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SUBJECTIVE EXPECTED UTILITY WITH MIXTURE SETS AND BOOLEAN ALGEBRAS

BY PETER C. FISHBURN

Research Analysis Corporation

1. Introduction. A standard version of the subjective expected-utility formulation includes a set C of consequences, a set S of states of the world, and a set of "acts" which are functions on S to C or on S to a set that includes C. With X denoting the set of elements into which the states are mapped by the acts, and letting f, g denote acts, an individual's preference relation $\succ$ on the acts is assumed to have certain properties that imply the existence of a real-valued function u on X and a finitely-additive probability measure P on an algebra $\mathscr{S}$ of subsets of S such that, for certain pairs of acts,

$$f \succ g \text{ iff } \int u(f(s))\,dP(s) > \int u(g(s))\,dP(s)\,. \tag{1}$$

In Savage's theory [5, 7] $X = C$, the acts are all functions on S to C, $\mathscr{S}$ is the set of all subsets of S, and (1) holds for all pairs of acts with u bounded and P such that if $A \subseteqq S$ and $0 \leqq \alpha \leqq 1$ then $P(B) = \alpha P(A)$ for some $B \subseteqq A$.

To avoid such a strong property for P, Fishburn [4] uses "extraneous" probabilities to form the set $\mathscr{P}$ of simple probability measures on C and takes $X = \mathscr{P}$. $\mathscr{S}$ is the set of all subsets of S, and the axioms in [4] imply (1) for all pairs of acts with $u(x) = \sum_c u(c)x(c)$ when $x \in \mathscr{P}$. No special properties arise for P, and u may or may not be bounded, depending in part on the nature of P.

This paper generalizes the approach of [4] in several ways. First, $\mathscr{P}$ is replaced by an arbitrary mixture set X as defined by Herstein and Milnor [6]. Second, we shall carry out the analysis with an arbitrary Boolean algebra $\mathscr{S}$ of subsets of S. This change requires special considerations for acts (functions on S to X) and special properties, including measurability as defined in Section 6, must be considered in connection with (1). A third change from [4] is that the final axiom used there is considerably weakened here without affecting the conclusions of the theory. In terms that are clarified later, our main conclusions from the axioms are that (1) holds for all pairs of acts in the convex closure of the set of "measurable" acts, and that every such act is "bounded." In addition to our main axioms we shall comment on a preference axiom that implies that P in (1) is countably additive.

Perhaps the best introduction to our use of an arbitrary Boolean algebra $\mathscr{S}$ rather than the set of all subsets of S is pages 8–10 in Dubins and Savage [1]. As they note, the use of the largest algebra avoids problems of measurability and integrability that arise in our discussion. However, they recognize the difficulty an individual may have in trying to visualize certain sets in this algebra.

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The use of a judiciously chosen $\mathscr{S}$ can alleviate this difficulty. With this in mind we proceed to our formulation.

2. Mixture sets and formulation. A *mixture set* [6] is a nonempty set M and a function that assigns an element $\alpha f + (1 - \alpha)g$ in M to each $\alpha \in [0, 1]$ and $(f, g) \in M \times M$ such that, for all $f, g \in M$ and $\alpha, \beta \in [0, 1]$:

M 1. $1f + 0g = f$,
M 2. $\alpha f + (1 - \alpha)g = (1 - \alpha)g + \alpha f$,
M 3. $\alpha[\beta f + (1 - \beta)g] + (1 - \alpha)g = \alpha\beta f + (1 - \alpha\beta)g$,
M 4. $\alpha f + (1 - \alpha)f = f$.

M 4 is implied by M 1, M 2 and M 3. Any nonempty interval of real numbers is a mixture set when $\alpha f + (1 - \alpha)g$ is interpreted in the usual way. If M is a set of probability measures on a fixed algebra of subsets of some nonempty set and if M is closed under direct convex combinations of measures then M is a mixture set.

An *Archimedean weak ordered mixture set* is a mixture set M and a binary relation $\succ$ on M that satisfies the following three axioms for all $f, g, h \in M$:

A 1. *$\succ$ on M is a weak order: that is, $\succ$ is asymmetric, and $f \succ g \Rightarrow f \succ h$ or $h \succ g$.*

A 2. *$f \succ g$ and $0 < \alpha < 1 \Rightarrow \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$.*

A 3. *$f \succ g$ and $g \succ h \Rightarrow \alpha f + (1 - \alpha)h \succ g$ for some $\alpha \in (0, 1)$, and $g \succ \beta f + (1 - \beta)h$ for some $\beta \in (0, 1)$.*

A real-valued function w on M is linear iff $w(\alpha f + (1 - \alpha)g) = \alpha w(f) + (1 - \alpha)w(g)$ for all $(\alpha, f, g) \in [0, 1] \times M^2$. The following theorem, a complete proof of which is given in [5] (pages 110–115), is similar to the von Neumann–Morgenstern expected-utility theorem [9].

