The Annals of Mathematical Statistics
1972, Vol. 43, No. 3, 823-831

LOCALLY MOST POWERFUL RANK TESTS FOR THE TWO-SAMPLE PROBLEM WITH CENSORED DATA[1]

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1. Introduction. Suppose that two samples of size m and n, respectively, are placed on life test. Here the observations are the time to failure, and they are naturally ordered. In some statistical applications it may be necessary and can be desirable, not to observe all of the failure times but to reach a decision on the basis of a part of a sample.

Incomplete data situations arise naturally in many fields. For example, they occur in experiments where the measuring device may fail to record very large or very small values. In medical experiments they can occur when it is not possible to wait until all the observations are available. Further, in life tests with costly electronic equipment, units which have not failed can be used in the future.

In this paper, the procedure of Rao, Savage and Sobel (1960) has been further studied for the case when only the smallest r failure times, in the combined sample, are observed. From this procedure, we obtain l.m.p.r. tests for both location and scale alternatives. The results can be extended directly to include left censoring when the first r^* observations are not available or when both first r_1 and last r_2 observations are not available.

It is also shown how to obtain the asymptotic distribution of the test statistics from the extension of the Chernoff-Savage theorem by Pyke and Shorack (1968) or by the method of Dupač and Hájek (1969).

We use the following notation throughout. Let $X_1, \cdots, X_m$ be a random sample of size m from a population with cumulative distribution function (cdf) $F(x)$ and $Y_1, \cdots, Y_n$ a random sample of size n from a population with cdf $G(x)$. The ordered observations in the combined sample of size $N = m + n$ are denoted by $W_1 \leqq W_2 \leqq \cdots \leqq W_N$. Only the first r (fixed) order statistics are observed and these are identified as coming from the first or the second population by the vector $\mathbf{Z} = (Z_1, \cdots, Z_r)$ where $Z_i = 1$ if the ith ordered observation, W_i, is from $F(x)$ and $Z_i = 0$ if W_i is from $G(x)$; $i = 1, 2, \cdots, r$. Let us denote the density function (pdf) corresponding to $F(x)$ by $f(x)$ and that corresponding to $G(x)$ by $g(x)$. Further, set,

$$m_r = \sum_{i=1}^{r} Z_i ,$$

so that m_r denotes the random number of observations from the first sample among the r observations. Set $n_r = r - m_r$. Then Rao, Savage and Sobel (1960) have proved that

[1] This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-69-1789.

$$\begin{aligned}
(1.1)\qquad \Pr(\mathbf{Z} = \mathbf{z}) &= \Pr[(Z_1, \cdots, Z_r) = (z_1, \cdots, z_r)] \\
&= \frac{m!\, n!}{(m - m_r)!\,(n - n_r)!} \int \cdots \int_{-\infty < w_1 < \cdots < w_r < \infty} \\
&\quad \times \{\prod_{i=1}^r f^{z_i}(w_i) g^{1-z_i}(w_i)\, dw_i\} \\
&\quad \times \{1 - F(w_r)\}^{m-m_r}\{1 - G(w_r)\}^{n-n_r} .
\end{aligned}$$

As a special case, when $F(x) \equiv G(x)$, the above probability reduces to

$$(1.2)\qquad \Pr(\mathbf{Z} = \mathbf{z}) = \binom{N - r}{m - m_r} \Big/ \binom{N}{m} .$$

In the reference above, the likelihood ratio, which is ratio of (1.1) to (1.2), is denoted by $L(Z; F, G)$ and in the particular case $G(x) = F(x, \theta)$ it is written as $L(Z, \theta)$ where θ is some parameter. In the usual manner, the locally most powerful rank (l.m.p.r.) test for $\theta > 0$, small, rejects the null hypothesis $F(x) = G(x)$ for those $\mathbf{Z} = \mathbf{z}$ for which $L'(\mathbf{z}, 0)$ is large. $L'(\mathbf{z}, 0)$ denotes the derivative of $L(\mathbf{z}, \theta)$ at $\theta = 0$.

In the following, we obtain some explicit results by specializing θ to either a location or a scale parameter.

