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MOMENTS OF OSCILLATIONS AND RULED SUMS[1]

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0. Introduction. Let $\{X_i\}$ be a sequence of independent identically distributed random variables for which $EX_1 = 0$ if it exists and $S_n \equiv \sum_{i=1}^n X_i$. Let $\{b(n)\}$ be a sequence of real numbers and $N_\infty(b(n)) \equiv$ number of times $|S_n| \geqq b(n)$.

In [4] Slivka and Severo show that if $b(n) \equiv \delta n$, then for $\beta \geqq 1$, $E(X^{\beta+1}) < \infty$ implies $E[N_\infty{}^\beta(\delta n)] < \infty$ for all $\delta > 0$. Their proof of this result depends heavily on a paper of Katz [1]. We will show here how the use of symmetrization and a fuller use of the above mentioned paper of Katz, not only gives the converse of the above result, but more generally gives:

RESULT 1. For $\beta \geqq 1$ and $\alpha > 0$; $E(X^{(\beta+1)/\alpha}) < \infty$ iff $E[N^\beta(\delta n^\alpha)] < \infty$ for all $\delta > 0$.

In an earlier paper [3], Slivka showed that if $EX_1^2 < \infty$ and $EX_1 = 0$, and if one chooses the sequence $\{b(n)\}$ that appears in the law of the iterated logarithm, i.e., $b(n) = [(1+\delta)2n \log\log n]^{\frac{1}{2}}$, then $N_\infty(b(n))$ has no moments for all $\delta > 0$. A perusal of the proof used shows that he actually proved that if $b(n)/(2n \log n)_n^{\frac{1}{2}} \to 0$, then $N_\infty(b(n))$ has no moments for all $\delta > 0$. We will show here that the sequence $b(n) \equiv (2(1+\delta)n \log n)^{\frac{1}{2}}$ is more sensitive to the moments of $N_\infty(b(n))$ than any of the above mentioned sequences. More precisely:

RESULT 2. If X_1 is symmetric, if $E(X^{2m}) < \infty$ with $m \geqq 1$, and $E(X) = 0$, then:

$$E[N_\infty{}^\beta(2(1+\delta)n \log n)^{\frac{1}{2}}] < \infty \qquad \text{if} \quad 1 \leqq \beta < \min(m, 1+\delta)\,;$$
$$= \infty \qquad \text{if} \quad \beta > 1+\delta\,.$$

Finally we will let $(\ ) : I^+ \to 2^{I^+}$, where (n) is a subset of I^+ with n elements, be called a rule and its corresponding ruled sum be defined by $S_{(n)} \equiv \sum_{i \varepsilon (n)} X_i$. We will show the moments of the function corresponding to $N_\infty(b(n))$, very much depends on the "rule" ( ).

1. On S_n. The only if part of Result 1 and the boundedness part of Result 2 are obtained by using the argument of Slivka and Severo in [4], with the following modifications:

RESULT 1. Use Theorem 3 instead of Theorem 1 from Katz [1].

RESULT 2. Incorporate the fact

$$(***) \qquad E|X|^{2+\varepsilon} < \infty \quad \text{implies} \quad \sup_x |\Phi(x) - P[S_n \leqq x]| < n^{-\varepsilon/2}$$
for all $x > 0$,

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where $\Phi(\cdot)$ is the cummulative normal distribution, along with Theorem 3 of Katz [1].

We will now establish the other parts of Results 1 and 2.

Let I_A be the indicator of the event A, and let

$$N_m(b(n)) \equiv \sum_{j=1}^{m} I_{[|s_j| \leq b(j)]} \, .$$

$N_m(b(n)) \uparrow {}_m N_\infty(b(n))$ and thus by the monotone convergence theorem, $\lim {}_m E[N_m{}^\beta(b(n))] = E[N_\infty{}^\beta(b(n))]$ for all $\beta > 0$. (Convergence to ∞ is allowed.)

Let $X_i{}^{(s)} = X_i - X_i'$ be a symmetrization of X_i ([2] page 247) (We adopt the convention that all quantities relative to X_i' and $X_i{}^{(s)}$ will be denoted by $'$ and ${}^{(s)}$ respectively.)