THEOREM 1. *Suppose that $(M, \succ)$ is an Archimedean weak ordered mixture set. Then there is a linear real-valued function w on M such that*

$$f \succ g \text{ iff } w(f) > w(g) \qquad \text{for all} \quad f, g \in M .$$

Moreover, a linear real-valued function w' on M satisfies this in place of w iff *there are numbers $a > 0$ and b such that $w'(f) = aw(f) + b$ for all $f \in M$.*

The final sentence in this theorem is often abbreviated by saying that w (linear and order-preserving) is unique up to a positive linear transformation.

We now turn to the formulation that will be used throughout the paper. First, X, with elements $x, y, \cdots$, will denote a mixture set associated directly with the consequences. In [4], X was taken as the set of simple probability measures on the consequences. In the case where the consequences constitute an interval of real numbers (such as when the consequences are dollar amounts) X could be this interval with $\alpha x + (1 - \alpha)y$ interpreted in the usual way. Under this interpretation our axioms imply that the utility of x is directly proportional to x.

Although this is often unrealistic it is frequently used in decision-theoretic studies (such as [1]) to simplify the analysis.

Second, with S the set of states of the world [7], $\mathscr{S}$ will denote a Boolean algebra of subsets of S. The actual choice of $\mathscr{S}$ in a particular situation may depend on the ability of the individual to visualize various subsets of S and on certain considerations of preference. Elements in $\mathscr{S}$ will be denoted as $A, B, \cdots$. A^c is the *complement* of A.

We shall call $\mathscr{A}$ an *$\mathscr{S}$-partition* iff $\mathscr{A}$ is a set of mutually disjoint subsets of S each of which is in $\mathscr{S}$, and $\bigcup_{\mathscr{A}} A = S$.

The largest class of "acts" that we shall consider is

$$F = \{f\colon f \text{ is a function on } S \text{ to } X \text{ and there is an } \mathscr{S}\text{-partition on each element of which the function is constant}\}.$$

If $\mathscr{S}$ contains every unit subset of S then F is the set of all functions on S to X. Otherwise there may be non-measurable functions on S to X that are not in F, but these would apparently be of no real interest. (If they were then $\mathscr{S}$ could be enlarged to account for them.)

With $\alpha f + (1 - \alpha)g$ the function that assigns $\alpha f(s) + (1 - \alpha)g(s)$ in X to state s, it follows that F is a mixture set. In later sections we shall discuss several subsets of F that are also mixture sets.

For $f, g \in F$, $f = g$ on A iff $f(s) = g(s)$ for all $s \in A$. For $f \in F$ and $x \in X$, $f = x$ on A iff $f(s) = x$ for all $s \in A$. $\bar{x}$ is the "act" in F for which $\bar{x}(s) = x$ on S. That is, $\bar{x}$ is a constant "act."

The *preference relation* $\succ$ will be applied to F or a subset of F. *Indifference* $\sim$ is defined by $f \sim g$ iff not $f \succ g$ and not $g \succ f$. We shall let $\mathscr{N}$ denote the "null" events in $\mathscr{S}$, defined as follows:

$$\mathscr{N} = \{A\colon A \in \mathscr{S} \text{ and } f \sim g \text{ whenever } f, g \in F \text{ and } f = g \text{ on } A^c\}.$$

The preference relation $\succ$ will be used with elements in X as well as F under the following definitions: $x \succ f$ iff $\bar{x} \succ f$, $x \succ y$ iff $\bar{x} \succ \bar{y}$ and so forth.

3. Simple acts. With $F' \subseteqq F$ suppose that F' is a mixture set. Then $(F', \succ)$ is *regular* if and only if the following hold throughout F':

A4. *$x \succ y$ for some $x, y \in X$,*

A5. *$(A \in \mathscr{S} - \mathscr{N}$ and $f = x$ on A and $g = y$ on A and $f = g$ on $A^c) \Rightarrow (f \succ g$ iff $x \succ y)$.*

These are similar to A4 and A5 in [4] and to P5 and P3 in [7].

Let F_0 be the simple acts in F so that

$$F_0 = \{f\colon f \in F \text{ and } f \text{ is constant on each element in some finite } \mathscr{S}\text{-partition}\}.$$

THEOREM 2. *Suppose that $F_0 \subseteqq F' \subseteqq F$ and that $(F', \succ)$ is a regular Archimedean weak ordered mixture set. Then there is a linear real-valued function u on X and a* *finitely-additive probability measure P on $\mathscr{S}$ such that, for all $f, g \in F_0$ and $A \in \mathscr{S}$,*

$$f \succ g \quad \text{iff} \int u(f(s))\, dP(s) > \int u(g(s))\, dP(s)\,, \tag{2}$$

$$P(A) = 0 \quad \text{iff}\ A \in \mathscr{N}\,. \tag{3}$$

Moreover, u' (linear) on X and P' on $\mathscr{S}$ satisfy (2) *in place of u and P* iff $P' = P$ *and $u' = au + b$ with $a > 0$.*

PROOF. Much of this proof is essentially the same as the proofs indicated for Theorems 2 and 3 in [4]. However, the early part of the derivation of (2) requires some comment.

