The Annals of Mathematical Statistics
1972, Vol. 43, No. 3, 748-776

ASYMPTOTIC THEORY FOR SUCCESSIVE SAMPLING WITH VARYING PROBABILITIES WITHOUT REPLACEMENT, II

BY BENGT ROSÉN

Royal Institute of Technology, Stockholm

This paper is a direct continuation of the corresponding paper [7], in which our main results were formulated in Section 3. Some of these results were proved and we prepared ourselves for the proof of the asymptotic normality of the sample sum. Our main concern in this part is to carry through the remaining proofs.

9. A Hilbert space. In this section we shall introduce some notation and concepts concerning a Hilbert space, whose relevance will become clear in the next section.

H will denote the real separable Hilbert space which somewhat loosely can be described as the twofold product Hilbert space of l_2 with itself. Elements in H will be denoted as doubly infinite sequences in the following way (* denotes transposition),

$$u = (x_1, x_2, \cdots \mid y_0, y_1, y_2, \cdots)^* \,, \tag{9.1}$$

where $x_1, x_2, \cdots, y_0, y_1, y_2, \cdots$ are real numbers which satisfy

$$\sum_{h=1}^{\infty} x_h^2 + \sum_{l=0}^{\infty} y_l^2 < \infty \,. \tag{9.2}$$

The "unsymmetric" way of indexing in (9.1), i.e. to index the *first sequence* (the x's) by $1, 2, \cdots$ and the *second sequence* (the y's) by $0, 1, 2, \cdots$ will turn out to be convenient later on.

Addition and scalar multiplication in H are defined as componentwise operations in the natural way. The inner product $\langle \cdot, \cdot \rangle$ is defined as follows. Let $u \in H$ be according to (9.1) and let

$$u' = (x_1', x_2', \cdots \mid y_0', y_1', y_2', \cdots)^* \,. \tag{9.3}$$

Then,

$$\langle u, u' \rangle = \sum_{h=1}^{\infty} x_h x_h' + \sum_{l=0}^{\infty} y_l y_l' \,. \tag{9.4}$$

The corresponding norm will be denoted by $\| \ \|$.

In the sequel f_ν and g_μ will be the following elements in H.

(9.5) $f_\nu = (0, 0, \cdots, 0, 1, 0, \cdots \mid 0, 0, \cdots)^*$ where the 1 is in component ν in the first sequence, $\nu = 1, 2, \cdots$.

(9.6) $g_\mu = (0, 0, \cdots \mid 0, 0, \cdots, 0, 1, 0, \cdots)^*$ where the 1 is in component μ in the second sequence, $\mu = 0, 1, 2, \cdots$.

Received June 22, 1971; revised October 21, 1971.

It is easily seen that the set $f_1, f_2, \cdots, g_0, g_1, g_2, \cdots$ is an ON-basis in H. This ON-basis will be called *the natural* ON-*basis*

Next we settle some notation concerning bounded linear operators A on H. The adjoint of A will be denoted by A^*. When we talk of matrix representations of operators we always mean the matrix representation relative to the natural ON-basis. By the matrix representation of the operator A we thus mean the following "doubly infinite" matrix.

$$A \sim \left[\begin{array}{l|l} a^{(1)}_{\nu\mu} & a^{(2)}_{\nu\mu} \\ \nu, \mu = 1, 2, \cdots & \nu = 1, 2, \cdots \\ & \mu = 0, 1, 2, \cdots \\ \hline a^{(3)}_{\nu\mu} & a^{(4)}_{\nu\mu} \\ \nu = 0, 1, 2, \cdots & \nu, \mu = 0, 1, 2, \cdots \\ \mu = 1, 2, \cdots & \end{array}\right] \tag{9.7}$$

where

$$a^{(1)}_{\nu\mu} = \langle Af_\mu, f_\nu\rangle , \qquad a^{(2)}_{\nu\mu} = \langle Ag_\mu, f_\nu\rangle , \qquad a^{(3)}_{\nu\mu} = \langle Af_\mu, g_\nu\rangle , \\ a^{(4)}_{\nu\mu} = \langle Ag_\mu, g_\nu\rangle . \tag{9.8}$$

The four parts of the matrix (9.7) will be called its *blocks*. We have the following formulas. Let u and u' be the elements in (9.1) and (9.3) and suppose they are related according to $u' = Au$. Then

$$x_h' = \textstyle\sum_{i=1}^{\infty} a^{(1)}_{hi} x_i + \sum_{j=0}^{\infty} a^{(2)}_{hj} y_j , \qquad h = 1, 2, \cdots . \tag{9.9}$$

$$y_l' = \textstyle\sum_{i=1}^{\infty} a^{(3)}_{li} x_i + \sum_{j=0}^{\infty} a^{(4)}_{lj} y_j , \qquad l = 0, 1, 2, \cdots . \tag{9.10}$$

Furthermore, the matrix representation of a product of operators is obtained by multiplying the corresponding matrices according to the natural rules.

10. Main steps in the proof of Theorems 3.2 **and** 3.3. In this section we shall present the broad lines in the proof of Theorems 3.2 and 3.3. In order not to conceal the fundamental ideas by too many details some of the basic lemmas are only stated while their proofs are postponed. First some notation, assumptions and preliminary results.

