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SOME ASYMPTOTIC RESULTS ON RANDOM RANK STATISTICS

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This paper deals with two different problems. The first one deals with asymptotic normality of simple linear rank statistics based on random number of observations X_i, henceforth called random rank statistics, under the alternative where each X_i has a different distribution F_i. The second problem deals with showing that the random rank statistics as a function of the regression parameter in the simple linear regression model is asymptotically uniformly linear (hence continuous) in that parameter. Obviously the two problems are different and could be solved in separate papers but for certain lemmas which are common to the solution of both of these problems. It is suggested not to try to apply the result of Section 3 to Section 4, unless mentioned explicitly. The results of Section 2 are the results which are common, to some extent, to the solution of these two problems.

Pyke and Schorack [11] proved asymptotic normality of a class of two sample random rank statistics under two sample alternatives. Our theorem 3.1 could be thought of as a generalization of the result of [11] to more than two samples situation. Our score function φ is in smaller class than that of [11]. On the other hand our methods yield the asymptotic normality for random rank-sign statistics. This is also contained in Section 3. Section 2 proves a basic lemma about weak convergence of random weighted empirical cumulatives to a tied down continuous Gaussian process.

In [5] and [7] asymptotic uniform linearity of rank statistics based on nonrandom number of observations was proved. In [5] conditions are very general on φ and underlying distribution F whereas conditions in [7] are quite stringent. But in [7] we do not need any artificial condition like (2.1) of [5] on underlying regression constants. However in both of these references, regression scores were assumed to be bounded. In Section 4 here we extend the results of [7] to random rank and random rank-sign statistics and to the case where regression scores need not be bounded. In Section 5 we show how the results of Section 4 can be used to construct a bounded length confidence interval for a regression parameter using rank sign statistics with asymptotic (as length $\to 0$) coverage probability achieved.

Apart from applying Theorem 3.1 to the i.i.d. case, as is mentioned in the remark at the end of Section 5, it is hoped that Theorem 3.1 can be found applicable in some other interesting situations.

1. Assumptions and notation. Suppose X_i, $i \geqq 1$ are independent random variables with cdf's F_i, $i \geqq 1$, c_i, $i \geqq 1$ are some real numbers and $\{N_r\}$ a sequence of positive integer valued random variables. All the random variables are assumed to be defined on the same probability space $(\Omega, \mathscr{A}, P)$ and all probability in this paper will be computed under the probability measure P, unless otherwise specified. Let

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$$(1.1)\qquad u(x) = 1 \quad \text{if } x \geqq 0$$
$$= 0 \quad \text{if } x < 0\,.$$

Define for an integer n

$$(1.2)\qquad R_{in} = \sum_{j=1}^{n} u(X_i - X_j)\,.$$

Let, for a real number $b > 0$,

$$(1.3)\qquad \begin{aligned} \mathscr{C} = \{\varphi:\ & \varphi: [0, 1] \to \text{ Real line, } \varphi \text{ absolutely continuous,} \\ & \varphi' \text{ exists a.e. such that } \|\varphi'\|_1 = \textstyle\int_0^1 |\varphi'(u)|\, du \leqq b < \infty\,, \\ & \varphi(u) = \textstyle\int_0^u \varphi'(v)\, dv \qquad \forall 0 \leqq u \leqq 1\,, \text{ and} \\ & \varphi' \text{ continuous on } [0, 1]\}\,. \end{aligned}$$

Clearly $\varphi \in \mathscr{C} \Rightarrow \varphi$ is bounded, uniformly continuous, square integrable and φ' is uniformly continuous.

Define for a $\varphi \in \mathscr{C}$

$$(1.4)\qquad S_n = n^{-1} \sum_{i=1}^{n} c_i \varphi(R_{in}/(n+1))$$

$$(1.5)\qquad \mu_n = n^{-1} \sum_{i=1}^{n} c_i \int \varphi(n^{-1} \sum_{j=1}^{n} F_j(x))\, dF_i(x)\,.$$

Our problem is to show that under suitable conditions on the underlying objects $\mathscr{L}(N_r^{\frac{1}{2}}(S_{N_r} - \mu_{N_r})s_r^{-1}) \to N(0, 1)$ for some sequence $\{s_r\}$ of positive real numbers.

We begin by stating certain assumptions. To begin with all the limits in the sequel will be taken as $r \to \infty$, unless otherwise specified. By $\mathscr{L}(X)$ we will mean the law of random variable X.

About $\{N_r\}$ we assume that there exist positive integers $\{a_r\}$ and $\{b_r\}$ such that $a_r < b_r$,

$$(1.6)\qquad a_r \to \infty\,, \qquad b_r \to \infty\,, \qquad b_r/a_r \to 1$$

and if $A_r = [a_r \leqq N_r \leqq b_r]$, then

$$(1.7)\qquad P_r[A_r] \to 1\,.$$

A_r^c will denote complement of A_r.

About $\{c_i\}$ and $\{F_i\}$ we assume that all $\{F_i\}$ are continuous and that

$$(1.8)\qquad \lim_{n\to\infty} \max_{1\leqq i\leqq n} c_i^2 / \sum_{i=1}^{n} c_i^2 \to 0\,.$$

Let $\sigma_{nc}^2 = n^{-1} \sum_{i=1}^{n} b_i^2$.

We further assume that $\forall x, y$

$$(1.9)\qquad \lim_{n\to\infty} n^{-1}\sigma_{nc}^{-2} \sum_{i=1}^{n} c_i^2\{F_i(x) \wedge F_i(y)\}\{1 - F_i(x) \vee F_i(y)\} = K_c(x, y)$$

exists. Let $G_i(u) = F_i(\tan \pi(u - \frac{1}{2}))$ for $0 \leqq u \leqq 1$. Assume, for $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, that

$$(1.10)\qquad \lim_{\delta\to 0} \max_{1\leqq i\leqq\infty} \sup_{|u-v|\leqq\delta} |G_i(u) - G_i(v)| = 0\,.$$

Furthermore assume that

$$(1.11)\qquad \sigma_{b_r c}^2 / \sigma_{a_r c}^2 \to 1$$

or equivalently

$$\sum_{i=1}^{b_r} c_i^2 / \sum_{i=1}^{a_r} c_i^2 \to 1 . \tag{1.12}$$

This implies in view of (1.6) and (1.7) that

$$\sigma^2_{N_r c}/\sigma^2_{b_r c} \quad \text{and} \quad \sigma^2_{a_r c}/\sigma^2_{N_r c} \to 1 \quad \text{in probability.} \tag{1.13}$$

We also remark here that

$$P[\max_{1\le i\le N_r} c_i^2/\max_{1\le i\le b_r} c_i^2 \leqq 1] \to 1 . \tag{1.14}$$

A condition like (1.11) will be used in Section 2 whereas conclusions like (1.14) and (1.13) will be used in the proof of Theorem 3.1.

Next we introduce some notation so that we can represent S_{N_r} as a Chernoff-Savage type statistic.

For a real number x and an integer n define

$$m_n(x) = n^{-1} \sum_{i=1}^{n} c_i u(x - X_i) , \qquad \bar{m}_n(x) = n^{-1} \sum_{i=1}^{n} c_i F_i(x) \tag{1.15}$$

and

$$H_n(x) = n^{-1} \sum_{i=1}^{n} u(x - X_i) , \qquad \bar{H}_n(x) = n^{-1} \sum_{i=1}^{n} F_i(x) . \tag{1.16}$$

Under the above notation we have

$$S_{N_r} = \int \varphi\left(\frac{N_r}{N_r + 1} H_{N_r}(x)\right) dm_{N_r}(x) ,$$

and

$$\mu_{N_r} = \int \varphi(\bar{H}_{N_r}(x))\, d\bar{m}_{N_r}(x) .$$

Also define

$$L_n = n^{\frac{1}{2}}(m_n - \bar{m}_n) , \qquad Z_n = n^{\frac{1}{2}}(H_n - \bar{H}_n) . \tag{1.17}$$

Right away, since $N_r/(N_r + 1) \to 1$ in probability, we may without loss of generality write

$$S_{N_r} = \int \varphi(H_{N_r}(x))\, dm_{N_r}(x) . \tag{1.18}$$

Also define

$$l_{ir}(x) = a_r^{-1} \sum_{j=1}^{a_r} (c_j - c_i) \int [u(y - x) - F_i(y)]\varphi'(\bar{H}_r(y))\, dF_j(y) \tag{1.19}$$

where $\bar{H}_r = \bar{H}_{a_r}$.

