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A NOTE ON SUFFICIENCY AND INVARIANCE[1]

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Under certain conditions, it is shown that the invariant and almost-invariant σ-fields are equivalent if and only if the invariant σ-field is independent of an appropriate sufficient σ-field. This result helps unify work of Hall, Wijsman and Ghosh and of Pfanzagl, who dealt with the forward implication and work of Berk and Bickel, who treated the reverse implication. The conditions required are that the sufficient and invariant σ-fields be essentially disjoint and together generate the σ-field of the original data.

1. Introduction and summary. The various papers cited below have dealt with the relation between sufficiency and invariance. Basu's theorem ([1] and [6] page 162) and the results in [4] and [7] are directed toward concluding the independence of sufficient and (almost) invariant σ-fields. Closely related work in [5] deals with their conditional independence. (When the conditioning σ-field is trivial, the results in [5] specialize to give the result in [7].) In common, the work in [5] and [7] assumes that invariance and almost-invariance are equivalent for the situation under discussion. Proceeding in the other direction, the work in [3] uses the independence to deduce the equivalence of invariance and almost invariance.

In this note we examine a bit more the logical relations between the various conditions used in the above cited papers. It is seen that under certain conditions, the independence of the almost-invariant and sufficient σ-fields is necessary and sufficient for the equivalence of invariance and almost-invariance.

2. Notation and preliminaries. Let $(\mathscr{X}, \mathscr{A})$ be a measurable sample space, $\mathscr{P}$ a family of probability distributions on $\mathscr{A}$ and G a group of $1-1$ bimeasurable transformations of $\mathscr{X}$ onto itself. We suppose that $\mathscr{P}$ is invariant under G. That is, for $P \in \mathscr{P}$ and $g \in G$, $Pg^{-1} \in \mathscr{P}$. Let $\mathscr{A}_S \subset \mathscr{A}$ be a sufficient σ-field. We denote by $\mathscr{A}_A \subset \mathscr{A}$ the σ-field of all $\mathscr{P}$-almost-G-invariant set and by $\mathscr{A}_I \subset \mathscr{A}$, the σ-field of G-invariant sets in $\mathscr{A}$. The reader is referred to [2], [3], [5] and [6] for further discussion of these notions. We always have $\mathscr{A}_A \supset \mathscr{A}_I$ and a fundamental question is whether the inclusion is proper (modulo null sets). A discussion of this question may be found in [3]; we return to it below.

For $A, B \in \mathscr{A}$, we write $A \sim B$ if A and B are $\mathscr{P}$-equivalent. A similar notation is used for random variables and σ-fields. Let $\mathscr{A}_{SI} = \{A \in \mathscr{A}_I : \exists B \in \mathscr{A}_S, B \sim A\}$. $\mathscr{A}_{SI}$ is an essential intersection of $\mathscr{A}_S$ and $\mathscr{A}_I$. The

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definition is unsymmetric in that always $\mathscr{A}_{SI} \subset \mathscr{A}_I$ but not necessarily $\mathscr{A}_{SI} \subset \mathscr{A}_S$. (Of course modulo null sets, this last relation is also true.) This definition has the advantage of giving the same σ-field for all equivalent choices of $\mathscr{A}_S$. In many examples of interest, $\mathscr{A}_S$ is chosen so that $\mathscr{A}_{SI} = \mathscr{A}_S \cap \mathscr{A}_I$. The essential intersection $\mathscr{A}_{SA} \subset \mathscr{A}_A$ is defined similarly.

Following Pfanzagl [7], we say a statistic (= measurable mapping of $\mathscr{X}$) S is equivariant for G if for all x and y in $\mathscr{X}$ and g in G, $Sx = Sy$ entails $Sgx = Sgy$. (In [2], S was then said to commute with G.) An equivariant mapping induces on its range a group of $1 - 1$ transformations $\bar{G}$, defined by $\bar{g}S = Sg$ and the correspondence $g \to \bar{g}$ is a homomorphism. Henceforth, S is assumed to be measurable for a given σ-field $\mathscr{B}$ on range S and we require the elements of $\bar{G}$ to be $(\mathscr{B}, \mathscr{B})$ measurable, hence bimeasurable. (If no σ-field on range S is present a priori, taking $\mathscr{B}$ to be the largest σ-field rendering S measurable automatically makes the $\bar{g}$ bimeasurable.) S is almost-equivariant if for all g in G, there is a bimeasurable transformation $\bar{g}$ on range S so that $Sg \sim \bar{g}S$.

We establish some lemmas.

1. LEMMA. *If f is a bounded real-valued statistic, then for $g \in G$, $P \in \mathscr{P}$,*

$$E_P(f \mid g\mathscr{A}_S) \sim E(fg \mid \mathscr{A}_S)g^{-1}\,. \tag{1}$$

The proof is omitted. See equation (2.1) of [5] and also the last line on page 1573 of [3].

