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SOME REMARKS ON ASYMMETRIC PROCESSES[1]

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This paper contains three results: (a) an exact measure function for the zero set of a real-valued asymmetric Cauchy process, (b) an upper bound for the Hausdorff dimension of the zero set of a real process with stationary independent increments, and (c) an exact lower function at infinity for the completely asymmetric Cauchy process.

1. Summary. Proposition 1 gives the exact Hausdorff measure function for the zero set of a real-valued asymmetric Cauchy process. Proposition 2 gives an upper bound for the Hausdorff dimension of the zero set of a real process with stationary independent increments. Finally, in Proposition 3 is an exact lower function at infinity for the completely asymmetric Cauchy process.

2. Results. Let $X = \{X(t),\ t \geqq 0\}$ be a real-valued stochastic process with stationary independent increments having no Gaussian part. Then $Ee^{iux(t)} = \exp\{t\psi(u)\}$, where

$$\psi(u) = au + \int_{-\infty}^{\infty} [e^{iux} - 1 - iux/(1+x^2)]\nu(dx)\,. \tag{2.1}$$

If support $\nu \subset (0, \infty)$ and if $\int_0^1 |x|\nu(dx) = \infty$, it will be convenient to call X a (completely) asymmetric process. If

$$\psi(u) = |u|[1 + i(\operatorname{sign} u)h \log |u|] \tag{2.2}$$

where $h = 2b/\pi$, $b = p - q$, $q = 1 - p$, $1 \geqq p \geqq 0$, $b > 0$, then X is called an asymmetric Cauchy process; if $b = 1$, X is a completely asymmetric Cauchy process, and in this case formula (2.1) holds with $a = 0$ and $\nu\,(dx) = \text{const.}\ x^{-2}\,dx$, $x > 0$. If $b = 0$, then X is the symmetric Cauchy process. Assume henceforth that X is a standard Markov process, with $X(0) = 0$. Refer to Blumenthal and Getoor [2] for definitions of terms used below.

At the symposium on Markov processes and potential theory at Madison in 1967, Orey ([11], page 125) asked whether 0 were regular $\{0\}$ for the completely asymmetric Cauchy process. Port and Stone [12] subsequently showed that indeed 0 is regular for $\{0\}$ whenever X is any asymmetric Cauchy process. This is a very curious fact, since, if X is the symmetric Cauchy process, 0 is not regular for itself, and hence in certain cases singletons are more likely to be regular for X than they are for the symmetrization of X. Very recently, Bretagnolle [5] has given the precise criteria (for general X) under which 0

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will be regular for itself. In particular, this will be the case whenever X is a completely asymmetric process (as defined above; see [5], page 33). Here is an amusing proof of this last fact that seems to shed additional light and to give some intuition as to what is going on. According to Shtatland [14], if X is completely asymmetric, then almost surely $\limsup_{t\to 0} X(t)/t = +\infty$ and $\liminf_{t\to 0} X(t)/t = -\infty$, implying that X passes from above 0 to below 0 "infinitely often" as $t \to 0$. Since the process has only upward jumps, it must therefore hit 0 whenever it passes from above to below. Since it does this for arbitrarily small times, 0 is regular for itself. A more vigorous argument using the same idea will show for completely asymmetric X that every λ-excessive function is continuous—a fact established by other methods and in greater generality by Bretagnolle [5].

Since 0 is regular for itself when X is asymmetric Cauchy, but not if X is symmetric Cauchy, one might conjecture that perhaps as X becomes "more symmetric" (e.g., as $b \downarrow 0$), then the zero set $Z(\omega) = \{t \geqq 0 : X_t(\omega) = 0\}$ should become "smaller." That this is not so in the sense that the correct measure function for the zero set changes with b is a consequence of the following result.

PROPOSITION 1. *Let X be an asymmetric Cauchy process. Then an exact measure function for the zero set of X is*

$$Q(t) = \log\log|\log t|/|\log t| .$$

The exact measure function for the zero set of the real stable process with index $\alpha > 1$ was calculated by Taylor and Wendel [16].

PROOF. Let $\{T_t, t \geqq 0\}$ be the subordinator associated with inverse local time. According to a theory that is now standard (this theory is explained in [2] or [3]), we need only find the correct measure function for the range of $\{T_t\}$. It is well known that $Ee^{-\alpha T_t} = \exp\{-tg(\alpha)\}$, where $g(\alpha) = [u^\alpha(0)]^{-1} - \gamma$ (see [3] or [2], Chapter VI). Here $u^\alpha(0) = \int_0^\infty e^{-\alpha t} f_t(0)\, dt$, where $f_t(x)$ is the density of X_t. By Lemma 2.1 of [8],

$$\lim_{\alpha\to\infty} u^\alpha(0) \log \alpha = 2p/\pi h^2 \tag{2.3}$$

in the asymmetric Cauchy case.