Furthermore let

$$\hat{S}_n = n^{-1} \sum_{i=1}^{n} l_{ir}(X_i) \tag{1.20}$$

$$s_r^2 = \operatorname{Var}(a_r^{\frac{1}{2}}\hat{S}_{a_r}) = a_r^{-1} \sum_{i=1}^{a_r} \operatorname{Var}(l_{ir}(X_i)) . \tag{1.21}$$

We will need these definitions in proving our main theorem. Before closing this section we state one of the important conditions that is needed.

We need an existence of a constant $0 < k < \infty$ such that

$$\limsup \max_{1\le i\le b_r} c_i^2/s_r^2 \leqq k < \infty . \tag{1.22}$$

It may be noted that (1.22) implies that

$$\limsup_{r\to\infty} \max_{1\le i\le a_r} c_i^2\sigma_{a_rc}^{-2} < \infty \tag{1.23}$$

which in turn implies (1.8).

For any cdf G we define

$$G^{-1}(u) = \inf\{x; G(x) \geqq u\} \qquad 0 \leqq u \leqq 1 .$$

Finally for any function f on the real line, $||f||$ will denote the usual "sup" norm and $||fG^{-1}||$ will stand for $\sup_{0\le u\le 1} |f(G^{-1}(u))|$.

2. Some weak convergence results. Let

$$W_n(x) = n^{-\frac{1}{2}} \sum_{i=1}^n d_i[u(x - X_i) - F_i(x)] = n^{-\frac{1}{2}} V_n(x) \qquad \text{(say)} \tag{2.1}$$

where n is an integer, $\{d_i\}$ are some constants.

From [8] we recall the following

LEMMA 2.1. *If* $\{d_i\}$ *and* $\{F_i\}$ *satisfy conditions like* (1,8), (1.9) *and* (1.10) *then* $\forall \varepsilon > 0$

$$\lim_{\delta\to 0} \lim_{n\to\infty} P[\sup_{|x-y|\le\delta} |W_n(x) - W_n(y)| > \varepsilon\sigma_{nd}] = 0 \tag{2.2}$$

and

$$\lim_n \mathscr{L}(\sigma_{nd}^{-1} W_n(x), -\infty \leqq x \leqq +\infty) = \mathscr{L}(W_d^0(x), -x \leqq x \leqq +\infty) \tag{2.3}$$

where W_d^0 *is a continuous Gaussian process with* $W_d^0(\pm\infty) = 0$ wp 1, $EW_d^0 = 0$ *and* $\operatorname{Cov}(W_d^0(x), W_d^0(y)) = K_d(x, y)$.

REMARK. In [8], the above lemma was proved for the processes defined on [0, 1]. The above W_n processes may be transformed to the ones on [0, 1] by taking X_i to $U_i = (1/\pi)\tan^{-1} X_i + \frac{1}{2}$. With this transformation the weak convergence of the processes on [0, 1] is equivalent to that of the processes defined on the extended real line, above. Also the condition (2.9) of [8] is implied by the condition (1.10) above because F_i get transformed to G_i defined above.

Our objective is to show that the process W_{N_r} are relatively compact as $r \to \infty$. Before we proceed further we state an inequality here, the proof of which is straightforward generalization of one given in real random variable case on page 45 in [1]. Also see [2] for measurability considerations.

INEQUALITY. *Let* $D[-\infty, +\infty]$ *be the space of functions on* $[-\infty, +\infty]$, *the elements of which are right continuous and have left limit. Let* $\{Y_i\}$ *be sequence of independent* rv's *in* $D[-\infty, +\infty]$ *and* $T_j = \sum_{i=1}^j Y_i$. *Then* $\forall \varepsilon > 0$

$$\text{Prob}[\max_{1\le j\le n} ||T_j|| > 2\varepsilon] \leqq \text{Prob}[||T_n|| > \varepsilon](1 - \eta_n)^{-1} \tag{2.4}$$

where

$$\eta_n = \max_{1\le j\le n} \text{Prob}[||T_n - T_j|| > \varepsilon] . \tag{2.5}$$

LEMMA 2.2. *In addition to the condition of Lemma* 2.1 *assume that* $\{a_r\}$, $\{b_r\}$ *and* $\{N_r\}$ *satisfy* (1.6) *and* (1.7) *and that* $\{d_i\}$ *satisfies a condition like* (1.11). *Then* $\forall\, \varepsilon > 0$

$$\lim_{\delta\to 0} \lim_{r\to\infty} P[\sup_{|x-y|\leq\delta} |W_{N_r}(x) - W_{N_r}(y)| > \varepsilon\sigma_{N_r d}] = 0 \tag{2.6}$$

and

$$\mathscr{L}(\sigma_{N_r d}^{-1} W_{N_r}(x), -\infty \leqq x \leqq +\infty) \to \mathscr{L}(W_d^0(x), -\infty \leqq x \leqq +\infty)\,. \tag{2.7}$$

Consequently

$$\mathscr{L}(\sigma_{N_r d}^{-1}||W_{N_r}||) \to \mathscr{L}(||W_d^0||) \tag{2.8}$$

where W_d^0 *is the same as in Lemma* 2.1 *and* $||W_d^0||$ *is bounded* rv'*s.*

PROOF. The proof consists of comparing random processes with nonrandom processes.

In what follows $\sigma_r \equiv \sigma_{a_r d}$.

First we show that

$$a_r^{-\frac{1}{2}}\sigma_r^{-1}||V_{N_r} - V_{a_r}|| \to 0 \qquad \text{in probability.} \tag{2.9}$$

Let

$$Y_i(x) = d_i[u(x - X_i) - F_i(x)] \qquad a_r < i \leqq b_r\,,$$
$$T_j(x) = \sum_{i=a_r+1}^{j} d_i[u(x - X_i) - F_i(x)] \qquad a_r < j \leqq b_r\,.$$

Then $\{Y_i\}$ are independent rv's and $Y_i \in D[-\infty, +\infty]$ and (2.4) is applicable to

$$T_j(x) = V_j(x) - V_{a_r}(x)\,.$$

Therefore

$$\begin{aligned} P[||V_{N_r} - V_{a_r}|| > 2\varepsilon a_r^{\frac{1}{2}}\sigma_r] &\leqq P[\max ||V_j - V_{a_r}|| > 2\varepsilon a_r^{\frac{1}{2}}\sigma_r] + P[A_r^c] \\ &\leqq P[||T_{b_r}|| > \varepsilon a_r^{\frac{1}{2}}\sigma_r](1 - \eta_r)^{-1} + P(A_r^c) \end{aligned} \tag{2.10}$$

where "max" in the first inequality is taken over $a_r \leqq j \leqq b_r$ and

$$\eta_r = \max_{a_r \leqq j \leqq b_r} P[||V_{b_r} - V_j|| > \varepsilon a_r^{\frac{1}{2}}\sigma_r]\,.$$

Using (1.11) it is easy to show that $\operatorname{Var}(a_r^{-\frac{1}{2}}\sigma_r^{-1}T_{b_r}(x)) \to 0$ for each fixed x. Using (2.2) and (1.11) one shows that

$$\sup_{|x-y|\leq\delta} |T_{b_r}(x) - T_{b_r}(y)|a_r^{-\frac{1}{2}}\sigma_r^{-1} \to 0 \qquad \text{in probability}$$

as $r \to \infty$ and then $\delta \to 0$.

Consequently $a_r^{-\frac{1}{2}}\sigma_r^{-1}||T_{b_r}|| \to 0$ in probability.

Similarly one can show that $\eta_r \to 0$. Combining these conclusions with (1.7) and (2.10) we have (2.9).