Let $\{B_1, \cdots, B_n\}$ be a finite $\mathscr{S}$-partition and let F_B be the subset of F_0 of functions constant on each element of this partition. $(F_B, \succ)$ is easily seen to be an Archimedean weak ordered mixture set. Hence, by Theorem 1 there is a linear real-valued function w on F_B such that

$$f \succ g \quad \text{iff}\ w(f) > w(g) \qquad \text{for all} \quad f, g \in F_B\,. \tag{4}$$

We represent $f \in F_B$ by the n-tuple $(f(B_1), \cdots, f(B_n))$. Fix $y \in X$, define $w_i(y)$ for $i = 1, \cdots, n$ so that $\sum_{i=1}^n w_i(y) = w(\bar{y})$, and define w_i on X by

$$w_i(x_i) = w(y, \cdots, y, x_i, y, \cdots, y) - \textstyle\sum_{j \neq i} w_j(y)\,.$$

Summation gives $\sum w_i(x_i) = \sum w_i(y, \cdots, y, x_i, y, \cdots, y) - (n-1)w(\bar{y})$.

Using M 2 and M 4 on the mixture set X (for each component $i < n$), $\frac{1}{2}(x_1, \cdots, x_i, y, \cdots, y) + \frac{1}{2}(y, \cdots, y, x_{i+1}, y, \cdots, y) = \frac{1}{2}(x_1, \cdots, x_i, x_{i+1}, y, \cdots, y) + \frac{1}{2}\bar{y}$. Since the two sides of this are equal, they are indifferent, and hence have the same w value from (4). Linearity of w then gives $w(x_1, \cdots, x_i, y, \cdots, y) + w(y, \cdots, y, x_{i+1}, y, \cdots, y) = w(x_1, \cdots, x_{i+1}, y, \cdots, y) + w(\bar{y})$. Summing this from 1 to $n-1$ and cancelling gives $w(x_1, \cdots, x_n) = \sum w(y, \cdots, y, x_i, y, \cdots, y) - (n-1)w(\bar{y})$. Comparing this with the result of the preceding paragraph we see that

$$w(x_1, \cdots, x_n) = \textstyle\sum_{i=1}^n w_i(x_i) \qquad \text{for all} \quad (x_1, \cdots, x_n) \in F_B\,. \tag{5}$$

Suppose that $B_i \in \mathscr{S} - \mathscr{N}$. Then by A 5, (4) and (5),

$$x \succ y \quad \text{iff}\ w_i(x) > w_i(y)\,, \qquad \text{for all} \quad x, y \in X\,.$$

By linearity and (5)

$$\begin{aligned} w(\alpha(y, \cdots, y, x, y, \cdots, y) &+ (1-\alpha)(y, \cdots, y, z, y, \cdots, y)) \\ &= \alpha w_i(x) + (1-\alpha) w_i(z) + \textstyle\sum_{j \neq i} w_j(y)\,. \end{aligned}$$

Using (5), $w(\alpha(y, \cdots, y, x, y, \cdots, y) + (1-\alpha)(y, \cdots, y, z, y, \cdots, y)) = w_i(\alpha x + (1-\alpha)z) + \sum_{j \neq i} w_j(\alpha y + (1-\alpha)y)$. Since $y = \alpha y + (1-\alpha)y$ by M 4, $w_i(\alpha x + (1-\alpha)z) = \alpha w_i(x) + (1-\alpha)w_i(z)$ and therefore w_i is linear.

Thus, it follows from Theorem 1 (applied to X) that if $B_i, B_j \in \mathscr{S} - \mathscr{N}$ then w_i is a positive linear transformation of w_j. By methods indicated previously [3, 4] there is a linear real-valued function u_B on X and nonnegative numbers

$P_B(B_i)$ that sum to one such that, for all $f, g \in F_B$,

$$f \succ g \quad \text{iff } \sum_{i=1}^{n} P_B(B_i)u_B(f(B_i)) > \sum_{i=1}^{n} P_B(B_i)u_B(g(B_i)),$$

with $P_B(B_i) = 0$ iff $B_i \in \mathscr{N}$.

The rest of the proof is similar to the proof of Theorem 3 in [4].

4. Countable additivity. Before examining the extension of (2) to more general acts we note a preference axiom that implies that P in (2) is countably additive. Since $\mathscr{S}$ is usually assumed to be a σ-algebra when countable additivity is considered, we shall follow this tradition.

ACA (*Axiom for countable additivity*). *Suppose that each A_i is in $\mathscr{S}$, that $A_1 \subseteqq A_2 \subseteqq \cdots$, that $A = \bigcup_{i=1}^{\infty} A_i$ and $B \in \mathscr{S}$, that $x \succ y$ and that*

$$\begin{aligned} &f_n = x \ \text{ on } \ A_n, && f_n = y \ \text{ on } \ A_n^c, && \text{for } \ n = 1, 2, \cdots \\ &f_A = x \ \text{ on } \ A, && f_A = y \ \text{ on } \ A^c, \\ &f_B = x \ \text{ on } \ B, && f_B = y \ \text{ on } \ B^c. \end{aligned}$$

Then if $f_A \succ f_B$ there is some n such that $f_n \succ f_B$.

