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IRREDUCIBLE MARKOV SHIFTS

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This paper classifies irreducible finite state Markov shifts up to isomorphism, showing that such a shift is isomorphic to a direct product of a rotation and a Bernoulli shift. This extends the result of Friedman Ornstein [3] that a mixing Markov shift is isomorphic to a Bernoulli shift.

The definition and various properties of Markov shifts are given in Billingsley [1] and will be summarized here.

Suppose S is a finite set, say $S = \{1, \cdots, s\}$, and $P = (P_{ij})$, $i, j \in S$, is a stochastic transition matrix, that is, P is a nonnegative matrix each of whose rows sums to one. A path (of length n) from i to j in S is a sequence $i_0, i_1, \cdots, i_n$, such that $i_0 = i$, $i_n = j$, and

$$P_{i_0 i_1} P_{i_1 i_2} \cdots, P_{i_{n-1} i_n} > 0 .$$

It will be assumed throughout this paper that P is *irreducible*, that is, given any i, j is S, there is a path from i to j.

A Markov shift τ is constructed from P as follows: Let X be the set of all functions from the integers Z into S, and $\mathfrak{B}_0$ the σ-algebra generated by the cylinder sets

$$C = C(i_k, \cdots, i_n) = \{x \mid x_i = i_j, k \leqq j \leqq n\} \tag{1}$$

for $k, n \in Z$, $k \leqq n$. The matrix P, being irreducible, determines a unique probability vector π such that $\pi P = \pi$. There is a unique measure m on $\mathfrak{B}$ such that if C is given by (1), then

$$m(C) = \pi_k \prod_{j=k}^{n-1} P_{i_j i_{j+1}} .$$

The shift τ, defined by $(\tau x)_n = x_{n+1}$, $n \in Z$, is an invertible, ergodic, measure-preserving transformation on $(X, \mathfrak{B}, m)$, where $\mathfrak{B}$ is the m-completion of $\mathfrak{B}_0$. This transformation τ is called the *Markov shift* with transition matrix P and stationary vector π. In the case where the rows of P are identical (and hence equal to π), the shift τ is called a *Bernoulli shift* with distribution π.

The *period* $\nu = \nu(P)$ of P is the greatest common divisor of the lengths of cycles, that is, paths from i to i, $i \in S$. If $\nu(P) = 1$, then τ is mixing, while if $\nu > 1$, then τ^ν is not ergodic. N.A. Friedman and D.S. Ornstein [3] have proved the following theorem:

THEOREM 1. *A mixing Markov shift is isomorphic to a Bernoulli shift.*

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If ν is a positive integer, then Z_ν will denote the additive group of integers mod ν, and m_ν is the measure on Z_ν defined by $m_\nu(i) = 1/\nu$, $0 \leqq i \leqq \nu - 1$. The rotation $\rho = \rho_\nu$ on Z_ν is defined by

$$\rho(z) = z + 1 , \qquad \bmod \nu .$$

Suppose that for $i = 1, 2$, τ_i is a measure-preserving transformation on $(X_i, \mathfrak{B}_i, m_i)$. The direct product $\tau_1 \times \tau_2$, is defined on the direct product space $(X_1 \times X_2, \mathfrak{B}_1 \times \mathfrak{B}_2, m_1 \times m_2)$ by

$$\tau_1 \times \tau_2(x_1, x_2) = (\tau_1 x_1, \tau_2 x_2) .$$

The following theorem and Theorem 1 classify irreducible Markov shifts up to isomorphism.

THEOREM 2. *If τ is an irreducible Markov shift of period $\nu > 1$, then τ is isomorphic to $\rho_\nu \times \beta$ where β is a Bernoulli shift.*

First we note that

$$S = \bigcup_{r=0}^{\nu-1} S_r$$

where $S_r \cap S_t = \varphi$ if $r \neq t$ and if $i \in S_r$, then

$$\text{(2)} \qquad \begin{aligned} P_{ij} &= 0 \quad \text{unless} \quad j \in S_{r+1} \qquad && 0 \leqq r < \nu - 1 \\ P_{ij} &= 0 \quad \text{unless} \quad j \in S_0 && r = \nu - 1 \end{aligned}$$

(see Feller [2] pages 360–362). Let

$$Y = \{x \mid x_0 \in S_0\}$$

The conditions (2) imply that $Y, \tau Y, \tau^2 Y, \cdots, \tau^{\nu-1} Y$ is a partition of X into disjoint sets.