Let $r(t) = (\pi p/2b^2)Q(t)$. According to Fristedt and Pruitt ([7], Theorem 3), the exact measure function f for the range of a subordinator having exponent g is given by $f(t) = h^{-1}(t)$, where $h(t) = \log|\log t|/g^{-1}(t^{-1}\log|\log t|)$. Using (2.3), one then verifies that $r(h(t)) \sim t$ as $t \downarrow 0$. Since $h(t)$ is strictly increasing and continuous for t near 0, and $h(0) = 0$, $r(u) \sim h^{-1}(u)$ as $u \to 0$, completing the proof of Proposition 1.

The following corollary is an immediate application of Theorem 2 of Fristedt and Pruitt [7].

COROLLARY. *Let* $\{L_t, t \geqq 0\}$ *be the local time at* 0 *for an asymmetric Cauchy process. Then there is a positive, finite constant c such that*

$$\limsup_{t\to 0} L_t|\log t|/\log\log|\log t| = c \quad \text{a.e.}$$

According to Proposition 1, the Hausdorff dimension of zero set of an asymmetric Cauchy process is zero. The following proposition gives an upper bound to the dimension of the zero set Z of an arbitrary process with stationary independent increments, and includes as a special case the fact stated in the preceding sentence. Let $\beta = \inf\{\alpha \geqq 0 : \int_{|x|<1} |x|^\alpha \nu(dx) < \infty\}$, where ν is the Lévy measure of the process X.

PROPOSITION 2. *Let X be a one-dimensional process with stationary independent increments, and having no Gaussian part. Suppose* (a) 0 *is regular for* $\{0\}$ *and* (b) $\nu(R) = +\infty$. *Then* $\dim Z \leqq 1 - 1/\beta$ a.e.

This bound was obtained by Blumenthal and Getoor [3] under an additional integrability assumption on the exponent ψ.

PROOF. It follows from (a), (b), and the hypothesis of no Gaussian part that $\beta \geqq 1$. Perhaps the easiest way to see this is to refer to the result of Kesten on the hitting probabilities of single points ([9], Theorem 1) and his remarks on pages 7—8 of the same memoir. Let $G(\omega) = \{(t, X_t(\omega) : 0 \leqq t < \infty\}$ be the graph of the process X. To prove the proposition, it is enough to prove

$$1 + \dim Z \leqq \dim G \leqq 2 - 1/\beta \quad \text{a.e.} \tag{2.4}$$

Under hypotheses (a) and (b), it follows from the results of Kesten that the process X hits points. Using this fact together with (a) and (b), one need only make fairly routine modifications in a proof given by Blumenthal and Getoor to obtain the inequality on the left ([4], see pages 313–314). To prove the right-hand inequality, it will be enough to prove it for the graph of X with t restricted to [0, 1]. Moreover, it can be assumed without loss of generality that the Lévy measure of X is concentrated on a finite interval. (This truncation procedure is explained, for example, in [10].) Define

$$M_{k,\varepsilon} = \sup_{0\leqq t\leqq \varepsilon} |X(t + (k-1)\varepsilon) - X((k-1)\varepsilon)|\,, \qquad k = 1, 2, \cdots .$$

Then $M_{1,\varepsilon}, M_{2,\varepsilon}, \cdots$ are independent, indentically distributed and satisfy $E|M_{k,\varepsilon}|^\alpha \leqq M(\alpha)\varepsilon$, where either $\alpha = 2$, or $2 \geqq \alpha > \beta$, and where $M(\alpha)$ is a constant depending only on α and the exponent of the process ([10], Theorem 2.2). If $R(k, \varepsilon)$ is a rectangle with center $((k-1)\varepsilon, X((k-1)\varepsilon))$ and with sides 2ε, $2M_{k,\varepsilon}$, then $R(1, \varepsilon), \cdots, R([\varepsilon^{-1}] + 1, \varepsilon)$ is a cover of $\{(t, X_t(\omega)) : 0 \leqq t \leqq 1\}$. Each of these rectangles can be covered by $[\varepsilon^{-1}M_{k,\varepsilon}] + 1$ squares of side 2ε. Let $C_1(\varepsilon), C_2(\varepsilon), \cdots$ denote this cover of $\{(t, X_t(\omega)) : 0 \leqq t \leqq 1\}$ by squares. Then, using the letter c to denote a positive finite constant (whose value may change from one place to the next), and defining A_ε to be the sum $\sum_i [\operatorname{diam} C_i(\varepsilon)]^\gamma$, we see that

$$A_\varepsilon \leqq c \sum_{k=1}^{\varepsilon-1} (1 + M_{k,\varepsilon}\varepsilon^{-1})\varepsilon^\gamma \leqq c \sum_{k=1}^{\varepsilon-1} M_{k,\varepsilon}\varepsilon^{\gamma-1} + c\varepsilon^{\gamma-1} .$$