Within the setting of Theorem 2, each f_n and each of f_A and f_B is in F_0. Suppose that $A_1, A_2, \cdots$ is an increasing $\mathscr{S}$-sequence with limit $A \in \mathscr{S}$ (by σ-algebra) and that $B \in \mathscr{S}$ and $P(A) > P(B)$. Then, since $x \succ y$ for some $x, y \in X$ by previous assumption (A4), we can structure f_n, f_A and f_B as in ACA. Since $P(A) > P(B)$ iff $f_A \succ f_B$ and $P(A_n) > P(B)$ iff $f_n \succ f_B$ by (2), it follows from ACA that $P(A_n) > P(B)$ for some n. It then follows from Theorems 1 (page 1789) and 2 (page 1794) in Villegas [8] that P is countably additive.

Axiom ACA is not devoid of intuitive appeal. To paraphrase Feller ([2] page 106), when $A_1 \subseteqq A_2 \subseteqq \cdots$ and $A = \bigcup A_i$, one could argue for sufficiently large n that A_n is practically indistinguishable from A. If $(x, y) =$ (win \$1000, win nothing) and the individual would rather bet on the occurrence of A than on B for the \$1000, then ACA requires that, for some sufficiently large n, he would rather bet on A_n than on B for the \$1000.

A somewhat gloomy and admittedly porous example for the failure of ACA runs as follows. $A_n =$ "man will be extinct by day n", $A =$ "man will someday be extinct", and $B =$ "one flip of this penny will yield 'heads up' ". It seems conceivable that a person could consider A more probable than B and yet find B more probable than A_n for each n.

On balance, Savage ([7] page 43) seems to have a reasonable attitude toward an axiom such as ACA. He feels that it ought not to be adopted outright for all situations, although there may be special situations where it is applicable and in which its implications can be put to good use in decision analysis.

5. A bounding lemma. Throughout the rest of this paper u on X and P on $\mathscr{S}$ are assumed to have the properties specified in Theorem 2. We shall assume in this section that $(F, \succ)$ is a regular Archimedean weak ordered mixture set.

Then, along with (2) for F_0, we know from Theorem 1 that there is a linear real-valued function v on F such that

$$f \succ g \quad \text{iff}\ v(f) > v(g) \qquad \text{for all} \quad f, g \in F. \tag{6}$$

The restriction of v on F_0 is also linear and satisfies $f \succ g$ iff $v(f) > v(g)$, for $f, g \in F_0$. Defining $u(f) = \int u(f(s))\, dP(s)$ using (2) for $f \in F_0$, u also is linear and satisfies $f \succ g$ iff $u(f) > u(g)$, for $f, g \in F_0$. Since F_0 is an Archimedean weak ordered mixture set, it follows from Theorem 1 that v on F_0 is a positive linear transformation of u on F_0. Therefore, with no loss in generality we can specify that

$$v(f) = \int u(f(s))\, dP(s) \qquad \text{for all} \quad f \in F_0 \tag{7}$$

along with (2) for F_0 and (6).

The question then is whether $v(f) = \int u(f(s))\, dP(s)$ for $f \in F - F_0$. To approach this question we shall use one more axiom, namely

A 6. *If $f(s) \succ x$ for all $s \in S$ then not $x \succ f$. If $x \succ f(s)$ for all $s \in S$ then not $f \succ x$.*

This says that if the person prefers every conceivable element in X under f to a given element $x \in X$ then he does not prefer x to f. The second half has a dual interpretation. We note that A 6 is weaker (assumes less) than the final axiom in [4], which says that if $f(s) \succ g$ for all $s \in S$ then not $g \succ f$ (and its dual). Because of this we require a new proof of the following lemma which is used in extending (2) [or (7)] to acts not in F_0.

LEMMA 1. *Suppose that $(F, \succ)$ is a regular Archimedean weak ordered mixture set that satisfies* A 6. *If $A \in \mathscr{S}$, $P(A) = 1$, and if c and d are finite when $c = \inf\{u(f(s)) : s \in A\}$ and $d = \sup\{u(f(s)) : s \in A\}$, then*

$$c \leqq v(f) \leqq d. \tag{8}$$

PROOF. Under the hypotheses of the lemma let $g = f$ on A and $c \leqq u(g(s)) \leqq d$ on A^c. With $f \in F$ and $A \in \mathscr{S}$ it is easily seen that $g \in F$. Since $P(A) = 1$, $A^c \in \mathscr{N}$ by (3) and therefore $g \sim f$. Hence $v(g) = v(f)$ by (6). Thus it will suffice to show that $c \leqq v(g) \leqq d$. We show that $v(g) \leqq d$. The proof for $c \leqq v(g)$ is similar. Two cases are considered as follows.