In view of the fact that $a_r^{-\frac{1}{2}}\sigma_r^{-1}||V_{a_r}||$ is bounded in probability in the limit, which follows in view of Lemma 2.1, (2.9) implies that the rv's $\sigma_r^{-1}a_r^{-\frac{1}{2}}||V_{N_r}||$ are bounded in probability in the limit. Using this conclusion, (1.13) and (1.7) it is easy to conclude that

$$||\sigma_{N_r d}^{-1} W_{N_r} - a_r^{-\frac{1}{2}}\sigma_r^{-1}V_{N_r}|| \to 0 \qquad \text{in probability.}$$

Combining all the above arguments and using (1.11) we have essentially proved

$$||\sigma_{N_r d}^{-1} W_{N_r} - \sigma_r^{-1} W_{a_r}|| \to 0 \qquad \text{in probability.}$$

Therefore (2.7) and (2.6) are proved in view of Lemma 2.1. □

COROLLARY 2.1. *Under the conditions of Lemma* 2.2 *we have* $\forall \varepsilon > 0$

$$\lim_{\delta\to 0} \lim_{r\to\infty} P[\sup_{|x-y|\le\delta} |L_{N_r}(x) - L_{N_r}(y)| > \varepsilon\sigma_{N_r c}] = 0 , \tag{2.11}$$

$$\lim_{\delta\to 0} \lim_{r\to\infty} P[\sup_{|x-y|\le\delta} |Z_{N_r}(x) - Z_{N_r}(y)| > \varepsilon] = 0 , \tag{2.12}$$

$$\lim P[||L_{N_r} H_{N_r}^{-1} - L_{N_r} \bar{H}_{N_r}^{-1}|| > \varepsilon\sigma_{N_r c}] = 0 , \tag{2.13}$$

$$\lim P[||L_{N_r} - L_{a_r}|| > \varepsilon\sigma_{N_r c}] = 0 , \tag{2.14}$$

$$\lim P[||Z_{N_r} - Z_{a_r}|| > \varepsilon] = 0 , \tag{2.15}$$

and

$$\lim P[||L_{N_r} \bar{H}_{N_r}^{-1} - L_{N_r} \bar{H}_{a r}^{-1}|| > \varepsilon\sigma_{N_r c}] = 0 . \tag{2.16}$$

Furthermore $\forall \varepsilon > 0 \exists b(\varepsilon) = b$ *and* $r_0 \ni r \geqq r_0 \Longrightarrow$

$$P[||Z_{N_r}|| > b] < \varepsilon , \tag{2.17}$$

$$P[\sigma_{N_r c}^{-1} ||L_{N_r}|| > b] < \varepsilon . \tag{2.18}$$

Also

$$\mathscr{L}(\sigma_{N_r c}^{-1} ||L_{N_r}||) \to \mathscr{L}(||W_c^0||) \quad \textit{and} \quad \mathscr{L}(||Z_{N_r}||) \to \mathscr{L}(||W_1^0||) .$$

PROOF. Put $d_i = c_i$ in W_n, then $W_n = L_n$ and (2.11), (2.14) and (2.18) are consequences of Lemma 2.2 and its proof.

Put $d_i = 1$ in W_n to get $W_n = Z_n$ and then (2.12), (2.15) and (2.17) are consequences of Lemma 2.2 and its proof.

The proof of (2.13) uses (2.11) and (2.17) and is similar to that of Theorem A6 of [6].

The proof of (2.16) uses the fact

$$||\bar{H}_{N_r}^{-1} - \bar{H}_{a_r}^{-1}|| \to 0 \qquad \text{in probability}$$

(2.11) and is similar to that of (2.13). □

3. Asymptotic normality of random rank statistics. Here we will state and prove our main

THEOREM 3.1. *Under* (1.3), (1.6), (1.7), (1.9), (1.10), (1.11) *and* (1.22) *we have*

$$\mathscr{L}(s_r^{-1} N_r^{\frac{1}{2}} (S_{N_r} - \mu_{N_r})) \to N(0, 1) \tag{3.1}$$

uniformly in all $\varphi \in M$ *where* M *is a relatively compact subset of* $\mathscr{C}$ *with respect to the norm* $||\varphi'||_1$.

PROOF. Our method of proof is decomposition method of $S_{N_r} - \mu_{N_r}$ and then showing $S_{N_r} - \mu_{N_r}$ behaves like $\hat{S}_{a_r}$.

We have the following decomposition.

$$\begin{aligned} N_r^{\frac{1}{2}}(S_{N_r} - \mu_{N_r}) &= N_r^{\frac{1}{2}} \int \varphi(\bar{H}_{Nr})\, d(m_{N_r} - \bar{m}_{N_r}) + N_r^{\frac{1}{2}} \int \varphi'(\bar{H}_{N_r})(H_{N_r} - \bar{H}_{N_r})\, d\bar{m}_{N_r} \\ &\quad + N_r^{\frac{1}{2}} \int [\varphi(H_{N_r}) - \varphi(\bar{H}_{N_r})]\, d(m_{N_r} - \bar{m}_{N_r}) \\ &\quad + N_r^{\frac{1}{2}} \int [\varphi(H_{N_r}) - \varphi(\bar{H}_{N_r}) - (H_{N_r} - \bar{H}_{N_r})\varphi'(\bar{H}_{N_r})]\, d\bar{m}_{N_r} \\ &= B_{1N_r} + B_{2N_r} + E_{1N_r} + E_{2N_r} \qquad \text{(say).} \end{aligned}$$

Note that conditions (1.6), (1.7), (1.8), (1.9) and (1.10) allow us to apply results of Corollary 2.1 and we will invoke these results without going through mention of assumptions, etc.

We first shown that

$$(3.2) \qquad |s_r^{-1} E_{iN_r}| \to 0 \quad \text{in probability} \qquad i = 1, 2$$

and then show, with $B_{a_r} = B_{1a_r} + B_{2a_r}$, that

$$s_r^{-1}(B_{N_r} - B_{a_r}) \to 0 \qquad \text{in probability}$$

and then show that B_{a_r} is nothing but $a_r^{\frac{1}{2}} \hat{S}_{a_r}$.

Before proceeding note that since

$$P[\sigma^2_{N_r c} = N_r^{-1} \textstyle\sum_{i=1}^{N_r} c_i^2 \leqq \max_{1 \leqq i \leqq b_r} c_i^2] \to 1$$

we have in view of (1.22) that

$$(3.3) \qquad P[\sigma_{N_r c}/s_r \leqq k^{\frac{1}{2}}] \to 1\,.$$

Using $P[L_{N_r}(\pm\infty) = 0] = 1\ \forall r \geqq 1$ and integrating R_{1N_r} term by parts one gets

$$s_r^{-1}|E_{1N_r}| \leqq (s_r^{-1}\sigma_{N_r c})\sigma^{-1}_{N_r c} ||L_{N_r} H^{-1}_{N_r} - L_{N_r} \bar{H}^{-1}_{N_r}||\, ||\varphi'||_1 \to 0 \qquad \text{in probability}$$

in view of (3.3) and (2.13).

To prove (3.2) for E_{2N_r} term one proceeds as follows. Note that $P[\bar{m}_{N_r} \ll \bar{H}_{N_r}] = 1\ \forall r \geqq 1$,

$$(3.4) \qquad P[||d\bar{m}_{N_r}/d\bar{H}_{N_r}|| \leqq \max_{1 \leqq i \leqq b_r} |c_i|] \to 1\,,$$

and that $\forall \varepsilon > 0\ \exists b \equiv b(\varepsilon)$ and $r_0 \ni r \geqq r_0 \Longrightarrow$

$$(3.5) \qquad H_{N_r} = \bar{H}_{N_r} + z a_r^{-\frac{1}{2}}\ \forall |z| \leqq b$$

with probability at least $1 - \varepsilon$. (3.5) follows from (2.17).

Using (1.22), (3.4) and (3.5) along with an argument used in ([3] page 629), we conclude (3.2) for E_{2N_r} term.

Now consider B_{1N_r}. Denote L_{a_r} by L_r and $\bar{H}_{a_r}$ by $\bar{H}_r$. Integrating by parts we have

$$\lim P[\textstyle\int \varphi(\bar{H}_{N_r})\, dL_{N_r} = -\int L_{N_r}\, d\varphi(\bar{H}_{N_r})] = 1\,.$$

Consequently

$$\begin{aligned} s_r^{-1}|B_{1N_r} - B_{1a_r}| &\leqq s_r^{-1}||L_{N_r}\bar{H}^{-1}_{N_r} - L_r \bar{H}_r^{-1}||\, ||\varphi'||_1 \\ &\leqq s_r^{-1}\sigma_{N_r c}\sigma^{-1}_{N_r c}\{||L_{N_r}\bar{H}^{-1}_{N_r} - L_{N_r}\bar{H}_r^{-1}|| \\ &\qquad + ||L_{N_r}\bar{H}_r^{-1} - L_r\bar{H}_r^{-1}||\}||\phi'||_1\,. \end{aligned}$$