2. LEMMA. *If $\mathscr{A}_S$ is either minimal sufficient or equivalent to the sub-field induced by an almost-equivariant statistic S, then for all g in G, $g\mathscr{A}_S \sim \mathscr{A}_S$.*

PROOF. Assume first $\mathscr{A}_S$ is minimal sufficient. Let f be a bounded real-valued statistic. It follows from (1) that $E_P(f \mid g\mathscr{A}_S)$ does not depend on P. Thus $g\mathscr{A}_S$ is also sufficient and therefore $g\mathscr{A}_S \supset \mathscr{A}_S[\mathscr{P}]$. Since this is true for every g in G, $g\mathscr{A}_S \sim \mathscr{A}_S$.

Now let S be almost-equivariant and suppose $\mathscr{A}_S \sim S^{-1}\mathscr{B}$. Then $g\mathscr{A}_S \sim gS^{-1}\mathscr{B} \sim S^{-1}\bar{g}\mathscr{B} = S^{-1}\mathscr{B} \sim \mathscr{A}_S$. The second equivalence follows from almost-equivariance; the penultimate relation follows from the bimeasurability of $\bar{g}$. □

3. LEMMA. *Suppose for all g in G, $g\mathscr{A}_S \sim \mathscr{A}_S$. Then the following equivalent statements hold.* (i) *If f is a bounded real-valued almost-invariant statistic, $E(f \mid \mathscr{A}_S)$ is almost invariant.* (ii) *$\mathscr{A}_S$ and $\mathscr{A}_A$ are conditionally independent given $\mathscr{A}_{SA}$.*

Cf. Lemma 3.1 of [5] and the lemma in [7].

PROOF. $E(f \mid \mathscr{A}_S)g \sim E(fg \mid \mathscr{A}_S)g \sim E(f \mid g\mathscr{A}_S) \sim E(f \mid \mathscr{A}_S)$. The first equivalence holds because $fg \sim f$, the second is (1), and the third because $g\mathscr{A}_S \sim \mathscr{A}_S$. The equivalence of statements (i) and (ii) follows as in Lemma 3.3 of [5]. □

A consequence of this lemma is that if for all g in G, $g\mathscr{A}_S \sim \mathscr{A}_S$, then $\mathscr{A}_{SA}$ is sufficient for $\mathscr{A}_A$. This is easily checked; see also the discussion after Theorem 3.2 of [5]. Under the additional assumption $\mathscr{A}_{SA} \sim \mathscr{A}_{SI}$, it follows that $\mathscr{A}_{SI}$ is sufficient for $\mathscr{A}_A$. This is a slight improvement of the main result of Section II. 3 of [5].

3. Results and discussion. The following conditions and statements (or variants) appear in the papers cited above.

α) $\mathscr{A}_{SA} \sim \mathscr{A}_{SI}$.

β) $\mathscr{A}_{SI} \sim \{\phi, \mathscr{X}\}$.

$\alpha\beta$) $\mathscr{A}_{SA} \sim \{\phi, \mathscr{X}\}$.

$\alpha\beta\gamma$) $\mathscr{A}_S$ and $\mathscr{A}_A$ are independent.

$\alpha\delta$) $\mathscr{A}_A \sim \mathscr{A}_I$.

a) $\mathscr{A}_{SA}$ is sufficient for $\mathscr{A}_A$.

ab) $\mathscr{A}_S$ and $\mathscr{A}_A$ are conditionally independent given $\mathscr{A}_{SA}$.

abc) $\forall g \in G$, $g\mathscr{A}_S \sim \mathscr{A}_S$.

abcd) $\mathscr{A}_S$ is boundedly complete.

ae) $\mathscr{P}$ collapses on $\mathscr{A}_A$. Equivalently, any sub-σ-field of $\mathscr{A}_A$ is sufficient for $\mathscr{A}_A$.

aef) G generates $\mathscr{P}$.

g) $\mathscr{A}_S \vee \mathscr{A}_I \sim \mathscr{A}$.

The lettering of the statements is intended to indicate some of the logical relations among them. Thus abc $\Rightarrow$ ab $\Rightarrow$ a (Lemma 3) and $\alpha\beta \Leftrightarrow \alpha, \beta\ (= \alpha \cap \beta)$; write $\alpha\beta\delta$ for $\alpha\beta$, $\alpha\delta$, etc. If $\mathscr{A}_S$ is generated by an equivariant S, then β is essentially the requirement that $\bar{G}$ be transitive on range S. Condition g is used in [3]; aef is also found there and in [7]. The first condition of ae means that the restrictions of all measures in $\mathscr{P}$ to $\mathscr{A}_A$ coincide. Clearly aef $\Rightarrow$ ae, but the implication is not reversible in general. (Take X and Y to be independent, $X \sim N(\mu, \sigma^2)$, $Y \sim N(\mu, 1 - \sigma^2)$, $-\infty < \mu < \infty$, $0 < \sigma < 1$. The translation group leaves $\mathscr{P}$ invariant and a maximal invariant is $X - Y$. $\mathscr{P}$ collapses on $\mathscr{A}_I$ and since $\mathscr{A}_A \sim \mathscr{A}_I$ for this group, $\mathscr{P}$ collapses on $\mathscr{A}_A$. Nonetheless, G does not generate $\mathscr{P}$.)