2. Expectation of the hazard rate at the r-th ordered observation. Let $W_1 < \cdots < W_N$ be an ordered sample from a population which has continuous cdf $F(x)$ and pdf $f(x)$. Let E_F denote the expectation when the underlying cdf is $F(x)$. First, we prove two lemmas which relate the hazard rate $h(x) \equiv f(x)/\{1 - F(x)\}$ and $xh(x)$ to the scores of the unobserved variables $W_{r+1}, \cdots, W_N$. Let us denote $E_F(-f'(W_j)/f(W_j))$ by $a_N(j, f)$ and $E_F(-1 - W_j f'(W_j)/f_j(W_j))$ by $a_{1N}(j, f)$.

LEMMA 2.1. *If $f(x)$ is absolutely continuous and $\int_{-\infty}^{\infty} |f'(x)|\, dx < \infty$, then*

$$(N - r)Eh(W_r) = (N - r)E_F\left[\frac{f(W_r)}{1 - F(W_r)}\right] = \sum_{j=r+1}^N a_N(j, f) , \qquad r < N .$$

PROOF. Denote the density of the jth ordered observation in a sample of size N by $g_{j,N}(x)$. Explicitly,

$$g_{j,N}(x) = c_{jN} F^{j-1}(x) f(x) \{1 - F(x)\}^{N-j} \qquad -\infty < x < \infty$$

where $c_{jN} = N!/\{(j - 1)!\,(N - j)!\}$. Let $J(j, N)$ denote $E_F[f(W_j)/\{1 - F(W_j)\}]$; $j = 1, 2, \cdots, N - 1$. Then, for $j \leqq N - 1$,

$$\begin{aligned}
J(j, N) &= \int_{-\infty}^{\infty} [f(w_j)/\{1 - F(w_j)\}] g_{j,N}(w_j)\, dw_j \\
&= c_{jN} \int_{-\infty}^{\infty} f(x) F^{j-1}(x) f(x) \{1 - F(x)\}^{N-j-1}\, dx .
\end{aligned}$$

Using integration by parts, $J(j, N)$ can be written as

$$\begin{aligned}
J(j, N) = c_{jN}&[F^j(x) f(x) \{1 - F(x)\}^{N-j-1}|_{-\infty}^{\infty}] \\
&- c_{jN} \int_{-\infty}^{\infty} F(x) \cdot [(j - 1) F^{j-2}(x) f^2(x) \{1 - F(x)\}^{N-j-1} \\
&- (N - j - 1) F^{j-1}(x) f^2(x) \{1 - F(x)\}^{N-j-2} \\
&+ F^{j-1}(x) f'(x) \{1 - F(x)\}^{N-j-1}]\, dx .
\end{aligned}$$

The first term in the above equation is zero because lim $f(x)$ is zero as $x \to \pm\infty$ [see Hájek and Šidák (1967) pages 19–20]; $F(x) \to 0$ as $x \to -\infty$ and $1 - F(x) \to 0$ as $x \to \infty$. Now we rearrange the second term, and get

$$J(j, N) = -(j-1)J(j, N) + (c_{jN}/c_{j+1,N})(N-j-1)J(j+1, N) + (c_{jN}/c_{j+1,N}) \int_{-\infty}^{\infty} \{-f'(x)/f(x)\} g_{j+1,N}(x)\,dx .$$

Since, $(c_{jN}/c_{j+1,N}) = j/(N-j)$, the above equation simplifies to the following recurrence formula,

$$(N-j)J(j, N) = (N-j-1)J(j+1, N) + a_N(j+1, f) \tag{2.1}$$

which holds for any $j = 1, 2, \cdots, N-1$.
Applying (2.1) successively for $j = r, r+1, \cdots, N-1$ we get the desired result.

The following lemma gives a similar result for $xh(x)$, $x = W_r$, which appears below in the test statistic for a scale shift.

LEMMA 2.2. *If f is absolutely continuous and $\int_{-\infty}^{\infty} |xf'(x)|\,dx < \infty$, then*

$$(N-r)E[W_r h(W_r)] = (N-r)E_F\left(W_r \frac{f(W_r)}{1-F(W_r)}\right) = \sum_{j=r+1}^{N} a_{1N}(j, f) , \quad r < N .$$

PROOF. The proof is similar to that of Lemma 2.1 and integration by parts yields the same formula.

3. Locally most powerful rank tests: censored case. The results of the previous section yield the test statistics for the l.m.p.r. tests. Consider first the location parameter model.