$s_r^{-1}\sigma_{N_rc}$ is bounded in probability by (3.3). First term on the right-hand side of the above inequality tends to zero in probability by (2.15) while the second term tends to zero in probability by the proof of Lemma 2.1. Hence

$$s_r^{-1}|B_{1N_r} - B_{1a_r}| \to 0 \qquad \text{in probability.} \tag{3.6}$$

Next consider B_{2N_r}:

$$\begin{aligned} B_{2N_r} - B_{2a_r} &= \int \varphi'(\bar{H}_{N_r})Z_{N_r}\,d\bar{m}_{N_r} - \int \varphi'(\bar{H}_r)Z_r\,d\bar{m}_r \\ &= \int \{\varphi'(\bar{H}_{N_r}) - \varphi'(\bar{H}_r)\}Z_{N_r}\,d\bar{m}_{N_r} + \int \varphi'(\bar{H}_r)Z_{N_r}\,d(\bar{m}_{N_r} - \bar{m}_r) \\ &\quad + \int \varphi'(\bar{H}_r)(Z_{N_r} - Z_r)\,d\bar{m}_r \\ &= I_{1r} + I_{2r} + I_{3r}\,. \end{aligned}$$

Now observe that in view of (1.7)

$$||\bar{H}_{N_r} - \bar{H}_r|| \to 0 \qquad \text{in probability.}$$

Using this fact along with uniform continuity of φ', (1.22) and (2.17) it is easy to see that

$$s_r^{-1}|I_{1r}| \to 0 \qquad \text{in probability.} \tag{3.7}$$

Again, using (1.22), (2.14), the fact that $\bar{m}_{a_r} \ll \bar{H}_r$ and the assumption $||\varphi'||_1 < \infty$, it is easy to see that

$$s_r^{-1}|I_{3r}| \to 0 \qquad \text{in probability.} \tag{3.8}$$

Finally write $\bar{m}_r \equiv \bar{m}_{a_r}$,

$$\begin{aligned} I_{2r} &= \int \varphi'(\bar{H}_r)Z_{N_r}\,d(\bar{m}_{N_r} - \bar{m}_r) \\ &= \int_0^1 \varphi' Z_{N_r}\bar{H}_r^{-1}\,d(m_{N_r}\bar{H}_r^{-1} - \bar{m}_r\bar{H}_r^{-1}) \end{aligned}$$

so that

$$\begin{aligned} |s_r^{-1}I_{2r}| &\leqq ||Z_{N_r}\bar{H}_r^{-1}||s_r^{-1}\int_0^1 |\varphi'|\,|d(\bar{m}_{N_r}\bar{H}_r^{-1} - \bar{m}_r\bar{H}_r^{-1})| \\ &\to 0 \qquad \text{in probability} \end{aligned} \tag{3.9}$$

because of (2.17), the fact that

$$s_r^{-1}||\bar{m}_{N_r}\bar{H}_r^{-1} - \bar{m}_r\bar{H}_r^{-1}|| \to 0 \qquad \text{in probability}$$

and that $|\varphi'|$ is continuous.

Combining (3.9), (3.8), (3.7) and (3.6) one has proved

$$s_r^{-1}|B_{N_r} - B_{a_r}| \to 0 \qquad \text{in probability.} \tag{3.10}$$

Now integrating by parts the B_{2a_r} term and adjusting for constants arising thereby one gets

$$P[B_{2a_r} = -\int_{-\infty}^{\infty} \{\int_{-\infty}^{x} \varphi'(\bar{H}_r(y))\,d\bar{m}_r(y)\}\,dZ_{a_r}(x)] = 1 \qquad \forall r \geqq 1\,.$$

After some simple algebra one gets

$$B_{2a_r} = a_r^{-\frac{1}{2}} \sum_{i=1}^{a_r} \{a_r^{-1} \sum_{j=1}^{a_r} c_j \int [u(y - X_i) - F_i(y)]\varphi'(\bar{H}_r(y))\,dF_j(y)\} \qquad \text{wp 1} \quad \text{for all} \quad r \geqq 1\,. \tag{3.11}$$

Using $\varphi(u) = \int_0^u \varphi'(v)\,dv$, we have similarly

$$(3.12)\quad B_{1a_r} = -a_r^{-\frac{1}{2}} \sum_{i=1}^{a_r} c_i\{a_r^{-1} \sum_{j=1}^{a_r} \int_{-\infty}^{\infty} [u(y - X_i) - F_i(y)]\varphi'(\bar{H}_r(y))\, dF_j(y)\} \quad \text{wp } 1 .$$

Consequently

$$(3.13)\quad \begin{aligned} B_{a_r} &= B_{1a_r} + B_{2a_r} \\ &= a_r^{-\frac{1}{2}} \sum_{i=1}^{a_r} \{a_r^{-1} \sum_{j=1}^{a_r} (c_j - c_i) \int [u(y - X_i) - F_i(y)]\varphi'(\bar{H}_r(y))\, dF_j(y)\} \\ &= a_r^{-\frac{1}{2}} \sum_{i=1}^{a_r} l_{ir}(X_i) = a_r^{\frac{1}{2}} \hat{S}_{a_r} \end{aligned}$$

where $l_{ir}(x)$ and $\hat{S}_{a_r}$ are defined by (1.19) and (1.20). Therefore in view of (3.10) and (3.2) it is enough to show that

$$(3.14)\quad \mathscr{L}(s_r^{-1} a_r^{\frac{1}{2}} \hat{S}_{a_r}) \to N(0, 1) .$$

But it is not hard to show that the summands $s_r^{-1} a_r^{-\frac{1}{2}} l_{ir}(X_i)$ satisfy Lindberg-Feller condition, thereby enabling us to conclude (3.14). Consequently (3.1) is proved for each fixed $\varphi \in M$.

The fact about uniformity in $\varphi \in M$ follows because all the four terms E_{iN_r} and B_{iN_r} $i = 1, 2$, may be shown to satisfy Lipschitz condition in the norm $||\varphi'||_1$. This can be done in the same way as was done in [3]. □

Note. It may be noted that the condition (1.22) is similar to (2.16) of [4].

REMARK. 1. Theorem 3.1 is true also for random rank-sign statistics. To see this we introduce

$$(3.15)\quad S_n^{+} = n^{-1} \sum_{i=1}^{n} c_i \varphi(R_{in}^{+}/(n + 1)) \operatorname{sgn}(x_i)$$

where

$$(3.16)\quad R_{in}^{+} = \sum_{j=1}^{n} u(|X_i| - |X_j|)$$

Also let

$$(3.17)\quad \begin{aligned} \nu_n(x) &= n^{-1} \sum_{i=1}^{n} c_i u(x - X_i) \operatorname{sgn}(X_i) \\ \bar{\nu}_n(x) &= n^{-1} \sum_{i=1}^{n} c_i[\operatorname{sgn}(x) F_i(x) - 2F_i(0)u(x)] \end{aligned}$$

$$(3.18)\quad \begin{aligned} H_n^{+}(|x|) &= n^{-1} \sum_{i=1}^{n} u(|x| - |X_i|) \\ \bar{H}_n^{+}(|x|) &= n^{-1} \sum_{i=1}^{n} [F_i(|x|) - F_i(-|x|)] \\ &= n^{-1} \sum_{i=1}^{n} \operatorname{sgn}(x)\{F_i(x) - F_i(-x)\} \end{aligned}$$

$$(3.19)\quad \begin{aligned} L_n^{+}(x) &= n^{\frac{1}{2}}(\nu_n(x) - \bar{\nu}_n(x)) \\ Z_n^{+}(|x|) &= n^{\frac{1}{2}}(H_n^{+}(|x|) - \bar{H}_n^{+}(|x|)) . \end{aligned}$$

In view of these definitions and that of W_n in (2.1), we have, if $d_i = c_i$ in (2.1)

$$L_{N_r}^{+}(x) = W_{N_r}(x) \operatorname{sgn}(x) - 2W_{N_r}(0)u(x)$$

and, if $d_i = 1$ in (2.1),

$$Z_{N_r}^{+}(|x|) = W_{N_r}(|x|) - W_{N_r}(-(|x|)$$

so that all the conclusions of Corollary 2.1 remain valid also for $L^+_{N_r}$ and $Z^+_{N_r}$ process. Therefore we state the following theorem, the proof of which is precisely similar to that of Theorem 3.1 in view of the above remarks and hence omitted.