LEMMA 1. *Let β be a positive integer, then for $b \geq 0$,*

$$\begin{aligned} 2^{\beta+1} E[N_m{}^\beta(b(n))] E[N_m{}^\beta(b(n)) I_{[|S_m| \geq b]}] \\ + 2E[N_m{}^\beta(b(n))] P[|S_m| \geq b] \\ \geq E[(N_m{}^{(s)}(2b(n)))^\beta I_{[|S_m{}^{(s)}| \geq 2b]}] \, . \end{aligned}$$

PROOF. Since $I_{[|S_j| \geq a]} + I_{[S_j'| > a]} \geq I_{[|S_j{}^{(s)}| > 2a]}$ we have:

$$\begin{aligned} &E[(N_m{}^{(s)}(2b(n)))^\beta I_{[|S_m{}^{(s)}| \geq 2b]}] \\ &\quad \leq \sum_{j=0}^{\beta} \tbinom{\beta}{j} E[N_m{}^{\beta-j}(b(n))(N_m'(b(n)))^j \cdot (I_{[|S_m| \geq b]} + I_{[|S_m'| \geq b]})] \, . \end{aligned}$$

Now using the fact that the primed and nonprimed quantities are independent and identically distributed completes the proof.

Let $[\cdot]$ be the largest integer function and I^+ be the set of positive integers.

LEMMA 2. *Let $\beta \geq 1$. Then $E[N_\infty{}^\beta(b(n))] < \infty$ implies*

(i) *for $\beta \notin I^+$,* $\quad \sum_{m=1}^{\infty} m^{\beta-[\beta]-1} E[N_m{}^{[\beta]}(b(n)) I_{[|S_m| \geq b_m]}] < \infty$;

(ii) *for $\beta \in I^+$,* $\quad \sum_{m=1}^{\infty} E[N_m{}^{\beta-1}(b(n)) I_{[|S_m| \geq b_m]}] < \infty$.

PROOF. Letting $N_0 \equiv 0$, we see $N_{m+1}^\beta(b(n)) \equiv \sum_{i=0}^{m}(N_{i+1}^\beta(b(n)) - N_i^\beta(b(n)))$. But each term of this sum is positive and thus

$$(**) \qquad \begin{aligned} &E[N_{m+1}^\beta(b(n))] \\ &\quad \geq \sum_{i=0}^{m} E\{((N_i(b(n)) + 1)^\beta - N_i{}^\beta(b(n))) I_{[|S_i| \geq b(i)]}\} \, . \end{aligned}$$

$\beta \notin I^+$. Letting $\alpha \equiv \beta - [\beta]$, we see $\min_{1 \leq h \leq i}((h+1)^\alpha - h^\alpha) = (i+1)^\alpha - i^\alpha$ because $\alpha < 1$. But the mean value theorem shows $(i+1)^\alpha - i^\alpha \geq \alpha i^{\alpha-1}$, and thus we can use the fact that $N_i(b(n)) \leq i$ to see

$$\begin{aligned} &E\{((N_i(b(n)) + 1)^\beta - N_i{}^\beta(b(i))) I_{[|S_{i+1}| \geq b(i+1)]}\} \\ &\qquad \geq (\beta - [\beta]) E\{N_i{}^{(\beta)}(b(n)) i^{\beta-[\beta]-1} I_{[|S_{i+1}| \geq b(i+1)]}\} \, . \end{aligned}$$

Substituting this into (**) and applying the monotone convergence theorem, we obtain (i).

$\beta \in I^+$. The binominal theorem applied to (**) gives

$$\begin{aligned} E[N_{m+1}^\beta(b(n))] &= \sum_{i=1}^{m} \left(\sum_{j=0}^{\beta-1} \tbinom{\beta}{j} E[N_i{}^j(b(n)) I_{[|S_i| \geq b(i)]}\right) \\ &\geq \sum_{i=1}^{m} E[N_i{}^{\beta-1}(b(n)) I_{[|S_i| \geq b(i)]}] \end{aligned}$$

which combined with the monotone convergence theorem gives (ii).

LEMMA 3. *If X_1 is a symmetric random variable and $\beta \in I^+$, then for $0 < \alpha < 1$, there exists a constant $C(\alpha) > 0$ $m_\alpha \in I^+$ so that for $m \geqq m_\alpha$*

$$E[N_m{}^\beta(b(n))I_{[|S_m| \geqq b_m]}] \geqq C(\alpha)m^\beta P[S_{[\alpha m]} \geqq b_m] .$$

PROOF. Let $l \in I^+$ and $i_0 \leqq i_1 \leqq i_2 \leqq \cdots i_l$, then

$$I_{[\bigcap_{j=1}^l [|S_{i_j}| \geqq C]]} \geqq I_{[S_{i_0} \geqq C]} I_{[\bigcap_{j=1}^l [S_{i_j} - S_{i_{j-1}} \geqq 0]]} + I_{[S_{i_0} \leqq -C]} I_{\bigcap_{j=1}^l [S_{i_j} - S_{i_{j-1}} \leqq 0]} .$$