To the sampling situation ($\mathbf{p} = (p_1, \cdots, p_N), \pi = (a_1, \cdots, a_N)$) we associate the mass distribution in R^2 (the (x, y)-plane) whose distribution function is

$$H(x, y; \mathbf{p}, \pi) = \frac{1}{N} \cdot \#(s\colon Np_s \leqq x, a_s \leqq y) , \qquad -\infty < x, y < \infty . \tag{10.1}$$

The corresponding marginal mass distribution along the x-axis only depends on $\mathbf{p}$. Its distribution function will be denoted by

$$H(x; \mathbf{p}) = \frac{1}{N} \cdot \#(s\colon Np_s \leqq x) , \qquad -\infty < x < \infty . \tag{10.2}$$

We shall be concerned with the following conditions on the sequence $(\mathbf{p}_k, \pi_k)$, $k = 1, 2, \cdots$

$$(10.3) \qquad 0 < l \leqq \min_s N_k p_{ks} \leqq \max_s N_k p_{ks} \leqq L < \infty , \qquad k = 1, 2, \cdots$$

$$(10.4) \qquad \max_s |a_{ks}| \leqq M < \infty ,$$

$$(10.5) \qquad H(x, y; \mathbf{p}_k, \pi_k) \Rightarrow H(x, y) \qquad \text{as} \quad k \to \infty .$$

Note that condition (10.3) is equivalent with (3.22). When (10.5) holds, the corresponding marginal masses also converge, i.e.

$$(10.6) \qquad H(x; \mathbf{p}_k) \Rightarrow H(x) \qquad \text{as} \quad k \to \infty .$$

It is readily verified that $H(x, y; \mathbf{p}_k, \pi_k)$ all have total mass 1 and that these masses all lie on the rectangle $0 < l \leqq x \leqq L < \infty$, $-M \leqq y \leqq M$. Thus the limit mass $H(x, y)$ in (10.5) also has total mass 1 which lies on the rectangle above. Hence the marginal masses $H(x; \mathbf{p}_k)$ and $H(x)$ also have total mass 1 and these masses lie on the interval $0 < l \leqq x \leqq L < \infty$.

DEFINITION 10.1. When (10.3) and (10.6) are fulfilled we define the function $\rho(\alpha)$, $0 \leqq \alpha < 1$, implicitly by the following relation

$$(10.7) \qquad 1 - \alpha = \int_0^\infty e^{-\rho(\alpha)x} \, dH(x) , \qquad 0 \leqq \alpha < 1$$

where $H(x)$ is the limit mass in (10.6).

The above definition is always meaningful since, as $H(x)$ has total mass 1 and no point mass in $x = 0$, the integral

$$\int_0^\infty e^{-\rho x} \, dH(x) , \qquad \rho \geqq 0$$

decreases strictly from 1 to 0 as ρ increases from 0 to ∞.

The following properties of $\rho(\alpha)$ are easily verified.

LEMMA 10.1. *The function $\rho(\alpha)$ in Definition 10.1 satisfies.*

(i) *$\rho(\alpha)$ is strictly increasing for $0 \leqq \alpha < 1$ and $\rho(\alpha) \to \infty$ as $\alpha \to 1$.*
(ii) *$\rho(\alpha)$ is continuous for $0 \leqq \alpha < 1$.*
(iii) *$\rho(0) = 0$.*

LEMMA 10.2. *Let*

$$(10.8) \qquad \rho_k(\alpha) = t_k(\alpha N_k)/N_k , \qquad 0 \leqq \alpha < 1 , \quad k = 1, 2, \cdots$$

where $t_k(y)$, $0 \leqq y < N_k$, is according to Definition 3.1. Then, if (10.3) and (10.6) hold, we have

$$(10.9) \qquad \rho_k(\alpha) \to \rho(\alpha) \qquad \text{as} \quad k \to \infty , \quad 0 \leqq \alpha < 1 .$$

The convergence in (10.9) is uniform in α on every interval $0 \leqq \alpha \leqq \alpha_0 < 1$.

PROOF. As is easily seen, formula (3.2) can be written in the following way.

$$(10.10) \qquad 1 - \alpha = \int_0^\infty e^{-\rho_k(\alpha)x} \, dH(x; \mathbf{p}_k) , \qquad 0 \leqq \alpha < 1 , \quad k = 1, 2, \cdots .$$

The assertion in the lemma now follows readily from (10.10), (10.6) and (10.7). We omit the details.

DEFINITION 10.2. When (10.3)—(10.5) are fulfilled, the functions $\kappa_{ij}(\alpha)$, $0 \leqq \alpha < 1$, $i, j = 0, 1, 2, \cdots$ are defined by

$$\kappa_{ij}(\alpha) = \iint_{R^2} x^i y^j e^{-\rho(\alpha)x} \, dH(x, y) , \qquad 0 \leqq \alpha < 1 \tag{10.11}$$

where $H(x, y)$ is the limit mass in (10.5) and $\rho(\alpha)$ is according to the previous definition.

As the entire mass $H(x, y)$ lies on a bounded rectangle, we have no convergence problems for the integral (10.11).

We list some easily verified properties of $\kappa_{ij}(\alpha)$.