THEOREM 3.2. *Assume* (1.3), (1.6), (1.7), (1.8), (1.9) *and* (1.10) *and* (1.11) *are true. Then existence of a constant* k^+ *such that*

$$\limsup \max_{1\leq i\leq a_r} |a_i|^2/v_r^2 \leqq k^+ < \infty \tag{3.20}$$

entails

$$\lim \mathscr{L}(v_r^{-1}N_r^{\frac{1}{2}}(S^+_{N_r} - \mu^+_{N_r})) = N(0, 1) \tag{3.21}$$

where

$$\mu^+_{N_r} = N_r^{-1} \textstyle\sum_{j=1}^{N_r} c_j \int \varphi(\bar{H}_{N_r}(|y|)) \operatorname{sgn}(y)\, dF_j(y)$$
$$v_r^2 = a_r^{-1} \textstyle\sum_{i=1}^{a_r} \operatorname{Var}\{l^+_{ir}(X_i)\}$$

with

$$l^+_{ir}(x) = a_r^{-1} \textstyle\sum_{j=1}^{a_r} [(c_j - c_i \operatorname{sgn}(x))(\int [u(|y| - |x|) - G_i(|y|)] \times \operatorname{sgn}(y)\varphi'(\bar{H}_r^+(|y|))\, dF_j(y)] \tag{3.22}$$

where

$$G_i(|y|) = F_i(|y|) - F_i(-|y|)\,. \tag{3.23}$$

REMARK 2. It is not hard to see that in both these theorem s_r^2 and v_r^2 may be replaced by

$$s^2_{N_r} = N_r^{-1} \textstyle\sum_{i=1}^{N_r} \operatorname{Var}(l_{ir}(X_i))$$

and

$$v^2_{N_r} = N_r^{-1} \textstyle\sum_{i=1}^{N_r} \operatorname{Var}(l^+_{ir}(X_i))$$

respectively, which is slight improvement over having asymptotic variance entirely devoid of N_r.

REMARK 3. Conditions (1.6) and (1.7) are equivalent to saying that there exists a sequence $n_r \to \infty$ such that

$$N_r/n_r \to 1 \qquad \text{in probability.}$$

REMARK 4. All the above and following results remain valid if r is taken to be in set of positive numbers instead of in the set of positive integers.

REMARK 5. Finally all the above results remain valid if $\{d_i\}$, $\{X_i\}$ and $\{c_i\}$ are replaced by $\{d_{ir}\}$, $\{X_{ir}\}$ and $\{c_{ir}\}$ respectively so long as dependence on r is via a_r or $\{b_r\}$ only and not via N_r.

4. Asymptotic uniform linearity (AUL) **of random rank statistics in regression parameter.** In this section we take

$$F_i(y) = F(y + tx_i) \tag{4.1}$$

or equivalently $X_i = X_i' - tx_i$, where X_i' are i.i.d. F, F a cdf, $\{x_i\}$ some real numbers and t is the regression parameter of interest.

$S_n(t)$, $W_n(t, \cdot)$, $L_n(t, \cdot)$ etc. will denote the corresponding statistics and processes S_n, $W_n(\cdot)$, $L_n(\cdot)$ etc. based on the above $\{X_i\}$. Thus, for example, $S_n(t)$ is the rank statistic based on the ranks of $\{X_i' - tx_i\}$, $1 \leqq i \leqq n$ etc.

Our problem here is to show that $T_{N_r}(t) = S_{N_r}(tN_r^{-\frac{1}{2}}\sigma_{N_r x}^{-1})$, when suitably normalized, is asymptotically uniformly linear (hence continuous) (AUL) in t in a bounded set in probability.

The solution consists of first proving that the statistics $S_{N_r}(ta_r^{\frac{1}{2}}\sigma_{a_r x}^{-1})$ are AUL and then using this and (1.11) one gets the desired result. Note that the rv's that define $T_{N_r}(t)$ are $\{X_i' - tx_{iN_r}\}$, where

$$x_{in} = (x_i/n^{\frac{1}{2}}\sigma_{nx}) , \tag{4.2}$$

and therefore are not independent whereas the rv's that define $S_{N_r}(ta_r^{-\frac{1}{2}}\sigma_{a_r x}^{-1})$ are independent and this is the reason to proceed to solve the problem in the above mentioned fashion.

In what follows $\sup_{t,x} |g(t, x)|$ or $||g||_a$ will stand for sup being taken over all $|t| \leqq a$ and $-\infty \leqq x \leqq +\infty$ of a function $g(\cdot, \cdot)$, for a $0 < a < \infty$ fixed. Furthermore let

$$W_{rn}(t, y) = W_n(ta_r^{-\frac{1}{2}}\sigma_{a_r x}^{-1}, y)$$
$$L_{rn}(t, y) = L_n(ta_r^{-\frac{1}{2}}\sigma_{a_r x}^{-1}, y) \qquad \text{etc.}$$

Note that $W_{rn}(0, y) = W_n(0, y) \equiv W_n(y)$ as defined by (2.1) when $F_i \equiv F$. Similar statements may be applied to $L_{rn}(0, y)$, $Z_{rn}(0, y)$ etc.

We have the

LEMMA 4.1. *If F is absolutely continuous with a bounded and uniformly continuous* pdf f *and if $\{d_i\}$ and $\{x_i\}$ satisfy a condition like* (1.8) *then* $\forall\, \varepsilon > 0$ *and any* $0 < a < \infty$ *fixed,*

$$\lim_{n\to\infty} P[\sup_{t,x} |W_{rn}(t, x) - W_n(0, x)| > \varepsilon\sigma_{nd}] = 0 . \tag{4.3}$$

Consequently

$$\mathscr{L}(\sigma_{nd}^{-1}||W_{rn}||_a) \to \mathscr{L}(||W_d^0||) . \tag{4.4}$$

PROOF. If σ_{nd}^2 and σ_{nx}^2 were bounded in addition to the above assumptions then the proof is given in the Appendix of [6]. In order to carry that proof through under the current assumptions one needs to make the following modifications in Lemmas A1 through A6 of [6]:

(i) Premultiply everything by σ_{nd}^{-1} and replace $\{c_i\}$ by $\{d_i\}$.
(ii) Replace $\delta_{in}(t)$ by tx_{in}.
(iii) Replace $\lambda_{in}(\varepsilon')$ by $\varepsilon'|x_{in}|$.

Recall the definition of $\{x_{in}\}$ from (4.2) above. After these modifications all the details of Appendix [6] will go through in view of the fact that $(n^{-\frac{1}{2}} \sum |d_i x_{in}|)\sigma_{nd}^{-1} \leqq 1$ and that the processes $\{\sigma_{nd}^{-1} W_{nd}(t, x); -\infty \leqq x \leqq +\infty\}$ weakly converge to a continuous Gaussian process for each fixed t which follows from Lemma 2.1

above and which is basic to the method of proof in [6]. We leave out the details. □

Note. In [6] Lemma A1 pdf "f" is supposed to be *uniformly continuous* but not *absolutely continuous* as appears there.

LEMMA 4.2. *In addition to the conditions of Lemma* 4.1, *let* $\{d_i\}$ *satisfy* (1.11). *Then under* (1.6) *and* (1.7) $\forall \varepsilon > 0$ *and any* $0 < a < \infty$ *fixed,*

$$\lim P[\sup_t \sup_x |\sigma_{N_r d}^{-1} W_{rN_r}(t, x) - \sigma_{a_r d}^{-1} W_{ra_r}(t, x)| > 2\varepsilon] = 0 . \tag{4.5}$$

Consequently in view of (4.3) *and* (4.4) $\forall \varepsilon > 0$ *and any* $0 < a < \infty$ *fixed,*

$$\lim P[\sup_t \sup_x |W_{rN_r}(t, x) - W_{N_r}(0, x)| > \varepsilon\sigma_{N_r d}] = 0 \tag{4.6}$$

and

$$\mathscr{L}(\sigma_{N_r d}^{-1}||W_{rNr}||_a) \to \mathscr{L}(||W_d^0||) . \tag{4.7}$$

Furthermore in view of (4.6) *and* (2.6) *when adapted to the* i.i.d. *case, we have* $\forall \varepsilon > 0$

$$\lim_{\delta\to 0} \lim_{r\to\infty} P[\sup_t \sup_{|x-y|\leq\delta} |W_{rN_r}(t, x) - W_{rN_r}(t, y)| > \varepsilon\sigma_{N_r d}] = 0 . \tag{4.8}$$

PROOF. Let $\varepsilon_r = 2\varepsilon a_r^{\frac{1}{2}}\sigma_{rd}$, $\sigma_{rd} = \sigma_{a_r d}$. From (2.1) we have

$$\begin{aligned} |W_{rN_r}(t, x) - W_{ra_r}(t, x)| &\leq |(N_r^{-\frac{1}{2}} - a_r^{-\frac{1}{2}})a_r^{\frac{1}{2}}a_r^{-\frac{1}{2}}V_{rN_r}(t, x)| \\ &\quad + |V_{rN_r}(t, x) - V_{ra_r}(t, x)|a_r^{-\frac{1}{2}} . \end{aligned} \tag{4.9}$$

Let us drop first suffix r in the sequel to facilitate writing. Thus e.g. $V_{N_r} \equiv V_{rN_r}$ etc. Now

$$P[||V_{N_r} - V_{a_r}||_a > 2\varepsilon_1] \leq P[\max ||V_j - V_{a_r}||_a > 2\varepsilon_r] + P(A_r^c) \tag{4.10}$$

where "max" is over $a_r \leq j \leq b_r$.