The differences $[S_{i_j} - S_{i_{j-1}}]_{j=1}^l$ are symmetric independent random variables since the X_i's are, and so we have

$$(*) \qquad E[\textstyle\bigcap_{j=1}^l I_{[|S_{i_j}| \geqq C]}] = P[\bigcap_{j=1}^l [|S_{i_j}| \geqq C]] \geqq 2^{-l} P[|S_{i_0} \geqq C] .$$

Letting $1 > r > \alpha$, and noting that there are at most β distinct factors in each term of the binominal expansion of $(x_1 + x_2 + \cdots + x_n)^\beta$, we see by (*)

$$E[N_m{}^\beta(b(n))I_{[|S_m| \geqq b(m)]}] \geqq E[(\textstyle\sum_{j=[\alpha m]}^m I_{[|S_i| \geqq b(m)]})^\beta I_{[|S_m| \geqq b(m)]}]$$
$$\geqq 2^{-\beta}(1 - r - m^{-1})^\beta m^\beta P[|S_{[\alpha m]}| \geqq b(m)]$$

and so the lemma holds.

We now establish the if part of Result 1.

For X_1 symmetric, Lemma 2 and Lemma 3 show that $E[N_\infty{}^\beta(b(n))] < \infty$ implies $\sum_{m=1}^\infty m^{\beta-1} P[|S_{(\alpha m)}| \geqq b(m)]$ for $0 < \alpha < 1$. But because $\{[\alpha m]\}_m = I^+$, and $b(n) = \alpha n^\alpha$, we have by Theorem 3 of Katz [1], $E[|X|^{\beta+1/\alpha}] < \infty$.

If X_1 were not symmetric, then one considers the symmetrization of X_1 and notes that by Lemma 1, Lemma 2's conclusion holds for $N_m{}^{(s)}(b(n))$ in place of $N_m(b(n))$. Proceeding now as above we get $E[|X^{(s)}|^{\beta+1/\alpha}] < \infty$. But $E[|X^{(s)}|^r] < \infty$ implies $E[|X|^r] < \infty$ for all $r > 0$ and so the proof of Result 1 is complete.

The unbounded part of Result 2 is proved by noting that Lemma 2 and Lemma 3 imply for some number C

$$E[N_\infty{}^\beta(2(1+\delta)n \ln n)^{\frac{1}{2}}] \geqq C \textstyle\sum_{m=1}^\infty m^{\beta-1} P[|m^{-\frac{1}{2}} S_m| \geqq (2\alpha(1+\delta) \ln [\alpha m])^{\frac{1}{2}}] .$$

But this and (***) show there is a constant C' for which

$$E[N_\infty{}^\beta(2(1+\delta)n \ln n)^{\frac{1}{2}}] \geqq C' \textstyle\sum_{m=1}^\infty m^{\beta-1} m^{\alpha(1+\delta)}$$

and so the result follows.

2. On ruled sums. Let ( ) be a rule (refer back to the end of the Introduction) and let $N_m(b(n), (n)) \equiv$ number of times $|S_{(n)}| \geqq b(n)$ for $n \leqq m$, $(m = 1, 2, 3, \cdots, \infty)$. The rule $(n) \equiv \{1, 2, 3, \cdots, n\}$ will be denoted by n, and $\langle n \rangle$ will denote a rule for which $\langle n \rangle \cap \langle m \rangle = \varnothing$ if $n \neq m$ (i.e., $\{S_{\langle n \rangle}\}$ is a sequence of independent random variables).

It is clear that $E[N_\infty(b(n))] = E[N_\infty(b(n), (n))]$ for all rules ( ). However, higher moments can behave quite differently. For instance by Result 1 we see the existence of higher moments of $N_\infty(b(n))$ depends on how many moments

X_1 has, but this is not the case for $\langle n\rangle$. To see this, note that by (**) and by the independence between the sums,

$$E[N_\infty^2(b(n), \langle n\rangle)] = \lim_{m\to\infty} \textstyle\sum_{j=1}^m E[N_j(b(n), \langle n\rangle)]P[|S_j| \geqq b(n)]$$
$$\leqq E^2[N_\infty(b(n), \langle n\rangle)]$$

and, continuing by induction, one sees that

LEMMA 4. *If* $E[N_\infty(b(n))] < \infty$, *then* $E[N_\infty^\beta(b(n), \langle n\rangle)] < \infty$ *for all* $\beta \in I^+$.

The dichotomy in behavior between the rules n and $\langle\cdot\rangle$ illustrated by Lemma 4 and Result 1, combined with the fact that $S_{\langle n\rangle}$ unlike S_n, has "no memory of previous sums," gives credence to the following notion:

$N_\infty(b(n))$ is more likely to be determined by the number of consecutive times S_n is large as opposed to the number of "new" times S_n becomes large.