THEOREM 3.1. *If $\int_{-\infty}^{\infty} |f'(x)|\,dx < \infty$, then the test with critical region*

$$S_r^N \equiv_{\text{def}} \sum_{i=1}^{r} (1 - Z_i) a_N(i, f) + (n - n_r)\{(N-r)^{-1} \sum_{j=r+1}^{N} a_N(j, f)\} \geqq k \tag{3.1}$$

is the l.m.p.r. *test for testing $G(x) \equiv F(x)$ against the alternative $G(x) = F(x - \theta)$, $\theta > 0$, on the basis of the r smallest observations.*

PROOF. As mentioned in Rao, Savage and Sobel (1960), l.m.p.r. test is of the form $L'(z, 0) \geqq k$. In order to evaluate $L'(z, 0)$, it suffices to substitute $G(x) = F(x - \theta)$ in (1.1) and differentiate it with respect to θ because (1.2) is independent of θ. Moreover, the differentiation can be performed under the integral sign in the expression for $L(z, \theta)$ when $G(x) = F(x - \theta)$ since $\int_{-\infty}^{\infty} |f'(x)|\,dx < \infty$. We obtain

$$L'(z, 0) = [N!/(N-r)!] \int \cdots \int_{-\infty < w_1 < \cdots < w_r < \infty} [\prod_{i=1}^{r} f(w_i)\{1 - F(w_r)\}^{N-r}\,dw_i] \times \left[\sum_{i=1}^{r} -(1 - z_i)\frac{f'(w_i)}{f(w_i)} + (n - n_r)\frac{f(w_r)}{\{1 - F(w_r)\}}\right]$$
$$= \sum_{i=1}^{r} (1 - z_i) E_F\left[-\frac{f'(W_i)}{f(W_i)}\right] + (n - n_r) E_F\left[\frac{f(W_r)}{\{1 - F(W_r)\}}\right].$$

Using Lemma 2.1 and writing

$$E_F\left[-\frac{f'(W_i)}{f(W_i)}\right] = a_N(i, f)$$

in the above expression, we obtain

$$L'(z, 0) = \textstyle\sum_{i=1}^{r} (1 - z_i)a_N(i, f) + (n - n_r)[(N - r)^{-1} \sum_{r+1}^{N} a_N(i, f)]$$

and the theorem is proved.

A similar result holds for the scale parameter case.

THEOREM 3.2. *If* $\int_{-\infty}^{\infty} |xf'(x)|\, dx < \infty$, *then the test with critical region*

(3.2) $$S_{1r}^N \equiv_{\text{def}} \textstyle\sum_{i=1}^{r} (1 - Z_i)a_{1N}(i, f) + (n - n_r)[(N - r)^{-1} \sum_{j=r+1}^{N} a_{1N}(j, f)] \geqq k$$

is the l.m.p.r. *test for testing* $G(x) \equiv F(x)$ *against the alternative* $G(x) = F(x/\theta)$, $\theta > 1$, *on the basis of the r smallest observations.*

The proof is similar to that of Theorem 3.1. and therefore is omitted.

It is interesting to note how the hazard rate $h(W_r)(W_r h(W_r))$ in the scale model appears in the l.m.p.r. test for the censored case. A heuristic interpretation is that information contained in the functions $\{f'(W_i)/f(W_i)\}$'s $i = r+1, \cdots, N$, and which are unknown in the censored case, is represented through $h(W_r) = f(W_r)/\{1 - F(W_r)\}$, at least as far as expectation. This interpretation is also supported by Lemmas 2.1 and 2.2. The relationship between the expected hazard rate at the rth largest observation and $a_N(i, f)$'s $i = r + 1, \cdots, N$, has not been obtained previously. In Section 5, we compare our method of obtaining the statistics with other methods leading to the same answer.

From Lemmas 2.1 and 2.2, we note that the l.m.p.r. test for censored data averages the weights for the censored portion. This is exactly what is shown in [5] for the asymptotically most powerful test.

Finally, we remark that the assumptions $\int |xf'(x)|\, dx < \infty$ in Theorem 3.1 and Theorem 3.2, respectively, are the usual conditions for obtaining l.m.p.r. tests. They imply that the scores sum to zero and also control the tails of $f(x)$. (See Hájek and Šidák (1967) I 2.4 and II 4.3.)