LEMMA 10.3. *The functions $\kappa_{ij}(\alpha)$ in (10.11) satisfy,*

(i) $\kappa_{ij}(\alpha)$ *is continuous on* $0 \leqq \alpha < 1$, $i, j = 0, 1, 2, \cdots$,
(ii) $\kappa_{ij}(\alpha) \to 0$ *as* $\alpha \to 1$, $i, j = 0, 1, 2, \cdots$.

LEMMA 10.4. *Let $\psi_k(x, y)$, $k = 1, 2, \cdots$ be a sequence of functions such that*

$$\psi_k(x, y) \to \psi(x, y) \qquad \text{as} \quad k \to \infty , \quad (x, y) \in R^2 , \tag{10.12}$$

where $\psi(x, y)$ is continuous on R^2, and assume that the convergence in (10.12) *is uniform on every bounded region in R^2. Then, if* (10.3)—(10.5) *hold, we have*

$$\iint_{R^2} \psi_k(x, y) \, dH(x, y; \mathbf{p}_k, \pi_k) \to \iint_{R^2} \psi(x, y) \, dH(x, y) \qquad \text{as} \quad k \to \infty . \tag{10.13}$$

PROOF. The lemma follows from well-known results about relations between weak convergence of measures and convergence of integrals.

LEMMA 10.5. *Let $(\mathbf{p}_k, \pi_k)$, $k = 1, 2, \cdots$ satisfy* (3.21) *and* (10.3)—(10.5) *and let $v_n^{(k)}(i, j)$ be defined by* (8.10). *Let furthermore*

$$n_k/N_k \to \alpha , \qquad 0 \leqq \alpha < 1 . \tag{10.14}$$

Then, we have for $i, j = 0, 1, 2, \cdots$

$$v_{n_k}^{(k)}(i, j) \to \kappa_{ij}(\alpha) , \qquad 0 \leqq \alpha < 1 \tag{10.15}$$

where $\kappa_{ij}(\alpha)$ is defined in (10.11).

PROOF. Formula (8.11) can be written

$$v_{n_k}^{(k)}(i, j) = \iint_{R^2} x^i y^j \exp\left(-\rho_k\left(\frac{n_k}{N_k}\right)x\right) dH(x, y; \mathbf{p}_k, \pi_k) + r_k(n_k, i, j)/N_k^{\frac{1}{2}} , \qquad k = 1, 2, \cdots . \tag{10.16}$$

When (3.21), (10.3) and (10.4) are fulfilled, we get from (8.12) and Theorem 3.1

$$\limsup_{k\to\infty} |r_k(n_k, i, j)| \leqq L^i M^j \limsup_{k\to\infty} r_\Delta(n_k) = 0 . \tag{10.17}$$

(10.15) now follows from (10.16), (10.17) and Lemmas 10.1, 10.2 and 10.4.

We now enter the core of the proof of Theorems 3.2 and 3.3. Let as usual $(\mathbf{p}, \pi)$ be a sampling situation and let $Y_\nu(i, j)$ be as in (8.2). Furthermore, let γ be a positive number which is greater than L in (2.18). Then, as is easily realized, the following X's are random elements which take their values in the Hilbert space H, which was introduced in the previous section.

$$X_\nu = \frac{1}{N^{\frac{1}{2}}}\Big(Y_\nu(1,0)^c\cdot\frac{1}{\gamma},\ Y_\nu(2,0)^c\cdot\frac{1}{\gamma^2},\ \cdots\Big|\ Y_\nu(0,1)^c,\ Y_\nu(1,1)^c\cdot\frac{1}{\gamma},\ Y_\nu(2,1)^c\cdot\frac{1}{\gamma^2},\ \cdots\Big)^*, \qquad \nu = 1, 2, \cdots, N. \tag{10.18}$$

By (10.18) we get the following double sequence of random elements associated with our sequence $(\mathbf{p}_k, \pi_k)$, $k = 1, 2, \cdots$, of sampling situations

$$\begin{array}{l} X_1^{(1)}, X_2^{(1)}, \cdots, X_{N_1}^{(1)} \\ X_1^{(2)}, X_2^{(2)}, \cdots, X_{N_2}^{(2)} \\ \vdots \\ X_1^{(k)}, X_2^{(k)}, \cdots, X_{N_k}^{(k)} \\ \vdots \end{array} . \tag{10.19}$$

We shall assume that (10.3) is fulfilled, and we assume henceforth that γ is chosen (independently of k) larger than L in (10.3). We are now in the framework of the author's paper [4] and the basic tool in the subsequent analysis will be Theorem B in [4]. We adopt a great deal of terminology and notation from [4] without explanation. Unfortunately common notation in sampling theory has led us into notational disagreement with [4] in the following respect. The letter n_k in [4] here corresponds to N_k while n_k in this paper has nothing to do with n_k in [4].