By simply imitating the proof of Lemma 3.21 of [1] one can conclude that

$$P[\max ||V_j - V_{a_r}||_a > 2\varepsilon_r] \leq P[||V_{b_r} - V_{a_r}||_a > \varepsilon_r](1 - \eta_r')^{-1} \tag{4.11}$$

where $\eta_r' = \max P[||V_{b_r} - V_j||_a > \varepsilon_r]$.

In carrying out the details to prove (4.11) one needs to remember that the rv's $||V_j - V_{a_r}||_a$ and $||V_{b_r} - V_j||_a$ are independent for each $a_r \leq j \leq b_r$. We will leave out the detail. Now

$$\begin{aligned} a_r^{-\frac{1}{2}}\sigma_{rd}^{-1}(V_{b_r}(t, x) - V_{a_r}(t, x)) &= \lambda_r^{\frac{1}{2}}(\sigma_{rd}^{-1}\sigma_{b_r d})\sigma_{b_r d}^{-1} W_{b_r} - \sigma_{rd}^{-1} W_{a_r} \\ &= k_r \sigma_{b_r d}^{-1} W_{b_r}(t, x) - \sigma_{rd}^{-1} W_{a_r}(t, x) \end{aligned}$$

where $\lambda_r = (a_r/b_r)^{-1}$, $k_r = \lambda_r^{\frac{1}{2}}(\sigma_{rd}^{-1}\sigma_{b_r d})$. Therefore

$$\begin{aligned} &a_r^{-\frac{1}{2}}\sigma_{rd}^{-1}|V_{b_r}(t, x) - V_{a_r}(t, x)| \\ &\quad \leq k_r \sigma_{b_r d}^{-1}|W_{b_r}(t, x) - W_{b_r}(0, x)| + \sigma_{rd}^{-1}|W_{a_r}(t, x) - W_{a_r}(0, x)| \\ &\quad\quad + \sigma_{rd}^{-1}|W_{b_r}(0, x) - W_{a_r}(0, x)| + |\lambda_r^{\frac{1}{2}} - 1|\sigma_{rd}^{-1}|W_{b_r}(0, x)| . \end{aligned} \tag{4.12}$$

Using (1.11) it is easy to show that if $T_r(x) = \sigma_{rd}^{-1}(W_{b_r}(0, x) - W_{a_r}(0, x))$, then

$\mathrm{Var}\,(T_r(x)) \to 0$ for each fixed x. Moreover using (2.2) when adapted to the i.i.d. case one concludes that

$$\sup_{|x-y|\leq\delta} |T_r(x) - T_r(y)| \to 0 \qquad \text{in probability}$$

when $r \to \infty$ and then $\delta \to 0$.
Consequently $\sup_x \sigma_{rd}^{-1}|W_{b_r}(0, x) - W_{a_r}(0, x)| \to 0$ in probability. Furthermore observe that the sequence $\sup_x \sigma_{rd}^{-1}|W_{b_r}(0, x)|$ of rv's is bounded in probability. Combining these facts with (4.3) and that $k_r \to 1$, $|\lambda_r^{\frac{1}{2}} - 1| \to 0$, one concludes, in view of (4.12), that

$$\sigma_{rd}^{-1} a_r^{-\frac{1}{2}} ||V_{b_r} - V_{a_r}||_a \to 0 \qquad \text{in probability.} \tag{4.13}$$

Similarly one can show that $\eta_r' \to 0$. Consequently in view of (4.11), (4.10) and (1.7)

$$a_r^{-\frac{1}{2}} \sigma_{rd}^{-1} ||V_{N_r} - V_{a_r}||_a \to 0 \qquad \text{in probability.} \tag{4.14}$$

(4.14) and (4.4) $\Rightarrow$ that

$$\mathscr{L}(a_r^{-\frac{1}{2}} \sigma_{rd}^{-1} ||V_{N_r}||_a) \to \mathscr{L}(||W_d^0||) \tag{4.15}$$

and hence the rv's $a_r^{-\frac{1}{2}} \sigma_{rd}^{-1} ||V_{N_r}||_a$ are bounded in probability in the limit. Using this and (1.7) with (4.9) we conclude that

$$\sigma_{rd}^{-1} ||W_{rN_r} - W_{ra_r}||_a \to 0 \qquad \text{in probability} \tag{4.16}$$

which in turn implies that the rv's $\sigma_{rd}^{-1}||W_{rN_r}||_a$ are bounded in probability in the limit because of (4.4).

Again using this, (4.16) and (1.11) it is easy to conclude that

$$||\sigma_{N_r d}^{-1} W_{rN_r} - \sigma_{rd}^{-1} W_{ra_r}||_a \to 0 \qquad \text{in probability.} \qquad \square$$

If we put $d_i = c_i$ in W_{N_r}, we have $W_{N_r}(t, x) = L_{N_r}(t, x)\ \forall t, x$ wp 1 and if we put $d_i = 1$ in W_{N_r}, we have $W_{N_r}(t, x) = Z_{N_r}(t, x)\ \forall t$ and x wp 1. Using these relations with Lemma 4.2 we have

COROLLARY 4.1. *$\forall \varepsilon > 0$ and any $0 < a < \infty$ fixed*

$$\lim_{\delta\to 0} \lim_{r\to\infty} P[\sup_t \sup_{|x-y|\leq\delta} |L_{rN_r}(t, x) - L_{rN_r}(t, y)| > \varepsilon\sigma_{N_r c}] = 0\,, \tag{4.17}$$

$$\lim_{\delta\to 0} \lim_{r\to\infty} P[\sup_t \sup_{|x-y|\leq\delta} |Z_{rN_r}(t, x) - Z_{rN_r}(t, y)| > \varepsilon] = 0\,, \tag{4.18}$$

$$\lim P[\sup_{t,x} |L_{rN_r}(t, x) - L_{rN_r}(0, x)| > \varepsilon\sigma_{N_r c}] = 0\,, \tag{4.19}$$

$$\lim P[\sup_{t,x} |Z_{rN_r}(t, x) - Z_{rN_r}(0, x)| > \varepsilon] = 0\,. \tag{4.20}$$

Moreover $\exists\, b < \infty \ni$

$$P[||Z_{rN_r}||_a > b] \to 0\,. \tag{4.21}$$

Also $\forall \varepsilon > 0$

$$\lim P[\sup_{|t|\leq a} \sup_{0\leq u\leq 1} |L_{rN_r}(t, \bar{H}_{rN_r}^{-1}(t, u)) - L_{rN_r}(t, H_{rN_r}^{-1}(t, u))| > \varepsilon\sigma_{N_r c}] = 0\,. \tag{4.22}$$

We now state our

THEOREM 4.1. *Let $\{X_i'\}$ be* i.i.d. rv'*s with an absolutely continuous* cdf F *and uniformly continuous and bounded* pdf f. *Let $\{c_i\}$ and $\{x_i\}$ satisfy* (1.11) *and* (1.8). *Let $\{N_r\}$, $\{a_r\}$ and $\{b_r\}$ satisfy* (1.6) *and* (1.7). *Furthermore assume f is such that*

$$I(f) = \int_{-\infty}^{\infty} (f'/f)^2 \, dF < \infty . \tag{4.23}$$

Moreover assume that $\varphi \in \mathscr{C}_0$ where

$$\mathscr{C}_0 = \{\varphi : \varphi \in \mathscr{C}; \int |\varphi'|^2 < \infty; \|\varphi''\| \leqq K_1\} . \tag{4.24}$$

Then $\forall \varepsilon > 0$ and any $0 < a < \infty$

$$\lim P[\sup_{|t| \leqq a} N_r^{\frac{1}{2}} |S_{N_r}(ta_r^{-\frac{1}{2}}\sigma_{a_r x}^{-1}) - S_{N_r}(0) - tN_r^{-\frac{1}{2}}\bar{b}_{N_r}(\varphi, f)| > \varepsilon \sigma_{N_r c} \sigma_\varphi] = 0 \tag{4.25}$$

where

$$\sigma_\varphi{}^2 = \int [\varphi - \bar{\varphi}]^2 , \qquad \bar{\varphi} = \int \psi ,$$
$$\bar{b}_n(\varphi, f) = n^{-1} \sum_{i=1}^n (x_i - \bar{x}_n)(c_i - \bar{c}_n)\sigma_{a_r x}^{-1} \int_0^1 \varphi(u, f)\varphi(u) \, du$$

with $\bar{x}_n = n^{-1} \sum_{i=1}^n x_i$, $\bar{c}_n = n^{-1} \sum_{i=1}^n c_i$ *and* $\varphi(u, f) = -f'(F^{-1}(u))/f(F^{-1}(u))$.