Case 1. $c < d$. Let $x_1, x_2, \cdots$, in X satisfy $u(x_1) \leqq u(x_2) \leqq \cdots$ with $u(x_n) \to d$. (This is guaranteed by the linearity of u and $c < d$.) Fix $y \in X$ with $u(y) < d$. Then, since $d \geqq u(g(s))$ for all s,

$$u(x_n) > \alpha u(g(s)) + (1 - \alpha)u(y) \qquad \text{for all} \quad s \in S$$

whenever $\alpha < [u(x_n) - u(y)]/[d - u(y)]$. Using A 6, (6) and (7) we get $d \geqq v(x_n) \geqq v(\alpha g + (1 - \alpha)\bar{y}) = \alpha v(g) + (1 - \alpha)v(y)$, or $d \geqq \alpha v(g) + (1 - \alpha)v(y)$. Since α can approach 1 as $n \to \infty$ $[u(x_n) \to d]$, it follows that $d \geqq v(g)$.

Case 2. $c = d$. The case 1 proof applies if $u(y) < d$ for some $y \in X$. Henceforth assume that $u(x) \geqq d$ for all $x \in X$. Contrary to the desired result, suppose that $v(g) > d$. Then, since $x \succ y$ for some x, y (by A 4), there is a $z \in X$ such that $v(g) > v(z) > d$. But $z \succ g(s)$ for all s and therefore $v(z) \geqq v(g)$ by A 6 and (6), a contradiction. Hence $v(g) \leqq d$.

6. Measurable acts. Even under A6 it should be clear that $v(f) = \int u(f(s))\, dP(s)$ need not be true for all $f \in F$. For a simple example that is suggested by Example 1 (page 13) in Dubins and Savage [1], let S be the positive integers, take $\mathcal{S} = \{A : A$ is finite or contains all but a finite number of elements in $S\}$, and suppose that P is diffuse with $P(A) = 0$ if A is finite and $P(A) = 1$ otherwise and that $u(x) = x$ for all $x \in X = [0, 1]$. Now let f be such that f(even integer) $= 0$ and f(odd integer) $= 1$. Clearly $f \in F$ but f is not in F_0. The integral $\int u(f(s))\, dP(s)$ is not well defined, and there are extensions of P to algebras that include $\mathcal{S}$ and that contain $\{2, 4, 6, \cdots\}$ such that $\int u(f(s))\, dP'(s)$ can take on any value in $[0, 1]$ depending on the extension P'.

Because $\int u(f(s))\, dP(s)$ need not be well-defined for every $f \in F$ we shall concern ourselves with those f for which $\int u(f(s))\, dP(s)$ is well-defined. Some special definitions will be used in this. Because we are working with Boolean algebras and a probability measure that is not necessarily countably additive, our definition of a measurable act is tailored to this context. u and P are as given in Theorem 2.

DEFINITION 1. $f \in F$ is *measurable* iff $\{s : u(f(s)) \in I\} \in \mathcal{S}$ for every interval I of real numbers.

DEFINITION 2. $f \in F$ is *bounded* iff there is an $A \in \mathcal{S}$ and real numbers a and b such that $A \subseteqq \{s : a \leqq u(f(s)) \leqq b\}$ and $P(A) = 1$.

Clearly, f is measurable iff $\{s : u(f(s)) < a\} \in \mathcal{S}$ and $\{s : u(f(s)) > a\} \in \mathcal{S}$ for every real number a. Because $u(X)$ is an interval of real numbers (by linearity) it follows easily that f is measurable iff $\{s : f(s) \succ x\} \in \mathcal{S}$ and $\{s : x \succ f(s)\} \in \mathcal{S}$ for every $x \in X$. If $f \in F$ is of real interest to the individual then it seems sensible that the events $\{f(s) \succ x\}$ and $\{x \succ f(s)\}$ be in $\mathcal{S}$. If they are not in $\mathcal{S}$ then one may wish to extend $\mathcal{S}$ to include them. This is why we said earlier that the actual choice of $\mathcal{S}$ may depend on considerations of preference.

Let F^*, which includes F_0, be the set of all $f \in F$ that are measurable. Then F^* is not necessarily a mixture set. Using our former example, suppose that $S = \{1, 2, \cdots\}$, $\mathcal{S} = \{A : A \subseteqq S$ and A or A^c is finite$\}$, $u(x) = x$ for all $x \in [0, 1]$, and that

$$\begin{aligned} f(s) &= s/(1 + s) \qquad \text{for all } s\,, \\ g(s) &= 1/(1 + s) \qquad \text{for even } s \\ &= (s + 2)/[(s + 1)(s + 3)] \qquad \text{for odd } s\,. \end{aligned}$$

Then f is strictly increasing in s and g is strictly decreasing in s so that f and g are measurable. However, $\frac{1}{2}f + \frac{1}{2}g$ is not measurable since

$$\{s : u(\tfrac{1}{2}f(s) + \tfrac{1}{2}g(s)) < \tfrac{1}{2}\} = \{s : u(f(s)) + u(g(s)) < 1\} = \{1, 3, 5, \cdots\}\,,$$

which is not in $\mathcal{S}$. (We consider this further in Section 7.)

