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A NOTE ON APPROACHABILITY IN A TWO-PERSON GAME[1]

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In Hou [2] it was shown that the class of approachable sets in a two-person game depended only on the mean vectors of the distributions involved subject to certain moment conditions. In this note we show that this result is, in fact, true whenever the mean vectors exist.

1. Introduction. For the convenience of ready reference the following description of the problem (as well as notation and definitions) is essentially that of Hou [2].

Let $M = ||M(i, j)||$ be an $r \times s$ matrix whose elements $M(i, j)$ are probability distributions in a Euclidean k-space, E^k. We associate with M a game between two players, I and II, with the following infinite sequence of engagements: At the nth engagement, $n = 1, 2, \cdots$, player I selects $i = 1, \cdots, r$ with probability $p_n(i)$, $\sum_{i=1}^r p_n(i) = 1$, and player II selects $j = 1, \cdots, s$ with probability $q_n(j)$, $\sum_{s=1}^j q_n(j) = 1$. Each selection is made without either player knowing the choice of the other player. Once i and j have been chosen, a payoff $Y_n \in E^k$ is then determined according to the distribution $M(i, j)$. The point Y_n and probabilities

$$p_n = (p_n(1), \cdots, p_n(r)) \qquad \text{and} \qquad q_n = (q_n(1), \cdots, q_n(s))$$

are announced to both players after each engagement.

A strategy for player I is a sequence of functions $f = \{f_n\}$, $n = 0, 1, \cdots$, where f_n is defined on the $3n$-tuples $(p_1, q_1, Y_1; \cdots; p_n, q_n, Y_n)$ with value p_{n+1} in

$$P = \{p = (p(1), \cdots, p(r)) : \textstyle\sum_{i=1}^r p(i) = 1 \text{ and } p(i) \geqq 0\} ,$$

and $p_1 = f_0$ is a point of P. For player II, a strategy $g = \{g_n\}$ is similarly defined with

$$g_n(p_1, q_1, Y_1 : \cdots ; p_n, q_n, Y_n) = q_{n+1} \in Q \qquad \text{and} \qquad q_1 = g_0 \in Q ,$$

where

$$Q = \{q = (q(1), \cdots, q(s)) : \textstyle\sum_{j=1}^s q(j) = 1 \text{ and } q(j) \geqq 0\} .$$

For a given M, each pair of strategies determines a sequence $Y_1, Y_2, \cdots$ of random vectors corresponding to the payoffs.

Let $\bar{Y}_n = n^{-1} \sum_{m=1}^n Y_m$ and denote the Euclidean distance between $\bar{Y}_n$ and a nonempty set S in E^k by $\delta(\bar{Y}_n, S)$. For a given M, the set S is said to be approachable by I in M, if there exists a strategy f^* for I such that, for every strategy g for II,

$$\delta(\bar{Y}_n, S) \to 0 \quad \text{a.s.}$$

The above model was introduced by Blackwell [1] at which time the question of whether or not the class of all approachable sets S depended only on mean

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vectors of the $M(i, j)$ was proposed as an unsolved question. Hou [2] has shown (in his Theorem 5) that, under the assumption that the $M(i, j)$ are probability distributions with finite $E\|\cdot\|^\alpha$, for some $\alpha > 1$ ($\|\cdot\|$ denotes Euclidean norm), the class of approachable sets does depend only on the mean vectors of the $M(i, j)$. In this note we will show (in a quite different way) that the above result is true subject only to the condition that the mean vectors exist.

2. The proof. Let $\bar{M} = \|\bar{M}(i, j)\|$ be the $r \times s$ matrix whose elements $\bar{M}(i, j)$ are the mean values of the respective distributions $M(i, j)$. For each pair (i, j) we denote by $w_{i,j}(k)$, $k = 1, 2, \cdots$ the number of the engagement in which the distribution $M(i, j)$ was used, to determine a payoff, for the kth time. Thus, for each pair (i, j), the sequence of random vectors

$$Y_{w_{i,j}(1)}, Y_{w_{i,j}(2)}, \cdots$$

consists of all payoffs (in order of occurrence) generated by distribution $M(i, j)$.

The method of proof to be used is based almost entirely on the following observation. Although the distribution to be chosen at the nth engagement depends on the past, once the distribution is chosen the payoff generated does not depend on the past. An immediate consequence of this is

LEMMA 1. *The sequence of random vectors $Y_{w_{i,j}(1)}, Y_{w_{i,j}(2)}, \cdots$ are independent and identically distributed edch with distribution $M(i, j)$.*

Let, for each pair (i, j), $T_{i,j}(n)$ be the number of times in the first n engagements that the distribution $M(i, j)$ was used to generate a payoff. With this definition, $T_{i,j}(n)$ equals the largest integer k such that $w_{i,j}(k) \leqq n$. Lastly we define

$$B(n) = \{(i, j) : T_{i,j}(n) \geqq 1\} ,$$

that is, the pair (i, j) is in $B(n)$ if and only if distribution $M(i, j)$ has been used to generate at least one of the first n payoffs.