As in [4] we put

$$S_\alpha^{(k)} = \sum_{\nu=1}^{[\alpha N_k]} X_\nu^{(k)}, \qquad 0 \leqq \alpha \leqq 1, \quad k = 1, 2, \cdots. \tag{10.20}$$

Our interest in the stochastic process $S_\alpha^{(k)}$, $0 \leqq \alpha \leqq 1$, depends on the fact that our original random variable Z_n (see (3.1)) is "embedded" in the S_α-process in the following way

$$\frac{1}{N^{\frac{1}{2}}}(Z_n^{(k)} - EZ_n^{(k)}) = \langle S_{n/N_k}^{(k)}, g_0\rangle, \tag{10.21}$$

where g_0 is defined in (9.6). Formula (10.21) is realized from (10.18), (8.2), (8.3) and (8.5).

As stated before we shall apply Theorem B in [4]. However we will not be able to verify condition (C1) in this theorem, but only a slightly weaker condition, which we shall call (C1)′. This condition runs as follows.

(10.22) (C1)′ Condition (C1) in [4] is modified to the effect that $\chi(s)$ need only be defined for $0 \leqq s < 1$ and the inequality in [4] need only hold for $0 \leqq \alpha, \beta \leqq \delta_0 < 1$, but for every $\delta_0 \in [0, 1)$.

A scrutiny of the arguments in [4] shows that if the assumption (C1) is changed to (C1)′ in Theorem B in [4], then the conclusion still holds if $\alpha_1, \alpha_2, \cdots, \alpha_d, \alpha$ and β are confined to the half-open interval $[0, 1)$.

LEMMA 10.6. *We assume that the sequence* $(\mathbf{p}_k, \pi_k)$, $k = 1, 2, \cdots$, *satisfies* (3.21), (10.3)—(10.5). *Then, the corresponding sequence* $\{S_\alpha^{(k)}, 0 \leqq \alpha \leqq 1\}$, $k = 1, 2, \cdots$ *satisfies*

(a) *condition* (C0) *in* [4];
(b) *condition* (C1)′ (*see* (10.22));
(c) *condition* (C2d) *in* [4], $d = 1, 2, \cdots$, *for the operator function* $M(\alpha)$, $0 \leqq \alpha < 1$, *whose matrix representation is*

$$M(\alpha) = \left(\begin{array}{cc|cc} u(\alpha) & J(\alpha) & \multicolumn{2}{c}{0} \\ \hline v(\alpha) & 0 & 0 & J(\alpha) \end{array}\right) \tag{10.23a}$$

where, with $\kappa_{ij}(\alpha)$ *according to* (10.11),

$$u^*(\alpha) = \left(\frac{\kappa_{20}(\alpha)}{\kappa_{10}(\alpha)^2} \cdot 1, \frac{\kappa_{30}(\alpha)}{\kappa_{10}(\alpha)^2} \cdot \frac{1}{\gamma}, \frac{\kappa_{40}(\alpha)}{\kappa_{10}(\alpha)^2} \cdot \frac{1}{\gamma^2}, \cdots\right), \tag{10.23b}$$

$$v^*(\alpha) = \left(\frac{\kappa_{11}(\alpha)}{\kappa_{10}(\alpha)^2} \cdot \gamma, \frac{\kappa_{21}(\alpha)}{\kappa_{10}(\alpha)^2} \cdot 1, \frac{\kappa_{31}(\alpha)}{\kappa_{10}(\alpha)^2} \cdot \frac{1}{\gamma}, \cdots\right), \tag{10.23c}$$

(10.23d) $J(\alpha)$ *is the diagonal matrix with all diagonal entries equal to* $-\gamma/\kappa_{10}(\alpha)$;

(d) *condition* (C3d) *in* [4], $d = 1, 2, \cdots$, *for the operator function* $D(\alpha)$, $0 \leqq \alpha < 1$, *whose matrix representation is*

(10.24) $D(\alpha)$

$$\sim \left[\begin{array}{l|l}
\begin{aligned} &\sigma_{\nu\mu}^{(1)}(\alpha) = \gamma^{-(\nu+\mu)}\Big(\frac{\kappa_{\nu+\mu+1,0}(\alpha)}{\kappa_{10}(\alpha)} \\ &\qquad - \frac{\kappa_{\nu+1,0}(\alpha)\cdot\kappa_{\mu+1,0}(\alpha)}{\kappa_{10}(\alpha)^2}\Big) \\ &\nu, \mu = 1, 2, \cdots. \end{aligned} &
\begin{aligned} &\sigma_{\nu\mu}^{(2)}(\alpha) = \gamma^{-(\nu+\mu)}\Big(\frac{\kappa_{\nu+\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)} \\ &\qquad - \frac{\kappa_{\nu+1,0}(\alpha)\cdot\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)^2}\Big) \\ &\nu = 1, 2, \cdots \\ &\mu = 0, 1, 2, \cdots. \end{aligned} \\ \hline
\begin{aligned} &\sigma_{\nu\mu}^{(3)}(\alpha) = \sigma_{\mu\nu}^{(2)}(\alpha) \\ &\nu = 0, 1, 2, \cdots \\ &\mu = 1, 2, \cdots. \end{aligned} &
\begin{aligned} &\sigma_{\nu\mu}^{(4)}(\alpha) = \gamma^{-(\nu+\mu)}\Big(\frac{\kappa_{\nu+\mu+1,2}(\alpha)}{\kappa_{10}(\alpha)} \\ &\qquad - \frac{\kappa_{\nu+1,1}(\alpha)\cdot\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)^2}\Big) \\ &\nu, \mu = 0, 1, 2, \cdots. \end{aligned}
\end{array}\right];$$

(e) *condition* (C4) *in* [4].

