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EXPANSIONS FOR THE DENSITY OF THE ABSOLUTE VALUE OF A STRICTLY STABLE VECTOR

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Let q be the density function of the absolute value of a strictly stable random vector in R^N, N-dimensional Euclidean space. Asymptotic expressions for $q(r)$ for large r and for small r are found. The proofs use the Fourier inversion formula and contour integration. Bessel functions play a role occupied by the exponential and trigonometric functions when $N = 1$.

Let q be the density function of the absolute value of a strictly stable random vector in R^N, N-dimensional Euclidean space. Asymptotic expressions for $q(r)$ for large r and for small r are found. A harder problem not treated here is to find asymptotic expressions for $p(x)$, where p is the density function of a strictly stable random vector. This problem is discussed quite completely for $N = 1$ in [4]. Pruitt and Taylor [3] discuss the behavior of p for $N > 1$ and, in particular, they show by simple examples ([3] page 299) that the general situation is quite complicated.

We say that a random vector X, as well as its distribution, is *strictly stable* if for every positive integer n, there exists a number c_n such that $X = \sum_{k=1}^{n} X_{n,k}$ where $X_{n,k}$, $k = 1, \cdots, n$ are independent and each has the same distribution as $c_n X$. Let $\langle \cdot, \cdot \rangle$ denote the inner product in R^N and let S^{N-1} denote the unit $N-1$-dimensional sphere, the surface of the unit N-dimensional ball. If X is a strictly stable random vector in R^N, then it has a density p which can be written in terms of a number $\alpha \in (0, 2]$, a finite measure μ on S^{N-1}, and, in case $\alpha = 1$, a vector b via the following formulas [2]:

$$p(x) = (2\pi)^{-N} \int_{R^N} \exp[-i\langle u, x\rangle - |u|^\alpha g(u/|u|)]\, du \,, \tag{1}$$

$$\begin{aligned} g(\phi) &= \int_{S^{N-1}} [1 - i \operatorname{sgn}\langle \phi, \theta\rangle \tan \tfrac{1}{2}\pi\alpha] |\langle \phi, \theta\rangle|^\alpha \mu(d\theta) \,, \qquad \alpha \neq 1 \\ &= \int_{S^{N-1}} |\langle \phi, \theta\rangle| \mu(d\theta) + i\langle \phi, b\rangle \,, \qquad \alpha = 1 \,. \end{aligned} \tag{1 a}$$

Given α, μ such that

$$\int_{S^{N-1}} \theta \mu(d\theta) = 0$$

when $\alpha = 1$, and, in case $\alpha = 1$, a vector b, there exists a corresponding p which is the density of a strictly stable random vector; only if $\alpha = 2$, can different measures μ give rise to the same (normal) density. We assume that μ is not concentrated on a hyperplane of dimension $N - 1$; for, if it were, we could consider the random vector to lie in R^M for some $M < N$.

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THEOREM 1. *Let m be a positive integer. As* $r \to \infty$,

$$q(r) = \frac{2\Gamma(1+\alpha)\sin(\frac{1}{2}\pi\alpha)\mu(S^{N-1})}{\pi r^{1+\alpha}}$$
$$+ \pi^{-1-N/2} \sum_{k=2}^{m} \frac{(-1)^{k-1}\Gamma(\frac{1}{2}(N+\alpha k))\Gamma(\frac{1}{2}(2+\alpha k))2^{\alpha k}\sin(\frac{1}{2}\pi\alpha k)}{\Gamma(1+k)r^{1+\alpha k}}$$
$$\times \int_{S^{N-1}} g(\phi)^k \lambda(d\phi) + O(r^{-1-\alpha(m+1)}),$$

where λ *is Lebesgue measure on* S^{N-1}. *If* $\alpha < 1$, $q(r)$, *for* $r > 0$, *equals the infinite series obtained by letting* $m = \infty$.

Before proceeding with the proof we state two lemmas.

LEMMA 1. *Let* λ *be Lebesgue measure on* S^{N-1}, $\phi \in S^{N-1}$, *and* f *be a continuous complex function on* $[-1, +1]$. *Suppose* $N \neq 1$. *Then*

$$\int_{S^{N-1}} f(\langle\phi, \theta\rangle)\lambda(d\theta) = \frac{2\pi^{\frac{1}{2}(N-1)}}{\Gamma(\frac{1}{2}(N-1))} \int_0^\pi f(\cos y)(\sin y)^{N-2}\,dy\,.$$

PROOF. The proof is a straightforward calculation if one chooses one coordinate axis parallel to ϕ. Alternatively, the lemma is a special case of formula 4. 644 of [1].