4. Invariance considerations and unbiasedness. In this section, we consider the more general problem of testing $H_0: F = G$ against the alternative K: Y is stochastically larger than X. The problem is invariant under the class of continuous and strictly increasing transformations with the maximal invariant consisting of the complete set of ranks. However, in terms of the observable quantities $(W_1, W_2, \cdots, W_r)$ and $\mathbf{Z} = (Z_1, \cdots, Z_r)$, it is clear that $\mathbf{Z}$ is the observable maximal invariant. Further, consider the representation $(f_0(V_1), \cdots, f_0(V_m), f_1(V_{m+1}), \cdots, f_1(V_{m+n}))$ of the sample $(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ where the V_i are independent uniform random variables and $f_1(v) \geqq f_0(v)$. Any set of v's for which $[W_i \leqq w_i,\ i \leqq r]$ also satisfies the same inequalities for order statistics based on $f_0(V_i)$ $i \leqq m + n$. Therefore

(4.1) $$P_{H_0}(W_1 \leqq w_1, \cdots, W_r \leqq w_r) \geqq P_k(W_1 \leqq w_1 \cdots, W_r \leqq w_r)$$

with strict inequality for some $(w_1, \cdots, w_r)$.

Note that the l.m.p.r. tests derived above were obtained by finding the distribution of the observable maximal invariant **Z** under local alternatives. The test statistics depend on **Z** through $m_r = \sum (1 - Z_i)$ and $\sum_1^r (1 - Z_i)a_N(i, f)$.

The next result on the unbiasedness of the rank tests is slightly more general than the l.m.p.r. tests for scale or location alternatives considered above since in those cases the scores are ordered.

THEOREM 4.1. *If the scores satisfy* $a_N(1) \leqq \cdots \leqq a_N(N)$, *then the test which rejects* $H_0: F = G$ *when*

$$\sum_1^r (1 - Z_i)a_N(i) + (n - n_r)[(N - r)^{-1}\sum_{r+1}^N a_N(i)] \geqq k$$

is unbiased against alternatives where Y *is stochastically larger than* X.

PROOF. Consider the whole, though unobservable, sample $X_1, \cdots, X_m, Y_1, \cdots, Y_n$. The result follows an application of Lemma 2, page 187, Lehmann (1959).

5. Comparison with other methods. The statistics S_r^N and S_{1r}^N defined by (3.1) and (3.2) respectively, have been considered by Lochner (1968). He gives the following justification. A nonparametric test corresponding to the uncensored case is based on the statistic

$$S_N = \sum_{i=1}^N (1 - Z_i)a_N(i) \tag{5.1}$$

where the $a_N(i)$'s are some weights associated with ranks i, $i = 1, 2, \cdots, N$. In the censored situation the actual observations $W_{r+1}, \cdots, W_N$ are unknown; however, the associated weights $a_N(i)$, $i = r + 1, \cdots, N$, are known and can be used to obtain the nonparametric statistics. Even then we do not know $Z_{r+1}, \cdots, Z_N$. To circumvent this problem, Lochner (1968) suggested that the Z_i, $i = r + 1, \cdots, N$, be replaced by their conditional expected values under $H_0: F = G$. Specifically, $(1 - Z_i)$ is replaced by

$$E[(1 - Z_i) \mid z_1, \cdots, z_r] = \frac{n - n_r}{N - r}.$$

Therefore, one obtains the statistic S_r^N and S_N^{1r} in the location and scale case. Clearly, one cannot deduce any optimal property from this development. The special case of S_{1r}^N, when the weights $a_{1N}(i, f)$ correspond to the exponential distribution, has also been considered by Basu (1968). He justifies its use by a statement that the performance of Savage's test is very good for the uncensored case.

It should be pointed out that the test statistic $S_r^{(N)}$ proposed by Basu (1968) differs only by an additive constant from the test statistic in Corollary 3.4 of Rao-Savage-Sobel (1960). Hence the claim about the two tests being comparable made by Basu at the end of his Section 6, based on Table VI, holds exactly for all sample sizes.

The asymptotic versions of the l.m.p.r. test statistics have been obtained by

Gastwirth (1965). It is inherent in his condition (b) on page 1245 that, asymptotically, censoring at the rth largest observation is the same as truncating at the pth percentile, $p = \lim r/N$. By maximizing the limiting Pitman efficiency, Gastwirth obtains the asymptotic version of the weight function for the censored case from that of the asymptotically most powerful rank test based on the complete sample.

One special case of S_r^N, when the underlying distribution $F(x)$ is normal, was considered by Rao-Savage Sobel (1960). However, they failed to express the last term, which we write as $E_F f(W_r)/\{1 - F(W_r)\}$, in a suitable form and only remarked that this function should be tabulated. Since $E_F f(W_r)/\{1 - F(W_r)\}$ was not recognized as the average of the last $(N - r)$ weights, the distribution of the l.m.p.r. test could not be studied (see Corollary 3.3 and Section 4 of Rao-Savage-Sobel (1960)). The particular case of S_{1r}^N with Savage scores was obtained by Rao-Savage-Sobel (1960) using the Lehmann alternative.