LEMMA 2.

$$\delta(\bar{Y}_n, n^{-1} \sum_{i=1}^r \sum_{j=1}^s T_{i,j}(n)\bar{M}(i, j)) \rightarrow 0 \quad \text{a.s.}$$

PROOF.

$$\begin{aligned} \bar{Y}_n &= n^{-1} \sum_{k=1}^n Y_k \\ &= n^{-1} \sum_{(i,j) \in B(n)} \sum_{k=1}^{T_{i,j}(n)} Y_{w_{i,j}(k)} \qquad (1) \\ &= \sum_{(i,j) \in B(n)} n^{-1} T_{i,j}(n) T_{i,j}^{-1}(n) \sum_{k=1}^{T_{i,j}(n)} Y_{w_{i,j}}(k) , \end{aligned}$$

By Lemma 1 and the strong law of large numbers it follows that if $M_{i,j}$ is used infinitely often, i.e., $T_{i,j}(n) \rightarrow \infty$, then

$$T_{i,j}^{-1}(n) \sum_{k=1}^{T_{i,j}(n)} Y_{w_{i,j}(n)} \rightarrow \bar{M}_{i,j} \quad \text{a.s.}$$

On the other hand, if $M_{i,j}$ is not used infinitely often, i.e., $T_{i,j}(n)$ is bounded,

$$n^{-1} \sum_{k=1}^{T_{i,j}(n)} Y_{w_{i,j}(k)} \rightarrow \mathbf{0} \quad \text{a.s.}$$

where **0** is the vector of zeros. Thus, by (1),

$$\bar{Y}_n - n^{-1} \sum_{(i,j)\in B(n)} T_{i,j}(n)\bar{M}(i,j) \to \mathbf{0} \quad \text{a.s.}$$

The lemma now follows as

$$\sum_{(i,j)\in B(n)} T_{i,j}(n)\bar{M}(i,j) = \sum_{i=1}^{r} \sum_{j=1}^{s} T_{i,j}(n)\bar{M}(i,j) .$$

We may think of $\bar{M}(i,j)$ as a probability distribution putting all its mass on the point $\bar{M}(i,j)$ and $\bar{M}$ as the matrix whose elements are these distributions. Hence we may speak of sets as being approachable by I in $\bar{M}$.

LEMMA 3. *A set S in E^k is approachable by* I *in M if and only if S is approachable by* I *in $\bar{M}$.*

PROOF. Approachability of S by player I in $\bar{M}$ is equivalent to the existence of a strategy f^* for I such that, for every strategy g for II,

$$\delta(n^{-1} \sum_{i=1}^{r} \sum_{j=1}^{s} T_{i,j}(n)\bar{M}(i,j), S) \to 0 \quad \text{a.s.} \tag{2}$$

Say S is approachable by I in M and say (while playing the game with M) that to achieve $\delta(\bar{Y}_n, S) \to 0$ a.s. player I can use f^{**}. When playing the game with $\bar{M}$ player I can do the following:

(i) at the first engagement use f_0^{**}.

(ii) when the nth payoff ($n = 1, 2, \cdots$), Y_n, is announced (it will be an element of the matrix $\bar{M}$) chose a distribution function from the elements of M having Y_n as its mean. Use this distribution to generate a random variable Z_n.

(iii) at the $(n+1)$th engagement ($n = 1, 2, \cdots$), replace $Y_1, \cdots, Y_n$ by $Z_1, \cdots, Z_n$ and use strategy f_n^{**} (which is defined on $(f_0^{**}, q_1, Z_1, \cdots, f_{n-1}^{**}, q_n, Z_n)$). By Lemma 2 and the definition of f^{**} (2) must hold for the strategy just described.

Conversely, if S is not approachable by I in M, then for every strategy f for I there exists a strategy g_f for II such that $\delta(\bar{Y}_n, S)$ does not converge to zero almost surely. In this case player II, when playing the $\bar{M}$ game, would generate the quasi-payoffs and then use g_f on them. Thus S is not approachable by I in $\bar{M}$.

THEOREM. *The class of sets in E^k which are approachable by* I *in M depends only on $\bar{M}$.*

PROOF. Let M_1 and M_2 be any two $r \times s$ matrices whose elements are probability distributions in E^k such that $\bar{M}_1 = \bar{M}_2$. The proof now follows, as by Lemma 3, S is approachable by I in $M_1 \Leftrightarrow S$ is approachable by I in $\bar{M}_1 (= \bar{M}_2) \Leftrightarrow S$ is approachable by I in M_2

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