The proof of this lemma is given in the next section.

It is easily verified that the operator functions $M(\alpha)$ and $D(\alpha)$ are weakly continuous. Thus, Lemma 10.6 yields that Theorem B in [4] is applicable with the slight modification that was mentioned before Lemma 10.6. According to this theorem we have for $(\alpha_1, \alpha_2, \cdots, \alpha_d) \in [0, 1)^d$,

$$(10.25)\quad \mathscr{L}(S_{\alpha_1}^{(k)}, S_{\alpha_2}^{(k)}, \cdots, S_{\alpha_d}^{(k)}) \Rightarrow N(0, [\Lambda(\alpha_\nu, \alpha_\mu); \nu, \mu = 1, 2, \cdots, d]) \quad \text{as} \quad k \to \infty ,$$

where the symmetric operator function $\Lambda(\alpha, \beta)$, $(\alpha, \beta) \in [0, 1)^2$ is uniquely determined by the following conditions. (Here we make a notational change in comparison with [4] to the effect that we let α and β change places. The reason for this change is that we find the ordering $\alpha \leqq \beta$ in some sense more natural than $\beta \leqq \alpha$.)

$$(10.26)\quad \text{(i)}\ \Lambda(\alpha, \beta) \text{ is weakly continuous in } (\alpha, \beta) \text{ for } (\alpha, \beta) \in [0, 1)^2.$$

$$(10.27)\quad \text{(ii)}\ \frac{d}{d\alpha}\Lambda(\alpha, \alpha) = M(\alpha)\Lambda(\alpha, \alpha) + \Lambda(\alpha, \alpha)M(\alpha)^* + D(\alpha) , \qquad 0 \leqq \alpha < 1 .$$

$$(10.28)\quad \text{(iii)}\ \Lambda(0, 0) = 0 .$$

$$(10.29)\quad \text{(iv)}\ \frac{d}{d\beta}\Lambda(\alpha, \beta) = M(\beta)\Lambda(\alpha, \beta) , \qquad 0 \leqq \alpha < \beta < 1 .$$

$$(10.30)\quad \text{(v)}\ \Lambda(\beta, \alpha) = \Lambda(\alpha, \beta)^* , \qquad (\alpha, \beta) \in [0, 1) .$$

The matrix representation of $\Lambda(\alpha, \beta)$ will be denoted

$$(10.31)\quad \Lambda(\alpha, \beta) \sim \left[\begin{array}{l|l} \lambda_{\nu\mu}^{(1)}(\alpha, \beta) & \lambda_{\nu\mu}^{(2)}(\alpha, \beta) \\ \nu, \mu = 1, 2, \cdots & \nu = 1, 2, \cdots \\ & \mu = 0, 1, 2, \cdots . \\ \hline \lambda_{\nu\mu}^{(3)}(\alpha, \beta) & \lambda_{\nu\mu}^{(4)}(\alpha, \beta) \\ \nu = 0, 1, 2, \cdots & \nu, \mu = 0, 1, 2, \cdots \\ \mu = 1, 2, \cdots . & \end{array}\right] .$$

LEMMA 10.7. *When $M(\alpha)$ and $D(\alpha)$ are according to* (10.23) *and* (10.24), *the unique solution $\Lambda(\alpha, \beta)$ of* (10.26)—(10.30) *is determined by* (10.31) *and the following formulas* (10.32)—(10.35), *which hold for $0 \leqq \alpha \leqq \beta < 1$.*

$$(10.32)\quad \lambda_{\nu\mu}^{(1)}(\alpha, \beta) = \gamma^{-(\nu+\mu)} \iint_{R^2} \left(\frac{\kappa_{\mu+1,0}(\alpha)}{\kappa_{10}(\alpha)} - x^\mu\right)\left(\frac{\kappa_{\nu+1,0}(\beta)}{\kappa_{10}(\beta)} - x^\nu\right) \times (1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH(x, y) ,$$

$$(10.33)\quad \lambda_{\nu\mu}^{(2)}(\alpha, \beta) = \gamma^{-(\nu+\mu)} \iint_{R^2} \left(\frac{\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)} - yx^\mu\right)\left(\frac{\kappa_{\nu+1,0}(\beta)}{\kappa_{10}(\beta)} - x^\nu\right) \times (1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH(x, y) ,$$

$$(10.34)\quad \lambda_{\nu\mu}^{(3)}(\alpha, \beta) = \lambda_{\mu\nu}^{(2)}(\alpha, \beta) ,$$

$$(10.35)\quad \lambda_{\nu\mu}^{(4)}(\alpha, \beta) = \gamma^{-(\nu+\mu)} \iint_{R^2} \left(\frac{\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)} - yx^\mu\right)\left(\frac{\kappa_{\nu+1,1}(\beta)}{\kappa_{10}(\beta)} - yx^\nu\right) \times (1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH(x, y) .$$

Again we postpone the proof, and it is given in Section 12. The following theorem sums up our reasoning so far.