Thus, as mentioned earlier, the statistics obtained here as having the property of being locally most powerful rank test have been studied by Basu (1968) and Lochner (1968). By proving that these tests are l.m.p.r. tests, we have further justified the use of these statistics.

Finally, from an asymptotic point of view, Doksum (1967), has obtained some optimality results for the Savage and Savage-Gastwirth tests.

6. Asymptotic distribution. In this section, we use the notation L_{Ni} to denote the score for the ith ordered observation. The asymptotic distribution under local alternatives is given below. The null case follows as a special case. It is known from Gastwirth (1965) that asymptotic null distribution is normal if the score functions are generated from the limit weight function. The l.m.p.r. test statistic may be expressed as

$$mT_N = \sum_{i=1}^r Z_i L_{Ni} + \frac{m - m_r}{N - r} \sum_{i=r+1}^N L_{Ni}$$

which has the alternative representation

$$mT_N = \sum_{i=1}^N Z_i L_{Ni}^r \tag{6.1}$$

where

$$\begin{aligned} L_{Ni}^r &= L_{Ni} && i = 1, 2, \cdots, r; \\ &= \frac{1}{N - r} \sum_{j=r+1}^N L_{Nj} && i = r + 1, \cdots, N. \end{aligned} \tag{6.2}$$

Define

$$L_N(u) = L_{Ni} \qquad \text{for} \quad (i - 1)/N < u < i/N.$$

If the limit $L(u) = \lim_{N\to\infty} L_N(u)$ exists, then, clearly, $\lim_{N\to\infty} L_N^r(u) = L_p(u)$ where

$$L_p(u) = L(u) \qquad 0 < u \leqq p$$
$$= \frac{1}{1-p} \int_1^p L(u)\, du \qquad p < u < 1 \tag{6.3}$$

and

$$L_N^r(u) = L_{Ni}^r \qquad \text{for} \quad (i-1)/N < u \leqq i/N\,. \tag{6.4}$$

Note that the asymptotically most power test employs the limiting score L_p from the l.m.p.r. test. As previously defined, $p = \lim_{N\to\infty} {}_N r$. However, for convenience, we will assume that

$$r = pN\,, \qquad \text{if } pN \text{ is an integer}$$
$$= [pN] + 1\,; \qquad \text{otherwise.}$$

It was observed earlier that when L_{Ni} is an exponential score, the statistic T_N is the generalized Savage statistic considered by Rao-Savage-Sobel (1960) and Basu (1968). To obtain the asymptotic distribution of the generalized Savage-statistic under local alternatives Basu (1968) appealed to Chernoff and Savage (1958).

The conditions for the Chernoff and Savage theorem are stated below with a slightly stronger sufficient condition as (6.5ii) (see their Theorem 2).

(6.5) CONDITIONS.

(i) $\lim_{N\to\infty} J_N(u) = J(u)$ exists for $0 < u < 1$, $J(u)$ is not a constant and $|J(u)| \leqq k[u(1-u)]^{-\frac{1}{2}+\delta}$ some $\delta < 0$ and some k.

(ii) $N^{\frac{1}{2}} \int_0^1 |J_N(u) - J(u)|\, du = o(1)$

(iii) $\lim_{N\to\infty} N^{-\frac{1}{2}} J_N(1) = 0$

(iv) $\left|\frac{d^i Ju}{du^i}\right| \leqq k[(1-u)]^{-i-\frac{1}{2}-\delta}$ for $i = 1, 2$; $\delta < 0$ and some k.

The condition (6.5) (iv) was relaxed by Govindarajulu, LeCam and Raghavachari (1965). They proved the theorem without any condition on $J''(u)$ and only assumed that $J(u)$ is absolutely continuous. Since the limiting weight function $L_p(u)$ usually has a jump discontinuity at the point p, the conditions of this extended version of the Chernoff-Savage Theorem are not satisfied. The reference by Basu (1958), page 1594, does not provide a proof.