That this result depends critically on S not being a σ-algebra is shown by the following lemma.

LEMMA 2. *F^* is a mixture set if $\mathcal{S}$ is a σ-algebra.*

PROOF. We need to show that $f, g \in F^*$ and $0 < \alpha < 1$ imply that $\alpha f + (1 - \alpha)g \in F^*$. It will suffice, for a fixed a and α, to show that $A = \{s: \alpha u(f(s)) + (1 - \alpha)u(g(s)) > a\} \in \mathcal{S}$. Let Ra be the set of rational numbers. If $s \in A$ then there are $b, c \in Ra$ such that $b + c > a$ and $\alpha u(f(s)) > b$ and $(1 - \alpha)u(g(s)) > c$. For $b, c \in Ra$ such that $b + c > a$ let $B(b, c) = \{s: u(f(s)) > b/\alpha\} \cap \{s: u(g(s)) > c/(1 - \alpha)\}$. Clearly $B(b, c) \subseteqq A$ and A is the union of all such $B(b, c)$. Since f and g are measurable, $B(b, c) \in \mathcal{S}$ and therefore $A \in \mathcal{S}$ since $\mathcal{S}$ is a σ-algebra and the number of $B(b, c)$ is countable.

Along with Definition 2 we shall say that f is *bounded below* iff there is an $A \in \mathcal{S}$ and a real number a such that $A \subseteqq \{s: a \leqq u(f(s))\}$ and $P(A) = 1$. Similarly, f is *bounded above* iff there is an $A \in \mathcal{S}$ and a real number b such that $A \subseteqq \{s: u(f(s)) \leqq b\}$ and $P(A) = 1$. We omit the simple proof of the next lemma.

LEMMA 3. *$f \in F$ is bounded* iff *f is bounded below and above.*

With u and P as given by Theorem 2 and v on F satisfying (6) and (7) we now state our main theorem.

THEOREM 3. *Suppose that $(F, \succ)$ satisfies* A 1 *through* A 6: *that is, $(F, \succ)$ is a regular Archimedean weak ordered mixture set that satisfies* A 6. *Then every measurable f is bounded and $v(f) = \int u(f(s))\, dP(s)$ for every measurable f.*

The proof of Theorem 3 is carried by the following lemmas, in which A 1—A 6 hold for $(F, \succ)$.

LEMMA 4. *$v(f) = \int u(f(s))\, dP(s)$ if f is measurable and bounded.*

LEMMA 5. *If there is a denumerable $\mathcal{S}$-partition with $P(A) > 0$ for every A in the partition then u on X is bounded.*

LEMMA 6. *If f is measurable then it is bounded.*

PROOF OF LEMMA 4. Assume that f is measurable and bounded with $A \in \mathcal{S}$, $P(A) = 1$ and $A \subseteqq \{s: a \leqq u(f(s)) \leqq b\}$ with a and b finite.

Let $g = f$ on A and $g = y$ on A^c where $a \leqq u(y) \leqq b$. g is measurable since $\{s: u(g(s)) \in I\} = [\{s: u(f(s)) \in I\} \cap A] \cup C$ where $C = \emptyset$ if $u(y) \notin I$ and $C = A^c$ if $u(y) \in I$. Sinch $A^c \in \mathcal{N}$, $f \sim g$ and thus $v(f) = v(g)$ by (6). Moreover, since $P(A) = 1$ and both f and g are bounded and measurable, $\int u(f(s))\, dP(s) = \int u(g(s))\, dP(s)$. It suffices therefore to show that $v(g) = \int u(g(s))\, dP(s)$. This is immediate from Lemma 1 if $a = b$.

Henceforth suppose that $a < b$ and take $a = 0$ and $b = 1$ for notational convenience. For a positive integer n let

$$A_1 = \{s: 0 \leqq u(g(s)) \leqq 1/n\}$$
$$A_i = \{s: (i - 1)n < u(g(s)) \leqq i/n\} \qquad i = 2, \cdots, n\,.$$

$S = \bigcup A_i$, and each A_i is in $\mathscr{S}$ since g is measurable. Take $x_i \in X$ for $i = 1, \cdots, n$ and define measurable f_i, g_i and h_i thus:

$$f_i = g \text{ on } A_i\,, \qquad f_i = x_i \text{ on } A_i^c \qquad \text{for } 1, \cdots, n$$
$$g_i = x_{i+1} \text{ on } \textstyle\bigcup_{j=1}^{i} A_j\,, \qquad g_i = x_i \text{ on } \bigcup_{j=i+1}^{n} A_j \qquad \text{for } i = 1, \cdots, n-1$$
$$h_i = g \text{ on } \textstyle\bigcup_{j=1}^{i+1} A_j\,, \qquad h_i = x_{i+1} \text{ on } \bigcup_{j=i+2}^{n} A_j \qquad \text{for } i = 1, \cdots, n-1\,.$$