LEMMA 2. *Let* w *be a complex number with a nonpositive real part. For any nonnegative integer* m,

$$\left| e^w - \sum_{k=0}^{m} \frac{w^k}{k!} \right| \leqq \frac{|w|^{m+1}}{(m+1)!}\,.$$

PROOF. The assumption that $\mathscr{R}w \leqq 0$ yields the result for $m = 0$. An induction argument using

$$e^w - \sum_{k=0}^{m+1} \frac{w^k}{k!} = \int_0^w \left[e^z - \sum_{k=0}^{m} \frac{z^k}{k!} \right] dz$$

completes the proof.

PROOF OF THEOREM 1. We consider only the case $N \neq 1$. The proof when $N = 1$ is easier. In order, we use (1), Lemma 1 and the definite integral formula 8. 411-7 of [1]:

$$\text{(2)} \quad q(r) = r^{N-1} \int_{S^{N-1}} p(r\theta)\lambda(d\theta)$$
$$= (2\pi)^{-N} r^{N-1} \int_{S^{N-1}} \int_0^\infty \int_{S^{N-1}} s^{N-1} \exp[-irs\langle\phi, \theta\rangle - s^\alpha g(\phi)]\lambda(d\theta)\,ds\,\lambda(d\phi)$$
$$\text{(3)} \quad = (2\pi)^{-N/2} \int_{S^{N-1}} \int_0^\infty (rs)^{N/2} J_{\frac{1}{2}(N-2)}(rs)\exp(-s^\alpha g(\phi))\,ds\,\lambda(d\phi)\,,$$

where $J_{\frac{1}{2}(N-2)}$ denotes a Bessel function of the first kind. Let $t = rs$.

Since $\int \mathscr{I} g(\phi)\lambda(d\phi) = 0$,

$$\text{(4)} \quad q(r) = (2\pi)^{-N/2}(2/\pi)r^{-1}$$
$$\times \int_{S^{N-1}} \mathscr{R}[\exp(-\pi i N/4) \int_0^\infty t^{N/2} K_{\frac{1}{2}(N-2)}(-it)\exp(-(t/r)^\alpha g(\phi))dt]\lambda(d\phi),$$

where $K_{\frac{1}{2}(N-2)}$ denotes a modified Bessel function of the third kind, which, in particular, has the property ([1] formulas 8. 405-1 and 8. 407-1):

$$J_{\frac{1}{2}(N-2)}(t) = \mathscr{R}[(2/\pi)\exp(-\pi iN/4)K_{\frac{1}{2}(N-2)}(-it)] .$$

Let $\beta = i$ if $\alpha < 1$ and $\beta = \exp(\pi i/4\alpha)$ if $\alpha \geqq 1$. Using Lemma 2, the fact that, since $\mathscr{I}\beta > 0$, $K_{\frac{1}{2}(N-2)}(-it)$ has a negative exponential tail as $t \to \beta\infty$ along a ray ([1] formula 8. 451-6), and the fact that $\mathscr{R}e^{-ct^\alpha} < 0$ along the ray from 0 to $\beta\infty$, we have, as $c \to 0$,

$$\begin{aligned}\int_0^\infty & t^{N/2}K_{\frac{1}{2}(N-2)}(-it)e^{-ct^\alpha}\,dt \\ &= \int_0^{\beta\infty} t^{N/2}K_{\frac{1}{2}(N-2)}(-it)e^{-ct^\alpha}\,dt \\ &= \sum_{k=0}^m \frac{(-c)^k}{k!}\int_0^{\beta\infty} t^{\alpha k+N/2}K_{\frac{1}{2}(N-2)}(-it)\,dt + O(c^{m+1}) \\ &= \sum_{k=0}^m \frac{(-c)^k\beta^{1+\alpha k+N/2}}{k!}\int_0^\infty u^{\alpha k+N/2}K_{\frac{1}{2}(N-2)}(-i\beta u)\,du + O(c^{m+1}) .\end{aligned}$$

We use the definite integral formula 6. 561-16 of [1]: set $c = g(\phi)/r^\alpha$, insert the resulting expression into (4), and simplify (4) using $\int \mathscr{I}g(\phi)^k\lambda(d\phi) = 0$. The term with $k = 0$ is zero and the terms with $k \geqq 2$ check with those in Theorem 1. For $k = 1$, we must show

$$\pi^{-N/2}\Gamma(\tfrac{1}{2}(N+\alpha))\Gamma(\tfrac{1}{2}(2+\alpha))2^\alpha \int_{S^{N-1}} g(\phi)\lambda(d\phi) = 2\Gamma(1+\alpha)\mu(S^{N-1}) .$$