Recently, Pyke and Shorack (1968) (see Proposition 5.1, Corollary 5.1 and the discussion on page 769) proved the Chernoff and Savage theorem with condition (6.5iv) replaced by (6.6) defined below:
$J(u)$ is a nonconstant function of bounded variation on $(\varepsilon, 1-\varepsilon)$ for all $\varepsilon > 0$ with

$$|J_c'(u)| \leqq k[u(1-u)]^{-\frac{3}{2}+\delta}\,, \qquad u \neq a_i \tag{6.6}$$

where $J = J_c + J_d$ and J_d is a saltus function taking only a finite number of jumps and J_c has a continuous derivative J_c' on the intervals $(0, a_1), \cdots, (0, a_q)$.

Thus, as remarked by Pyke and Shorack, the statement of the Chernoff and Savage Theorem holds even if $J(u)$ fails to satisfy condition (iv) at a finite number of points. This would cover l.m.p.r. tests which are of the form (6.3).

Alternatively, consider the case of scores given by $L_{Ni} = E - [f'(X_{(i)})/f(X_{(i)})]$ or $L_{Ni} = E[-1 - X_{(i)}f'(X_{(i)})/f(X_{(i)})]$ when the weights $L_N(u)$ and $L(u)$, for the uncensored case, satisfy the Chernoff-Savage conditions. Then, the censored versions, $L_N{}^r$ and L_p clearly satisfy all the conditions except possibly (6.5ii). However

$$N^{\frac{1}{2}} \int_0^1 |L_N{}^r(u) - L_p(u)|\, du$$
$$\leqq N^{\frac{1}{2}} \int_0^p |L_N{}^r(u) - L_p(u)|\, du + N^{\frac{1}{2}} \left|\frac{1-p}{N-r} \sum_{r+1}^N L_{Ni} - \int_p^1 L(u)\, du\right| .$$

Since $|N^{\frac{1}{2}} \sum_{r+1}^N L_{Ni} - N^{\frac{1}{2}} \int_p^1 L_N(u)\, du| \leqq N^{\frac{1}{2}} \int_p^1 |L_N(u) - L(u)| du$, the assumption (6.5ii) implies that the r.h.s. converges to zero. Thus, if the conditions hold for the original scores, they also hold for the curtailed tests considered above. According to Pyke and Shorack (1968), Theorem 5.1 and Proposition 5.1, the satistics are asymptotically normal with mean μ_N given above and variance given by their Equation (4.4). They impose a weak regularity condition on the underlying distributions F and G by requiring that a function $K_0 = F[(\lambda_0 F + (1-\lambda_0)G)^{-1}]$ be differentiable. Summarizing the special case of Theorem 5.1 [10]

THEOREM 6.1. *Let* $m/N = \lambda_0 + O(N^{-\frac{1}{2}})$ *and r be defined as above. If the scores for the uncensored case satisfy the Conditions* (6.5) *and the underlying distributions are such that* $F[(\lambda_0 F + (1-\lambda_0)G)^{-1}]$ *is differentiable on* (0, 1), *then the statistics* T_N *of* (6.1) *satisfy*

$$\mathscr{L}\left[\frac{N^{-\frac{1}{2}}(T_N - \mu_N)}{\sigma_0}\right] \to \mathscr{N}(0, 1)$$

where $\mu_N = \int L_p[(m/N)F + (n/N)G]\, dF$ *and* σ_0^2 *is given by* (4.4) *of* [10].

A similar result may be obtained from Dupač and Hájek (1969), Theorem 5. If the uncensored scores satisfy $\int_0^1 [L_N(u) - L(u)]^2\, du \to 0$, then

$$\int_0^1 [L_N{}^r(u) - L_p(u)]^2\, du \leqq \int_0^p [L_N(u) - L_p(u)]^2\, du + \frac{1}{1-p}\left[\frac{(1-p)}{N-r} \sum_{r+1}^N L_{Ni} - \int_p^1 L(u)\, du\right]^2$$

where the r.h.s converges to zero by assumption since

$$\frac{1}{N} \sum_{r+1}^N L_{Ni} = \int_p^1 L_N(u)\, du \to \int_p^1 L(u)\, du$$

when L_N converges to L mean square.

In the particular case of contiguous regression alternatives, Theorem 2.4, Chapter 6, of Hájek and Sidák (1967) establishes asymptotic normality for a wide range of approximate scores. Thus, under the scale and location alternatives considered in the earlier sections asymptotically, it is possible to use the limit weight function as in Gastwirth (1965) or any other of a wide choice of approximative scores. Certainly, for large samples, it will usually be more convenient to score with the limit weight function since tables of scores will be unavailable.

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