Note that $g = h_{n-1}$. As is easily verified, $\frac{1}{2}f_1 + \frac{1}{2}f_2 = \frac{1}{2}g_1 + \frac{1}{2}h_1$ and $\frac{1}{2}h_{i-1} + \frac{1}{2}f_{i+1} = \frac{1}{2}g_i + \frac{1}{2}h_i$ for $i = 2, \cdots, n-1$. By linearity for v and (6), $v(p) + v(q) = v(r) + v(t)$ when $\frac{1}{2}p + \frac{1}{2}q = \frac{1}{2}r + \frac{1}{2}t$. Using this and summing and cancelling we find that

$$v(g) = \textstyle\sum_{i=1}^{n} v(f_i) - \sum_{i=1}^{n-1} v(g_i)\,. \tag{9}$$

Since the $x_i \in X$ are arbitrary, we can choose them so that $(i-1)/n \leqq u(x_i) \leqq i/n$ for $i = 1, \cdots, n$. Then, using Lemma 1, $(i-1)/n \leqq v(f_i) \leqq i/n$. Therefore

$$(n-1)/2 \leqq \textstyle\sum_{i=1}^{n} v(f_i) \leqq (n+1)/2\,.$$

Since linearity permits us to choose $u(x_i)$ arbitrarily close to $(i-1)/n$, and since $v(g_i) = \sum_{j=1}^{i} P(A_j)u(x_{i+1}) + \sum_{j=i+1}^{n} P(A_j)u(x_i)$ by (7) since $g_i \in F_0$, it follows that for an appropriate choice of the x_i we obtain $v(g_i) \leqq \sum_{j=1}^{i} P(A_j)(i/n) + \sum_{j=i+1}^{n} P(A_j)(i-1)/n + 1/n^2$, then

$$\textstyle\sum_{i=1}^{n-1} v(g_i) \leqq (n-1)/2 - \sum_{i=1}^{n} P(A_i)(i-1)/n + 1/n\,.$$

Using (9), this gives $v(g) \geqq (n-1)/2 - [(n-1)/2 - \sum_{i=1}^{n} P(A_i)(i-1)/n + 1/n] = \sum_{i=1}^{n} P(A_i)(i-1)/n - 1/n$. By picking $u(x_i)$ close to i/n we obtain the other half of the following bounds on $v(g)$:

$$\textstyle\sum_{i=1}^{n} P(A_i)(i-1)/n - 1/n \leqq v(g) \leqq \sum_{i=1}^{n} P(A_i)i/n + 1/n\,.$$

Since $\sum_i P(A_i)(i-1)/n \leqq \int u(g(s))\,dP(s) \leqq \sum_i P(A_i)i/n$, it follows by letting $n \to \infty$ that $v(g) = \int u(g(s))\,dP(s)$.

Proof of Lemma 5. This proof is essentially the same as the proof of Theorem 5 in [4].

Proof of Lemma 6. Let f be measurable. Contrary to the lemma, suppose that f is unbounded. Using Lemma 3 assume for definiteness that f is unbounded above. We can suppose (after a linear transformation on u if necessary) that $[0, \infty) \subseteqq u(X)$. Modify $f \in F$ as follows. For each element in an $\mathscr{S}$-partition that verifies that $f \in F$ where $u(f(s)) < 0$, replace $f(s)$ by y with $u(y) = 0$. The modified f is in F, it is unbounded above and has $u(f(s)) \geqq 0$ for all s, and it is easily seen to be measurable. We work with f thus modified.

Let $A_n = \{s\colon n-1 \leqq u(f(s)) < n\}$ for $n = 1, 2, \cdots$. $S = \bigcup A_n$ and each $A_n \in \mathscr{S}$ since f is measurable. Let $C_n = \bigcup_{i=1}^{n} A_i$. Since f is unbounded above, $P(C_n) < 1$ for all n, and thus $P(C_n^c) = P\{u(f(s)) \geqq n\} > 0$ for all n. We consider two cases, according to whether $P(C_n^c) \to 0$.

Case 1. $P(C_n^c) \to 0$ as $n \to \infty$. Then there are denumerably many A_n for which $P(A_n) > 0$. Let these be $A_{n_1}, A_{n_2}, \cdots$ and let $B_1 = \bigcup_1^{n_1} A_n$, $B_2 = \bigcup_{n_1+1}^{n_2} A_n, \cdots$ so that $\{B_1, B_2, \cdots\}$ is denumerable $\mathscr{S}$-partition with $P(B_i) > 0$ for every i. But then Lemma 5 implies that u on X is bounded, and this contradicts the supposition that f is unbounded.