THEOREM 10.1. *If* $(\mathbf{p}_k, \pi_k)$, $k = 1, 2, \cdots$, *satisfies* (3.21) *and* (10.3)—(10.5), *then for every* d, $d = 1, 2, \cdots$, *and every* $(\alpha_1, \alpha_2, \cdots, \alpha_d) \in [0, 1]^d$ *we have*

$$(10.36) \qquad \mathscr{L}(S_{\alpha_1}^{(k)}, S_{\alpha_2}^{(k)}, \cdots, S_{\alpha_d}^{(k)}) \Rightarrow N(0, [\Lambda(\alpha_\nu, \alpha_\mu); \nu, \mu = 1, 2, \cdots, d]) \quad \text{as} \quad k \to \infty$$

where the operator function $\Lambda(\alpha, \beta)$, $(\alpha, \beta) \in [0, 1]^2$ *is given by the previous lemma.*

REMARK. Earlier we said that we had to confine the α's to the half-open interval $[0, 1)$, while the above theorem is formulated for $[0, 1]$. We shall justify this extension. By letting $\beta \to 1$ in (10.32)—(10.35) we get that $\Lambda(\alpha, \beta) \to 0$ as $\beta \to 1$. By definition we put

$$(10.37) \qquad \Lambda(\alpha, 1) = 0, \qquad 0 \leqq \alpha \leqq 1 .$$

From the fact that the sample sum attains a constant value when the sample size equals the population size, we get

$$(10.38) \qquad S_1^{(k)} = \mathbf{0} , \qquad k = 1, 2, \cdots .$$

From (10.37) and (10.38) it is seen that the extension to $[0, 1]$ is correct.

The next theorem is only a special case of Theorem 10.1.

THEOREM 10.2. *If* $(\mathbf{p}_k, \pi_k)$, $k = 1, 2, \cdots$, *satisfies* (3.21) *and* (10.3)—(10.5), *then for every* d, $d = 1, 2, \cdots$, *and every* $(\alpha_1, \alpha_2, \cdots, \alpha_d) \in [0, 1]^d$, *we have*

$$(10.40) \qquad \mathscr{L}\left(\frac{1}{N_k^{\frac{1}{2}}} (Z_{\alpha_1 N_k}^{(k)})^c, \frac{1}{N_k^{\frac{1}{2}}} (Z_{\alpha_2 N_k}^{(k)})^c, \cdots, \frac{1}{N_k^{\frac{1}{2}}} (Z_{\alpha_d N_k}^{(k)})^c\right) \Rightarrow N(0, [B(\alpha_\nu, \alpha_\mu); \nu, \mu = 1, 2, \cdots, d]) \quad \text{as} \quad k \to \infty ,$$

where

$$(10.41) \qquad B(\alpha, \beta) = \iint_{R_2} \left(\frac{\kappa_{11}(\alpha)}{\kappa_{10}(\alpha)} - y\right)\left(\frac{\kappa_{11}(\beta)}{\kappa_{10}(\beta)} - y\right)(1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH(x, y) , \qquad 0 \leqq \alpha \leqq \beta \leqq 1$$

$$= B(\beta, \alpha) , \qquad 0 \leqq \beta < \alpha \leqq 1 .$$

PROOF. Follows from Theorem 10.1 and (10.21). The covariance function $B(\alpha, \beta)$ in (10.41) is obtained as $B(\alpha, \beta) = \lambda_{00}^{(4)}(\alpha, \beta)$, where $\lambda^{(4)}(\alpha, \beta)$ is given in (10.35).

We have now taken the essential steps in the proofs of Theorems 3.2 and 3.3. Only fairly standard arguments remain to deduce these theorems from Theorem 10.2.

LEMMA 10.8. *Let* $A_k(n_k^{(1)}, n_k^{(2)}, \cdots, n_k^{(d)})$ *be according to* (3.33). *If* $(\mathbf{p}_k, \pi_k)$, $k = 1, 2, \cdots$, *satisfies* (3.21) *and* (10.3)—(10.5) *and if*

$$(10.42) \qquad n_k^{(u)}/N_k \to \alpha_u \quad \text{as} \quad k \to \infty , \quad u = 1, 2, \cdots, d ,$$

then

$$(10.43) \qquad (1/N_k)A_k(n_k^{(1)}, n_k^{(2)}, \cdots, n_k^{(d)}) \to [B(\alpha_\nu, \alpha_\mu); \nu, \mu = 1, 2, \cdots, d] \quad \text{as} \quad k \to \infty ,$$

where $B(\alpha, \beta)$ *is according to* (10.41).

PROOF. (3.13)—(3.15) can be written as follows

(10.44) $$\xi_k(n) = \iint_{R^2} x \exp\left(-\rho_k\left(\frac{n}{N_k}\right)x\right) dH(x, y; \mathbf{p}_k, \pi_k), \qquad k = 1, 2, \cdots$$

(10.45) $$\eta_k(n) = \iint_{R^2} xy \exp\left(-\rho_k\left(\frac{n}{N_k}\right)x\right) dH(x, y; \mathbf{p}_k, \pi_k), \qquad k = 1, 2, \cdots$$