Now we show that

(3) The induced transformation $\tau_Y = \tau^\nu \mid Y$ is isomorphic to a mixing Markov shift, $\bar{\tau}$.

To constract $\bar{\tau}$, let $\bar{S} = S_0 \times S_1 \times \cdots \times S_{\nu-1}$ (ordered lexicographically) and

$$\bar{P}_{(i_0, i_1, \cdots, i_{\nu-1})(j_0, j_1, \cdots, j_{\nu-1})} = P_{i_{\nu-1} j_0} P_{j_0 j_1} \cdots P_{j_{\nu-2} j_{\nu-1}}$$
$$\bar{\pi}_{(i_0, i_1, \cdots, i_{\nu-1})} = \pi_{i_0} P_{i_0 i_1} \cdots P_{i_{\nu-2} i_{\nu-1}}$$

for $(i_0, i_1, \cdots, i_{\nu-1}), (j_0, j_1, \cdots, j_{\nu-1}) \in \bar{S}$. It is easy to see that $\bar{P}$ is a stochastic matrix with stationary vector $\bar{\pi}$. From (2) it follows that for $i, j \in S_0$ any path $i = i_0, i_1, \cdots, i_{n-1} = j$ must have length divisible by ν, say $n = k\nu$, and each block $(i_{j\nu}, i_{j\nu+1}, \cdots, i_{j\nu+\nu-1}) \in \bar{S}$, $0 \leqq j < k$. Furthermore, the greatest common divisor of such k must be 1. Thus $\bar{P}$ is irreducible and $\gamma(\bar{P}) = 1$ so that the associated Markov shift $\bar{\tau}$ is mixing on $(\bar{X}, \bar{\mathfrak{B}}, \bar{m})$. The mapping $\psi: Y \to \bar{X}$ defined by $(\psi y)_n = (y_{n\nu}, y_{n\nu+1}, \cdots, y_{n\nu+\nu-1})$ is an isomorphism which carries τ_Y onto $\bar{\tau}$. This establishes (3).

We are now in position to construct β and the isomorphism which carries τ

into $\rho_\nu \times \beta$. The transformation τ_Y is isomorphic to a Bernoulli shift (from (3) and Theorem 1), hence τ_Y has roots of all orders, all of which are isomorphic to Bernoulli shifts (Ornstein [4]). Therefore, there is a Bernoulli shift β such that β^ν is isomorphic to τ_Y. Thus there is transformation $\tilde{\beta}$ on Y such that $\tilde{\beta}^\nu = \tau_Y$ and $\tilde{\beta}$ is isomorphic to β. Our proof will be completed by showing that τ is isomorphic to $\rho_\nu \times \tilde{\beta}$.

Let $\varphi : X \to Z_\nu \times Y$ be defined by

$$\varphi(x) = (h(x), \tilde{\beta}^{h(x)} \tau^{-h(x)} x)$$

where $h(x)$ is the integer given by $x \in \tau^{h(x)} Y$. We have $h(\tau x) = h(x) + 1 \pmod{\nu}$. It is easy to see that φ is invertible and measure-preserving. It is an isomorphism between τ and $\rho_\nu \times \tilde{\beta}$ because

$$\varphi\tau(x) = (h(\tau x), \tilde{\beta}^{h(\tau x)} \tau^{-h(\tau x)}(\tau x))$$

and

$$(\rho_\nu \times \tilde{\beta})\varphi(x) = (h(x) + 1 \pmod{\nu}, \tilde{\beta}\tilde{\beta}^{h(x)} \tau^{-h(x)} x) .$$

This completes the proof of Theorem 2.

Two consequences of Theorem 2 are the following:

COROLLARY 1. *Two irreducible Markov shifts are isomorphic if and only if they have the same period and the same entropy.*

COROLLARY 2. *An irreducible Markov shift is isomorphic to its inverse.*

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