Case 2. $P(C_n^c) \to \alpha > 0$ as $n \to \infty$. Let x_n have $u(x_n) = n$. Let

$$g_n = f \quad \text{on} \quad C_n , \qquad g_n = x_n \quad \text{on} \quad C_n^c$$
$$h_n = x_n \quad \text{on} \quad C_n , \qquad h_n = f \quad \text{on} \quad C_n^c .$$

All g_n and h_n are measurable. Since g_n is bounded also, Lemma 4 and $P(C_n^c) \geqq \alpha$ give $v(g_n) \geqq n\alpha$. Since $h_n(s) \succ x_{n-1}$ for all $s \in S$, A 6 and (6) give $v(h_n) \geqq n - 1$. Since $\frac{1}{2}\bar{x}_n + \frac{1}{2}f = \frac{1}{2}g_n + \frac{1}{2}h_n$, $n + v(f) = v(g_n) + v(h_n)$ and therefore $v(f) \geqq n\alpha - 1$ for all n. But with $\alpha > 0$ this contradicts the finiteness of $v(f)$.

Thus f must be bounded above, and a symmetric proof shows that f must be bounded below. Therefore f is bounded.

7. **Integrable acts.** In concluding this study we note that $v(f) = \int u(f(s))\, dP(s)$ for every f in the convex closure F' of F^*. By Lemma 2, $F' = F^*$ if $\mathscr{S}$ is a σ-algebra. Therefore what we say here extends our previous results only if $\mathscr{S}$ is not a σ-algebra. Since an act in F' need not be measurable, we need to make clear the meaning of $\int u(f(s))\, dP(s)$ in this case.

DEFINITION 3. $f \in F$ is *integrable* iff $\int u(f(s))\, dP'(s)$ exists for every extension P' on the set of all subsets of S of P on $\mathscr{S}$ and takes the same value ($-\infty$ and $+\infty$ being admitted) for every P'. This common value is written as $\int u(f(s))\, dP(s)$.

The convex closure F' of the set F^* of measurable acts can be defined as follows. Let $F_1 = \{f : f = \alpha g + (1 - \alpha)h$ for some $\alpha \in [0, 1]$ and $g, h \in F^*\}$. For $n > 1$ let $F_n = \{f : f = \alpha g + (1 - \alpha)h$ for some $\alpha \in [0, 1]$ and $g, h \in F_{n-1}\}$. Then we define $F' = \lim F_n = \bigcup_{n=1}^{\infty} F_n$. Clearly F' is a mixture set with $F^* \subseteqq F' \subseteqq F$ and it is the minimal such mixture set. That is, F' is the intersection of all mixture sets in F that include F^*.

THEOREM 4. *Suppose that the hypotheses of Theorem* 3 *hold. Then every* $f \in F'$ *is bounded and* $v(f) = \int u(f(s))\, dP(s)$ *for evey* $f \in F'$.

PROOF. First, let $f = \alpha g + (1 - \alpha)h$ with $g, h \in F^*$. By Theorem 3, g and h are bounded. Take $A, B \in \mathscr{S}$, $P(A) = P(B) = 1$, $A \subseteqq \{s : a \leqq u(g(s)) \leqq b\}$ and $B \subseteqq \{s : c \leqq u(h(s)) \leqq d\}$. Then $A \cap B \in \mathscr{S}$, $P(A \cap B) = 1$, and $A \cap B \subseteqq \{s : \inf\{a, c\} \leqq \alpha u(g(s)) + (1 - \alpha)u(h(s)) \leqq \sup\{b, d\}\}$, so that f is bounded. Using Theorem 3 and linearity, $v(f) = \alpha v(g) + (1 - \alpha)v(h) = \int \alpha u(g(s))\, dP(s) + \int (1 - \alpha)u(h(s))\, dP(s) = \int [\alpha u(g(s)) + (1 - \alpha)u(h(s))]\, dP(s) = \int u(\alpha g(s) + (1 - \alpha)h(s))\, dP(s)$. The only step in this chain that may need further comment is the third equality. For this we use the definitions and results of Section 10.3 in [5]. For any f let $f'(s) = u(f(s))$. Then, with g and h measurable and bounded, there are sequences $r_1, r_2, \cdots$ and $t_1, t_2, \cdots$ of simple real-valued measurable functions on S that converge uniformly from below to g' and h' respectively. It follows that $\alpha r_1 + (1 - \alpha)t_1$, $\alpha r_2 + (1 - \alpha)t_2$, is a sequence of simple real-valued measurable functions on S that converges uniformly from below to $\alpha g' + (1 - \alpha)h' = f'$. It follows that f is integrable and that, for any extension P' of P, $\int \alpha g'(s)\, dP(s) + \int (1 - \alpha)h'(s)\, dP(s) = \int \alpha g'(s)\, dP'(s) + \int (1 - \alpha)h'(s)\, dP'(s) = \int [\alpha g'(s) + (1 - \alpha)h'(s)]\, dP'(s) = \int f'(s)\, dP(s)$.

Therefore if $f \in F_1$ then f is bounded and there is a sequence of simple real-valued measurable functions on S that converges uniformly from below to f'. From the proof just given the same thing must be true for every $f \in F_2$. Induction then yields the desired result.

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