(10.46) $$\frac{1}{N_k}\sigma_k(m, n) = \iint_{R^2} \left(y - \frac{\eta_k(m)}{\xi_k(m)}\right)\left(y - \frac{\eta_k(n)}{\xi_k(n)}\right)\left(1 - \exp\left(-\rho_k\left(\frac{m}{N_k}\right)x\right)\right) \times \exp\left(-\rho_k\left(\frac{n}{N_k}\right)x\right) dH(x, y; \mathbf{p}_k, \pi_k),$$
$$1 \leqq m \leqq n \leqq N, \quad k = 1, 2, \cdots.$$

From (10.44)—(10.46) and Lemmas 10.2 and 10.4 we get that if $m_k/N_k \to \alpha$ and $n_k/N_k \to \beta$ as $k \to \infty$, then

(10.47) $$\xi_k(n_k) \to \kappa_{10}(\beta), \qquad \text{as} \quad k \to \infty$$

(10.48) $$\eta_k(n_k) \to \kappa_{11}(\beta), \qquad \text{as} \quad k \to \infty$$

(10.49) $$(1/N_k)\sigma_k(m_k, n_k) \to B(\alpha, \beta) \qquad \text{as} \quad k \to \infty.$$

The assertion in the lemma now follows from (10.49), and the lemma is thus proved.

LEMMA 10.9. *Let* $\alpha_1, \alpha_2, \cdots, \alpha_d$ *be different numbers on the interval* (0.1). *Then the matrix* $[B(\alpha_\nu, \alpha_\mu); \nu, \mu = 1, 2, \cdots, d]$, *where* $B(\alpha, \beta)$ *is according to* (10.41), *is positive definite if and only if the following condition* (10.50) *is fulfilled.*

(10.50) $H(x, y)$ *does not have its entire mass along a line* $y = y_0$.

The proof of this lemma is given in Section 13.

We shall be concerned with the following normalizing conditions on our population sequence π_k, $k = 1, 2, \cdots$ (cf. (2.21) and (2.22))

(10.51) $$\mu_{\pi_k} = 0, \qquad k = 1, 2, \cdots$$

(10.52) $$\sigma^2_{\pi_k} = 1, \qquad k = 1, 2, \cdots.$$

LEMMA 10.10. *Let* $(\mathbf{p}_k, \pi_k)$, $k = 1, 2, \cdots$ *satisfy* (10.3)—(10.5), (10.51) *and* (10.52). *Then the limit mass* $H(x, y)$ *in* (10.5) *satisfies condition* (10.50).

PROOF. The assumptions (10.51) and (10.52) can be written

(10.53) $$\iint_{R^2} y \, dH(x, y; \mathbf{p}_k, \pi_k) = 0, \qquad k = 1, 2, \cdots$$

(10.54) $$\frac{N_k}{N_k - 1}\iint_{R^2} y^2 \, dH(x, y; \mathbf{p}_k, \pi_k) = 1, \qquad k = 1, 2, \cdots.$$

By letting $k \to \infty$ in (10.53) and (10.54) we get from Lemma 10.4

(10.55) $$\iint_{R^2} y \, dH(x, y) = 0 \quad \text{and} \quad \iint_{R^2} y^2 \, dH(x, y) > 0.$$

From (10.55) we conclude that the $H(x, y)$-mass is not concentrated along a line $y = y_0$. Thus the lemma is proved.

LEMMA 10.11. *Let* $(\mathbf{p}_k, \pi_k)$, $k = 1, 2, \cdots$, *satisfy* (3.21), (10.3)—(10.5), (10.51) *and* (10.52). *Let furthermore* (10.42) *be fulfilled with* $\alpha_1, \alpha_2, \cdots, \alpha_d$ *different numbers in* (0.1). *Then*

(10.56) (a) $\left(\frac{1}{N_k^{\frac{1}{2}}}(Z^{(k)}_{n_k^{(1)}})^c, \cdots, \frac{1}{N_k^{\frac{1}{2}}}(Z^{(k)}_{n_k^{(d)}})^c\right)$ *is asymptotically* $N(0, A_k(n_k^{(1)}, \cdots, n_k^{(d)}))$*-distributed as* $k \to \infty$, *where* $A_k(n_k^{(1)}, \cdots, n_k^{(d)})$ *is defined in* (3.33).

(b) *The assertion in* (b) *of Theorem* 3.3 *is true.*

PROOF. The claim (10.56) follows from Theorem 10.2, (10.43) and the fact that $[B(\alpha_\nu, \alpha_\mu); \nu, \mu = 1, 2, \cdots, d]$ is positive definite under the assumptions in the lemma (see Lemmas 10.9 and 10.10). The assertion (b) follows from (10.56) and the following formula (10.57), which is a consequence of (3.12), (3.10), Theorem 3.1 and (10.3).

$$\lim_{k\to\infty} \max_{\tau_1 N_k \leqq n \leqq \tau_2 N_k} \left| \frac{EZ_n^{(k)} - \mu_k(n)}{N_k^{\frac{1}{2}}} \right| = 0 . \tag{10.57}$$

Thereby the lemma is proved.