Reconsidering the proof used in the "only if" part of Result 1, one sees that it holds for all rules ( ). By Lemma 4 though, we see the converse certainly does not hold for all rules. So we will close this section by showing that if X_1 has at least two moments and $\alpha > 0$, then for each relation R (not outlawed by the above observation) between the moments of X_1 and the moments of $N(\varepsilon n^\alpha, (n))$, there is a rule ( ) so that R holds. More precisely we will show

RESULT 3. If $\beta \in I^+$, $2 \leqq r \leqq \alpha^{-1}(\beta + 1)$, and X_1 is symmetric, then there is a rule ( ) so that

$$E|X|^r < \infty \text{ iff } E[N^\beta(\delta n^\alpha, (n))] < \infty \qquad \text{for all} \quad \delta > 0 .$$

For convenience of exposition, we will indicate the construction of ( ) for $\alpha = 1$ and $\beta = 2$; however, the other cases are either just a matter of taking moments of this ( ) or constructing a rule in an analogous fashion to the way ( ) is constructed. So let $2 \leqq r \leqq 3$, $l \equiv r - 1$, $\{\pi_t\}_{t=1}^\infty$ be the portion of I^+ given by $\pi_t \equiv \{j \in I^+ : [t^l] \leqq j < [t^{l+1}]\}$, and let ( ) be any rule for which the following properties hold:

PROPERTY a. $(n) \cap (m) = \varnothing$ if $n \in \pi_t$, $m \in \pi_{t'}$, and $t \neq t'$

PROPERTY b. $(n) - (n-1)$ is a singleton set if $n, n-1 \in \pi_t$ for some $t \in I^+$.

For convenience of notation we let $v(t) \equiv [t^l]$, $d(t) \equiv v(t+1) - v(t)$, $h(t) = v(t) + 2^{-1}d(t)$, and $K = \sup_t v(t+1)/v(t) < \infty$.

We will show that for such a rule there are constants a, d, and f such that if $m = [l(t_0)]$ for some $t_0 \in I^+$, then

$$(+) \qquad a \textstyle\sum_{j=1}^m j^{l-1}P[|S_j| \geqq K^2\varepsilon j] \leqq C(m) \equiv E[N_m^2(\varepsilon n, (n))]$$
$$\leqq d \textstyle\sum_{j=1}^m j^{l-1}P[|S_j| \geqq \varepsilon j] + f .$$

Once these inequalities are established, then one only need let $t_0 \to \infty$, and refer once again to Theorem 3 of Katz [1] in order to prove our result. To show the above set of inequalities hold, we

(i) note by the mean value theorem that $l(t+1)^{l-1} \geqq d(t)$ and $h(t) \geqq lt^{l-1}$ and thus for $j \in \pi_t$: $d(t) \leqq l2^l j^{l-1}$ and $h(t) \geqq l2^{1-l}j^{l-1}$

(ii) define $D(m)$ by

$$C(m) = \sum_{t=1}^{t_0} \sum_{j=l(t)}^{l(t+1)} \sum_{k=l(t)}^{l(t+1)} P[|S_j| \geqq \varepsilon j] \cap [|S_k| \geqq \varepsilon k] + D(m)$$

and, by noting that $D(m) \uparrow$ and $D(m) \leqq E(N_m^2(\varepsilon n, \langle n \rangle))$, see that Lemma 4 gives $D(m)_m \to f$, where f is some number $< \infty$.

Combining (i) and (ii), we easily see that the righthand inequality of $(+)$ holds with $d \equiv l \cdot 2^{l-1}$.

To establish the lefthand inequality of $(+)$ we first note that (ii) and (*) imply

$$C(m) \geqq \sum_{t=1}^{t_0} \sum_{j=l(t)}^{h(t)} 4^{-1} h(t) P[|S_j| \geqq \varepsilon l(t+1)] .$$

Second we note by (i) and the definition of K

$$\sum_{j=l(t)}^{n(t)} h(t) P[|S_j| \geqq \varepsilon l(t+1)] \geqq \sum_{j=l(t)}^{h(t)} (l 2^{1-l}) j^{l-1} P[|S_j| \geqq (\varepsilon K) j]$$

and

$$\sum_{j=l(t+1)}^{h(t+1)} h(t+1) P[|S_j| \geqq \varepsilon l(t+2)] \geqq \sum_{j=h(t)}^{l(t+1)} (l 2^{1-l}) j^{l-1} P[|S_j| \geqq (K^2 \varepsilon) j] .$$

Thus the lefthand inequality of $(+)$ holds with $2 \equiv l 8^{-1} 2^{1-l}$.

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