PROOF. In view of Corollary 4.1 the proof of this theorem is analogous to that of Theorem 2.1, Lemma 2.3 and Theorem 2.2 of [7], where a result like (4.25) was proved for nonrandom sample size case, to which, results like Corollary 4.1, for a nonrandom sample size case, were basic. However there is a slight difference in our present assumptions and those of the above mentioned results. In Theorem 2.1 of [7] we assume F to be strictly increasing and σ_{nx}^2 and σ_{nc}^2 to be bounded. Boundedness of σ_{nx}^2 and σ_{nc}^2 were used in obtaining an analogue of Corollary 4.1 for nonrandom sample size case but we have seen it here that Corollary 4.1 is true without these assumptions. Next, strict monotonicity of F was used in showing that $\sup_t |R_{2n}(t)| \to 0$ in probability, where $R_{2n}(t)$ is either R_{2n} of Theorem 3.1 above when adapted to present situation or $R_n(t)$ of Theorem 2.1 of [7]. We will show here that under $\varphi \in \mathscr{C}_0$

$$\sigma_{N_r c}^{-1} \sup_{|t| \leqq a} |R_{2N_r}(t)| \to 0 \qquad \text{in probability.}$$

Let $g_r(u, z) = \{\varphi(u + a_r^{-\frac{1}{2}}z) - \varphi(u) - za_r^{-\frac{1}{2}}\varphi'(u)\}$. Now in view of (4.21) and that fact that $(N_r a_r^{-1}) \to 1$ in probability, we have that $\forall \varepsilon < 0 \, \exists b \equiv b(\varepsilon)$ and $r_0 \equiv r(\varepsilon) \ni r \geqq r_0 \Rightarrow$

$$P[H_{N_r}(t, y) = \bar{H}_{N_r}(t, y) + za_r^{-\frac{1}{2}} \, \forall |t| \leqq a; \; -\infty \leqq y \leqq +\infty \quad \text{and} \quad |z| \leqq b] \geqq 1 - \varepsilon .$$

Hence with probability at least $1 - \varepsilon$ for $r \geqq r_0$ we have

$$\begin{aligned} R_{2N_r}(t) &= N_r^{-1} \sum_{i=1}^{N_r} c_i \int N_r^{\frac{1}{2}} g_r(\bar{H}_{N_r}(t, y), z) \, dF(y + tx_{ir}) \\ &= N_r^{-1} \sum_{i=1}^{N_r} c_i Z_{ir} \qquad \text{(say)} \\ &\leqq \sigma_{N_r c} \cdot (N_r^{-1} \sum_{i=1}^{N_r} Z_{ir}^2)^{\frac{1}{2}} \qquad \text{(by Cauchy Schwarz).} \end{aligned}$$

But again by Cauchy-Schwarz inequality

$$\begin{aligned} N_r^{-1} \sum_{i=1}^{N_r} Z_{ir}^2 &\leqq N_r^{-1} \sum_{i=1}^{N_r} \int \{N_r^{\frac{1}{2}} g_r(\bar{H}_{N_r}(t, y), z)\}^2 \, dF(y + tx_{ir}) \\ &= \int \{N_r^{\frac{1}{2}} g_r(\bar{H}_{N_r}(t, y), z)\}^2 \, d\bar{H}_{N_r}(t, y) \\ &= \int_0^1 \{N_r^{\frac{1}{2}} g_r(u, z)\}^2 \, du \end{aligned}$$

so that with probability at least $1 - \varepsilon$ we have, for $r \geqq r_0$,

$$\begin{aligned} \sigma_{N_r c}^{-1} |R_{2N_r}(t)| &\leqq [\int_0^1 \{N_r^{\frac{1}{2}} g_r(u, z)\}^2 \, du]^{\frac{1}{2}} \\ &\leqq b[\int_0^1 \varphi_r^2(u) \, du]^{\frac{1}{2}} (N_r^{\frac{1}{2}} a_r^{-\frac{1}{2}}) \end{aligned}$$

where

$$\varphi_r(u) = \sup_{|z| \leq b} \left| \frac{\varphi(u + za_r^{-\frac{1}{2}}) - \varphi(u)}{za_r^{-\frac{1}{2}}} - \varphi'(u) \right|$$

$$\to 0 \qquad \text{for almost all } u.$$

Furthermore since $\int |\varphi'|^2 < \infty$ and since $\int |\varphi^2| < \infty$, we have

$$\int_0^1 \varphi_r^2(u) \, du \to 0 \qquad \text{also.}$$

Hence we have

$$\sup_t \sigma_{N_r c}^{-1} |R_{2N_r}(t)| \to 0 \qquad \text{in probability.}$$

The rest of the proof is precisely similar to that of the nonrandom sample size case, as given in [7]. □

Now an obvious implication of Theorem 4.1 above and conditions (1.6), (1.7) and (1.11) is

THEOREM 4.2. *Under the conditions of Theorem* 4.1 *we have* $\forall \varepsilon > 0$ *and any* $0 < a < \infty$ *fixed,*

$$\lim P[\sup_{|t| \leq a} N_r^{\frac{1}{2}} |T_{N_r}(t) - T_{N_r}(0) - tN_r^{-\frac{1}{2}} b_{N_r}(\varphi, f)| > \varepsilon \sigma_{N_r c} \sigma_\varphi] = 0 \tag{4.26}$$

where

$$b_n(\varphi, f) = \sigma_{a_r x} \sigma_{nx}^{-1} \bar{b}_n(\varphi, f) \tag{4.27}$$

and $T_n(t) = S_n(tn^{-\frac{1}{2}} \sigma_{nx}^{-1})$.

Because of relationships between $L_{N_r}^+$, $Z_{N_r}^+$ etc. with W_{N_r} as indicated in Remark 3 at the end of Section 3, in proving Theorem 4.2 we also have essentially proved

THEOREM 4.3. *In addition to the assumption of Theorem* 4.1 *assume* F *to be symmetric about zero. Then* $\forall \varepsilon > 0$ *and* $0 < a < \infty$,

$$\lim P[\sup_{|t| \leq a} N_r^{\frac{1}{2}} |T_{N_r}^+(t) - T_{N_r}^+(0) - N_r^{-\frac{1}{2}} b_{N_r}^+(\varphi, f)| > \varepsilon \sigma_{N_r c} \sigma_\varphi] = 0 \, , \tag{4.28}$$

where $T_n^+(t)$ *is the rank sign statistic based on the ranks of* $\{|X_i' - tx_{in}|; 1 \leqq i \leqq n\}$ *and signs of* $\{X_i' - tx_{in}; 1 \leqq i \leqq n\}$ *or equivalently*

$$T_n^+(t) = S_n^+(tn^{-\frac{1}{2}} \sigma_{nx}^{-1}) \tag{4.29}$$

where $S_n^+(t)$ *is* S_n^+ *as defined by* (3.18) *and* (3.19) *for* rv's $\{X_i' - tx_i;\ 1 \leqq i \leqq n\}$ *and where*

$$b_n^+(\varphi, f) = -2n^{-1}\sigma_{nx}^{-1} \sum_{i=1}^n c_i x_i \int_0^\infty f(x)\varphi'(G(x))\, dF(x) \tag{4.30}$$

and $G(x) = 2F(x) - 1$, $x \geqq 0$.

5. An application. Suppose $\{X_i'\}$ are independent rv's $\ni$

$$P[X_i' \leqq y] = F(y + tx_i) \qquad i \geqq 1 \tag{5.1}$$

where t is the regression parameter of interest. Given a number $d > 0$ and $0 < \alpha < 1$, we give here a class of confidence interval of prescribed length $2d$ which achieves prescribed coverage probability $1 - 2\alpha$ as $d \to 0$, using rank sign statistics.