We have

$$\int_{S^{N-1}} g(\phi)\lambda(d\phi) = \int_{S^{N-1}}\int_{S^{N-1}} |\langle\phi, \theta\rangle|^\alpha\lambda(d\phi)\mu(d\theta) ,$$

which, by Lemma 1, the definite integral formula 8. 380-2 of [1], and the relation $2\pi^{\frac{1}{2}}\Gamma(2z) = 4^z\Gamma(z)\Gamma(z+\frac{1}{2})$, equals the desired ratio.

If $\alpha < 1$, we want to show that the infinite series converges to $q(r)$. Our proof of the asymptotic result actually gives a remainder bound of the order of

$$[r^{-1-\alpha(m+1)}/(m+1)!]\int_0^{\beta\infty} |t|^{\alpha(m+1)+N/2}|K_{\frac{1}{2}(N-2)}(-it)|\,|dt| ,$$

which, as $m \to \infty$, behaves like ([1] formula 8. 451-6)

$$[r^{-1-\alpha(m+1)}/(m+1)!]\Gamma(\alpha(m+1) + \tfrac{1}{2}(N+1)) \to 0 .$$

In case $\alpha = 2$, Theorem 1 is a known result for a normal density—namely that $q(r) \to 0$ faster than any power of r as $r \to \infty$. Here is a more precise result.

THEOREM 2. *If* $\alpha = 2$, *then*

$$q(r) = \frac{r^{N-1}}{(4\pi)^{N/2}}\int_{S^{N-1}} g(\phi)^{-N/2}\exp(-r^2/4g(\phi))\lambda(d\phi) , \tag{5}$$

where λ *is Lebesgue measure on* S^{N-1}.

PROOF. The inner integral of (3) can be evaluated explicitly ([1] formula 6. 631-4). Formula (5) follows.

THEOREM 3. *Let m be positive integer. As $r \to 0$,*

$$q(r) = \frac{4r^{N-1}}{(4\pi)^{\frac{1}{2}(N+1)}\alpha} \sum_{k=0}^{m} \frac{(-1)^k\Gamma(\frac{1}{2}(1+2k))\Gamma((N+2k)/\alpha)r^{2k}}{\Gamma(1+2k)\Gamma(\frac{1}{2}(N+2k))} \times \int_{S^{N-1}} g(\phi)^{-(N+2k)/\alpha}\lambda(d\phi) + O(r^{N+2m+1}) , \tag{6}$$

where λ is Lebesgue measure on S^{N-1}. If $\alpha > 1$, $q(r)$ equals the infinite series obtained by letting $m = \infty$.

PROOF. As in the proof of Theorem 1 we assume that $N \neq 1$. Let $\rho(r) = q(r)r^{-N+1}$. The proof is a rather straightforward application of Taylor's formula to $\rho(r)$. From (2) we obtain

$$\frac{\rho^{(n)}(0)}{n!} = \frac{(-i)^n}{(2\pi)^N n!} \int_{S^{N-1}} \int_0^\infty \int_{S^{N-1}} s^{n+N-1} \exp(-s^\alpha g(\phi))\langle\phi, \theta\rangle^n \lambda(d\theta)\, ds\, \lambda(d\phi) .$$

We use Lemma 1 and formula 8. 380–2 of [1] to evaluate the inner integral. The middle integral becomes a complete gamma integral if $t = s^\alpha g(\phi)$, and formula (6) follows. If $\alpha > 1$, one obtains the equality of $q(r)$ and the infinite series by showing that the remainder term in Taylor's formula goes to zero as m approaches infinity.

REMARK. In Theorem 3 the term with $k = 0$ is positive if and only if $p(0) > 0$. In [5], Taylor showed by an indirect argument that $p(0) = 0$ if and only if $\alpha < 1$ and μ is concentrated on a hemisphere. I know of no direct proof giving these necessary and sufficient conditions for $\int g(\phi)^{-N/\alpha}\lambda(d\phi)$ to equal zero. If $p(0) = 0$, then using the one-dimensional asymptotic theorem [4] we can conclude that all terms in the expansion (6) are zero. In this case the problem remains of finding a function asymptotic to $q(r)$ as r approaches zero.

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