The last term goes to zero as $\varepsilon \to 0$, provided $\gamma > 1$. The expectation of the first term is less than

$$c\varepsilon^{\gamma-1} \sum_{k=1}^{\varepsilon-1} EM_{k,\varepsilon} \leqq c\varepsilon^{\gamma-1}\varepsilon^{-1}||M_{1,\varepsilon}||_\alpha$$

where either $\alpha = 2$ or $2 \geqq \alpha > \beta$,

$$\leqq c[M(\alpha)]^{1/\alpha}\varepsilon^{\gamma-2+1/\alpha} .$$

This last term goes to zero if $\gamma > 2 - \alpha^{-1}$. Hence, if $\gamma > 2 - \alpha^{-1}$, $A_\varepsilon \to 0$ in L_1 norm as $\varepsilon \to 0$. A subsequence therefore converges to zero a.e., and this implies $\dim G \leqq 2 - \alpha^{-1}$. But $\alpha \geqq \beta$, so the proof is complete.

REMARKS. (a) The argument giving the right-hand side of (2.4) is an adaptation of an argument of Blumenthal and Getoor ([4], pages 314–315), which in turn was an adaptation of an argument of Besicovitch and Ursell [1].

(b) The argument giving (2.4) fails if X is n-dimensional, $n \geqq 2$. Let γ be the index introduced by Pruitt [13], and let β be the index above. Then it is easy to show that in n-dimensions, $n \geqq 1$: $\max\{1, \gamma\} \leqq \dim G \leqq \max\{1, \beta\}$.

We close with a result closely related to the preceding, which gives a lower function at infinity for the completely asymmetric Cauchy process. This should be compared to the results obtained by Takeuchi and Watanabe [15] for the symmetric Cauchy process. Apparently the question for the other asymmetric Cauchy processes is still open.

PROPOSITION 4. *Let X be the completely asymmetric Cauchy process with $\psi(u) = \int_0^\infty [e^{iux} - 1 - iux/(1 + x^2)]x^{-2}\,dx$. Then* $\liminf_{t\to\infty} X(t)/(t \log t) = 1$ a.e.

PROOF. Write $X(t) = X^1(t) + X^2(t)$, where $X^2(t)$ is the sum of the jumps of X up to time t that exceed 1, and where $X^1(t) = X(t) - X^2(t)$. Then $X^1(t)$ has moments of all orders, $0 \leqq EX^1(t) < \infty$, and so $X^1(t)/t \log t \to 0$ a.e. as $t \to \infty$, by the strong law. It therefore suffices to consider only $X^2(t)$, which has Lévy measure concentrated on $(1, \infty)$, and which is a subordinator with $Ee^{-uX^2(t)} = \exp\{-tg(u)\}$, where $g(u) = \int_1^\infty [1 - e^{-ux}]x^{-2}\,dx$. Integrating by parts, $g(u) = 1 - e^{-u} + u \int_1^\infty e^{-ux}x^{-1}\,dx$. By a Tauberian theorem, $\int_1^\infty e^{-ux}x^{-1}\,dx \sim \log(u^{-1})$ as $u \to 0$, implying that $g(u) \sim u \log(u^{-1})$ as $u \to 0$. By a lemma of Fristedt and Pruitt ([7], Lemma 4), if $\gamma > 1$ then $\liminf_{t\to\infty} X^2(t)/h_\gamma(t) \geqq \gamma - 1$ a.e., where $h_\gamma(t) = \log|\log t|/g^{-1}(\gamma t^{-1} \log|\log t|)$. Using the asymptotic result for g above, one finds that $h_\gamma(t) \sim \gamma^{-1} t \log t$ $(t \to \infty)$ so that $\liminf_{t\to\infty} X^2(t)/(t \log t) \geqq (\gamma - 1)/\gamma$. Since $\gamma > 1$ is arbitrary, let $\gamma \uparrow +\infty$ to get $\liminf_{t\to\infty} X(t)/t \log t \geqq 1$. Also, by Lemma 5 of [7], if $\gamma < 1$, then $\liminf_{t\to\infty} X^2(t)/h_\gamma(t) \leqq \gamma$ a.e., so $\liminf X(t)/(t \log t) \leqq 1$ a.e. This completes the proof.

REMARKS. The integral test of Fristedt [6] shows that

$$\limsup_{t\to\infty} X^2(t)/(t \log t) = +\infty ,$$

and so $\limsup_{t\to\infty} X(t)/t \log t = +\infty$. The same integral test gives (for example) $\limsup_{t\to\infty} X(t)/t \log^2 t = 0$, etc.

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