Assume $\varphi \in \mathscr{C}_0$ and $\varphi \uparrow$ so that $S_n^+(t)$ is $\downarrow$ in t also see e.g. [6].

Let $z_\alpha = k_\alpha^{\frac{1}{2}}$ be $\ni \Phi(z_\alpha) = 1 - \alpha$, Φ the $N(0, 1)$ cdf. Define

$$\begin{aligned} \sigma_n &= \sigma_{nx}\sigma_\varphi \\ L_n &= \inf\{t;\ \sigma_n^{-1} n^{\frac{1}{2}} S_n^+(t) \leqq z_\alpha\} \\ U_n &= \sup\{t;\ \sigma_n^{-1} n^{\frac{1}{2}} S_n^+(t) \geqq -z_\alpha\}\,. \end{aligned} \tag{5.2}$$

Also define for $d > 0$

$$N_d = \min\{n \geqq 2;\ \sigma_{nx}(U_n - L_n) \leqq 2d\}\,. \tag{5.3}$$

From the methods in [10] and nonrandom sample size version of Theorem 4.3 above, when $c_i = x_i$, one can conclude that

$$n^{\frac{1}{2}}\sigma_{nx}(U_n - L_n) \to \{2z_\alpha/\sigma_\varphi^{-1} b^+(\varphi, f)\} \qquad \text{in probability} \tag{5.4}$$

where

$$b^+(\varphi, f) = 2\int_0^\infty f(x)\varphi'(G(x))\, dF(x)\,. \tag{5.5}$$

Consequently

$$\sigma_{nx}(U_n - L_n) \to 0 \qquad \text{in probability.} \tag{5.6}$$

Using (5.6) it is easy to show that

$$N_d/n_d \to 1 \qquad \text{in probability} \qquad \text{as} \quad d \to 0 \tag{5.7}$$

where

$$n_d^{\frac{1}{2}} = \{2z_\alpha/\sigma_\varphi^{-1} b^+(\varphi, f)\}\, d^{-1}\,. \tag{5.8}$$

Thus conditions (1.6) and (1.7) are satisfiedwith $r = 1/d$, $a_r \equiv a_d = n_d - 1$ and $b_r \equiv b_d = n_d + 1$, say, and $N_r \equiv N_d$.

THEOREM 5.1. *If F is symmetric about zero and is absolutely continuous with a bounded and uniformly continuous density f, $\{x_i\}$ satisfy* (1.11) *and* (1.8) *and $\varphi \in \mathscr{C}_0$ and is* $\uparrow$, *then*

$$\lim_{d\to 0} \mathscr{L}_t(\sigma_{N_d x} N_d^{\frac{1}{2}}(U_{N_d} - t)) = N(-z_\alpha B^{-1}, B^{-2}) \tag{5.9}$$

$$\lim_{d\to 0} \mathscr{L}_t(\sigma_{N_d x} N_d^{\frac{1}{2}}(L_{N_d} - t)) = N(z_\alpha B^{-1}, B^{-2}) \tag{5.10}$$

and

(5.11) $\lim_{d\to 0} \mathscr{L}_t(\sigma_{N_d x} N_d^{\frac{1}{2}}(U_{N_d} - t), \sigma_{N_d x} N_d^{\frac{1}{2}}(L_{N_d} - t)) =$ *Bivariate degenerate normal distribution with means* $-z_\alpha B^{-1}$, $z_\alpha B^{-1}$ *and variances* B^{-2} *where* $B \equiv B(\varphi, f) = \sigma_\varphi^{-1} b^+(\varphi, f)$ *and where* $\mathscr{L}_t(X)$ *is law of an* rv X *when* t *is the true parameter. Consequently, if* P_t *is probability when* t *is the true parameter, then*

$$P_t[L_{N_d} \leqq t \leqq U_{N_d}] \to 1 - 2\alpha . \tag{5.12}$$

PROOF. Without any further mention, results of Section 4 will be applied with $c_i = x_i$.

In view of (5.7) and results of [9] we conclude that

$$\mathscr{L}(\sigma_{N_d}^{-1} N_d^{\frac{1}{2}} S_{N_d}^+(0)) \to N(0, 1) . \tag{5.13}$$

Now from (4.30) and (5.5) we have $b_n^+(\varphi, f) = -\sigma_{nx} b^+(\varphi, f)$ so that

$$\sigma_{N_d x}^{-1} b_{N_d}^+(\varphi, f) = -b^+(\varphi, f) \qquad \text{wp } 1 . \tag{5.14}$$

Since conditions (1.6) and (1.7) are satisfied by $N_d \equiv N_r$ $(r = 1/d)$ we can conclude from Theorem 4.3, in view of our assumptions, that

$$\lim_{d\to 0} \mathscr{L}_0(\sigma_{N_d}^{-1} N_d^{\frac{1}{2}} T_{N_d}^+(t)) = \lim_{d\to 0} \mathscr{L}_0(\sigma_{N_d}^{-1} N_d^{\frac{1}{2}} S_{N_d}^+(0) - tB(\varphi, f)) . \tag{5.15}$$

Since φ is $\downarrow$, both U_n and L_n are well defined and we have in view of (5.2) and (4.29) that

$$\begin{aligned} &\lim_{d\to 0} P_t[\sigma_{N_d x} N_d^{\frac{1}{2}}(U_{N_d} - t) \leqq y] \\ &\quad = \lim_{d\to 0} P_0[\sigma_{N_d x} N_d^{\frac{1}{2}} U_{N_d} \leqq y] \\ &\quad = \lim_{d\to 0} P_0[\sigma_{N_d}^{-1} N_d^{\frac{1}{2}} S_{N_d}^+(y \sigma_{N_d x}^{-1} N_d^{\frac{1}{2}}) \leqq -z_\alpha] \\ &\quad = \lim_{d\to 0} P_0[\sigma_{N_d}^{-1} N_d^{\frac{1}{2}} T_{N_d}^+(y) \leqq -z_\alpha] \end{aligned} \tag{5.16}$$

which in view of (5.15) and (5.13) implies (5.9). (5.10) and (5.11) follow similarly. We have used translation invariance of $\{U_{N_d}\}$ in arriving at (5.16). □

REMARK. We could have concluded (5.14) from Theorem 3.1 above after putting $F_i \equiv F$ and assuming that $\limsup_{n\to\infty} \max_{1\leqq i\leqq n} x_i^2/n\sigma_{nx}^2 \leqq k < \infty$. But since the result is available without this boundedness condition in [9], we prefer not to use that. It is easy to see that this boundedness condition implies (1.11) and (1.8).

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REFERENCES

[1] BRIEMAN, L. (1968). *Probability*. Addison-Wesley, Reading.

[2] FERNANDEZ, P. (1970). On weak convergence of random sums of independent random elements. Ph. D. Dissertation, University of California, Berkeley.

[3] GOVINDARAJUL, Z., LECAM, L., and RAJAVACHERI, M. (1965). Generalizations of theorems of Chernoff and Savage on asymptotic normality of nonparametric test statistics. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* I, 609–638.

[4] HÁJEK, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. *Ann. Math. Statist.* **39** 325–346.

[5] JUREČKOVÁ, J. (1969). Asymptotic linearity of rank statistics in regression parameter. *Ann. Math. Statist.* **40** 1889-1900.

[6] KOUL, H. L. (1969). Asymptotic behavior of Wilcoxon type confidence regions in multiple linear regression. *Ann. Math. Statist.* **40** 1950-1979.

[7] KOUL, H. L. (1970a). A class of ADF tests for subhypothesis in multiple linear regression. *Ann. Math. Statist.* **41** 1273-1281.

[8] KOUL, H. L. (1970b). Some convergence theorems for ranks and weighted empirical cumulatives. *Ann. Math. Statist.* **41** 1768-1773.

[9] KOUL, H. L. (1970c). Asymptotic normality of random rank statistics. *Ann. Math. Statist.* **41** 2144-2149.

[10] KOUL, H. L. (1971). Asymptotic behavior of a class of confidence regions based on ranks in regression. *Ann. Math. Statist.* **42** 466-476.

[11] PYKE, R. and SHORACK, G. (1968). Weak convergence and a Chernoff-Savage theorem for random sample size. *Ann. Math. Statist.* **39** 1675-1685.