transitive, then the spectral measure of X is discontinuous.

PROPOSITION 5. *If the spectral measure of a stationary Gaussian process X on a group G is discontinuous, then for every closed subgroup K of G the spectral measure of X_K is discontinuous.*

PROOF. Let E be the annihilator of K. Then $\hat{K} \cong G/E$. If $df(\alpha) > 0$ and if αE is the α coset of E then

$$dF_K(\alpha E) = \int_{\gamma \in E} dF(\alpha\gamma) > dF(\alpha) > 0 ,$$

where F_K is the spectral function of X_K. ☐

The next proposition extends the Maruyama-Grenander result completely in the case of $G = R^2$. In this case X is known as a homogeneous random field.

PROPOSITION 6. *Let X be homogeneous Gaussian random field on R^2 with continuous covariance. Then X is metrically transitive if and only if its spectral measure is continuous.*

PROOF. If X is metrically transitive its spectral measure is continuous by Proposition 3.

Assume X is not metrically transitive. Then by Proposition 4 the process Y where $Y_t =_{\text{law}} X_{(t,at)}$ is not metrically transitive for each choice of a.

$$R_y(t) = R_x((t, at)) = \int \exp(i\lambda_1 t + i\lambda_2 at)\, dF_x(\lambda_1, \lambda_2) = \int \exp(i\mu t)\, dF_y(\mu) .$$

It follows that

$$dF_y(\mu) = \int_{\mu=\lambda_1+a\lambda_2} dF(\lambda_1, \lambda_2) .$$

Since Y is not metrically transitive $dF_y(\mu_0) > 0$ for some μ_0 by the Maruyama Theorem. Hence for each a, $\exists \mu_0$ with

$$\int_{\mu_0=\lambda_1+a\lambda_2} dF_x(\lambda_1, \lambda_2) > 0 .$$

If $a \neq a_1$, then the lines $\lambda_1 + a\lambda_2 = \mu$ and $\lambda_1 + a_1\lambda_2 = \mu_1$ intersect in at most one point. There are an uncountable number of such lines each of which has positive measure under F_x. But F_x is a finite measure. Hence there must exist a point with positive measure. ∎

For the case R^2 there is an ergodic theorem due to Wiener [9] to the effect that if T_t and U_s are commuting groups of unitary operators on L^2 induced by measure preserving transformation, then

$$\frac{1}{\pi M^2} \iint_{t^2+s^2 \leqq M^2} T_t U_s f \, ds \, dt$$

exists.

For a homogeneous Gaussian random field on R^2 we can thus say X is ergodic if and only if its spectral measure is continuous.

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