The following theorem gives some further logical relations between the above statements. The results in [1], [3], [4], [5], and [7] are discussed in the context of the theorem.

4. THEOREM. (i) $\alpha\beta\gamma \Leftrightarrow \alpha\beta$, ab. *Hence if* ab *holds*, $\alpha\beta\gamma \Leftrightarrow \alpha\beta$. (ii) g, $\alpha\beta\gamma \Leftrightarrow$ abg, $\alpha\beta\delta$. *Hence if* abg *holds*, $\alpha\beta \Leftrightarrow \alpha\beta\gamma \Leftrightarrow \alpha\beta\delta$.

The following verbal summary may prove helpful. If $\mathscr{A}_S$ can be chosen so that ab holds (e.g., take $\mathscr{A}_S$ to be minimal), then $\mathscr{A}_S$ and $\mathscr{A}_A$ are independent if $\mathscr{A}_{SA}$ is trivial. In turn, this means $\mathscr{A}_{SA} \sim \mathscr{A}_{SI}$ and $\mathscr{A}_{SI} \sim \{\phi, \mathscr{X}\}$. As previously noted, each of these last two statements has an interesting interpretation. If, in addition, g holds, a third equivalent statement is $\mathscr{A}_A \sim \mathscr{A}_I$ and $\mathscr{A}_{SI} \sim \{\phi, \mathscr{X}\}$.

The result given in [7] (see also [4]) is essentially: α, β, abc, aef $\Rightarrow \alpha\beta\gamma$. Referring to (i) in the theorem, the condition aef is seen to be superfluous. However, a$\alpha\beta \Rightarrow$ ae, so in fact, $\alpha\beta$ab $\Leftrightarrow \alpha\beta\gamma \Leftrightarrow \alpha\beta$abe. So although aef and ae are superfluous as sufficient conditions, the latter is nevertheless necessary in the context of [7]. (The implication a$\alpha\beta \Rightarrow$ ae cannot, in general, be reversed. See the example on page 601 of [5]: the $N(c\sigma, \sigma^2)$ family.) Basu's theorem in this context may be stated as: abcde $\Rightarrow \alpha\beta\gamma$. (The implication is immediate: abcde $\Rightarrow \alpha\beta$, for $\mathscr{P}$ collapses on $\mathscr{A}_{SA}$, which therefore is trivial by bounded completeness. Thus abcde $\Rightarrow$ ab$\alpha\beta \Rightarrow \alpha\beta\gamma$ (by (i) of the theorem).) In [3], the relation g$\alpha\beta\gamma \Rightarrow \alpha\delta$ is established. This follows easily from (ii) of the theorem.

We note one further equivalence. The result in [5] and [7] require only α. However, $\alpha\delta$ is the more familiar condition. One may therefore wonder if it can happen that α holds while $\alpha\delta$ fails. (This can happen for pathological choices of $\mathscr{A}_S$.) We see from the theorem that under abg, $\alpha\beta \Leftrightarrow \alpha\beta\delta$. In fact, we may remove the degeneracy condition β and establish that under abg, $\alpha \Leftrightarrow \alpha\delta$. We need only show αabg $\Rightarrow \alpha\delta$, for which we adapt an argument from [3]. If $B \in \mathscr{A}_A$, we show that $E(1_B \mid \mathscr{A}_I) \sim 1_B$. Since $\mathscr{A} \sim \mathscr{A}_I \vee \mathscr{A}_S$, it is enough to show for any $C \in \mathscr{A}_I$ and $D \in \mathscr{A}_S$ that $E1_{BCD} = E1_{CD}E(1_B \mid \mathscr{A}_I)$. By conditioning on $\mathscr{A}_{SI} \sim \mathscr{A}_{SA}$ and using the conditional independence, we see that $E1_{BCD} = E\{E(1_{BC} \mid \mathscr{A}_{SI})E(1_D \mid \mathscr{A}_{SI})\}$. Similarly, $E1_{CD}E(1_B \mid \mathscr{A}_I) = E\{E(1_D E(1_{BC} \mid \mathscr{A}_I) \mid \mathscr{A}_{SI})\} = E\{E(1_D \mid \mathscr{A}_{SI})E(1_{BC} \mid \mathscr{A}_{SI})\}$, which is the same. Thus under abg, the problem of equivalence for invariance and almost-invariance is no easier to resolve by making a sufficiency reduction.

PROOF OF THEOREM. The implications in (i) are immediate. The forward implication in (ii) is obtained using the above-mentioned result from [3]: g$\alpha\beta\gamma \Rightarrow \alpha\delta$, together with (i): $\alpha\beta\gamma \Leftrightarrow \alpha\beta$ab. For the reverse implication, we again use (i): in fact, abg$\alpha\beta \Leftrightarrow$ g$\alpha\beta\gamma$. $\square$

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