Thus we have proved (b) in Theorem 3.3 under the extra conditions (10.4), (10.5), (10.51), (10.52) and (10.42). Our next aim is to show that these extra conditions are superfluous for the validity of (b) in Theorem 3.3. Conditions (10.5) and (10.42) can be removed by a compactness argument which is quite analogous to that on page 217 in [5]. We only give a sketch. Under (10.3) and (10.4) all the masses in the family $\mathscr{H} = \{H(x, y; \mathbf{p}_k, \pi_k), k = 1, 2, \cdots\}$ lie on the rectangle $l \leqq x \leqq L$, $-M \leqq y \leqq M$. This implies that $\mathscr{H}$ is sequentially compact under weak convergence. Furthermore, under (3.29) and (3.30) it is always possible to pick a subsequence for which (10.42) is fulfilled. By using these facts it is straightforward to show that (10.5) and (10.42) are superfluous.

That (10.51), (10.52) and (10.4) can be removed if instead (3.23) is assumed, is a consequence of the fact that Theorem 3.3 (b) is "invariant under linear transformations of the population". This is seen from the following formulas, whose verification is left to the reader. When checking (10.59) remember (3.2). Let $\pi = (a_1, a_2, \cdots, a_N)$ and $\pi' = (a_1', a_2', \cdots, a_N')$, where

$$a_s' = \tau a_s + \lambda , \qquad s = 1, 2, \cdots, N . \tag{10.58}$$

Let $'$ denote that a quantity relates to π'. Then, with $\mu(\cdot)$, $\sigma(\cdot, \cdot)$ and Z as in (3.11), (3.15) and (3.1), we have

$$\mu'(n) = \tau\mu(n) + n\lambda , \tag{10.59}$$

$$\sigma'(m, n) = \tau^2\sigma(m, n) , \tag{10.60}$$

$$Z_n' = \tau Z_n + n\lambda . \tag{10.61}$$

Thus, (b) in Theorem 3.3 is completely proved. To deduce (a) in Theorem 3.3

and Theorem 3.2 from what is already proved we shall need the results in the following two lemmas. These results are well known and therefore not proved. In the sequel $|| \ ||$ denotes Euclidean norm.

LEMMA 10.12. *Let D_k be the covariance matrix of the random vector $\mathbf{U}_k$, $k = 1, 2, \cdots$. Suppose that $\mathbf{U}_k$ is asymptotically $N(0, A_k)$-distributed as $k \to \infty$. If for some $\delta > 0$ we have*

$$\limsup_{k\to\infty} E||A_k^{-\frac{1}{2}}\mathbf{U}_k||^{2+\delta} < \infty , \tag{10.62}$$

then $\mathbf{U}_k$ is also asymptotically $N(0, D_k)$-distributed as $k \to \infty$.

LEMMA 10.13. *Let $\mathbf{U}_k$ be asymptotically $N(0, A_k)$-distributed as $k \to \infty$. Then $\mathbf{U}_k$ is also asymptotically $N(0, B_k)$-distributed as $k \to \infty$, if and only if*

$$B_k A_k^{-1} \to I \qquad \text{as} \quad k \to \infty . \tag{10.63}$$

We are now prepared to finish the proof of Theorems 3.3 and 3.2. We start by showing that (a) in Theorem 3.3 is true. We first prove it under the assumptions (10.51), (10.52) and (10.4). Let the sequence $(n_k^{(1)}, n_k^{(2)}, \cdots, n_k^{(d)})$, $k = 1, 2, \cdots$, satisfy (3.29) and (3.30), and let A_k be the corresponding matrix according to (3.33). Further let D_k be the covariance matrix of $(Z^{(k)}_{n_k^{(1)}}, \cdots, Z^{(k)}_{n_k^{(d)}})$. We have

$$\begin{aligned} E||A_k^{-\frac{1}{2}}((Z^{(k)}_{n_k^{(1)}})^c, \cdots, (Z^{(k)}_{n_k^{(d)}})^c)||^4 &\leqq \frac{C_d}{||A_k||^2} \sum_{u=1}^d E|(Z^{(k)}_{n_k^{(u)}})^c|^4 \\ &\leqq \frac{C_d}{||A_k||^2} \cdot d \cdot (n_k^{(d)})^2 \cdot M_k^4 \cdot C\left(\rho_k, \frac{n_k^{(d)}}{N_k}\right) \end{aligned} \tag{10.64}$$

according to Lemma 8.5.

From (10.43), Lemmas 10.9 and 10.10 and a simple compactness argument we conclude that

$$\liminf_{k\to\infty} \frac{||A_k||^2}{N_k^2} > 0 . \tag{10.65}$$

Now, (10.64) and (10.65) yield that

$$\limsup_{k\to\infty} E||A_k^{-\frac{1}{2}}((Z^{(k)}_{n_k^{(1)}})^c, \cdots, (Z^{(k)}_{n_k^{(d)}})^c)'||^4 < \infty . \tag{10.66}$$

From Lemma 10.11 (a), Lemma 10.12 and (10.66) we conclude that (a) in Theorem 3.3 is true under the assumptions (10.51), (10.52) and (10.4). That these conditions can be replaced by (3.23) follows from the easily verified fact that also (a) in Theorem 3.3. is "invariant under linear transformations of the population." We omit the details of this verification. Thus Theorem 3.3 is completely proved.

Next we shall prove Theorem 3.2. From (a) and (b) in Theorem 3.3 and from Lemma 10.13 we conclude that if

$$0 < \liminf_{k\to\infty} \frac{n_k}{N_k} \leqq \limsup_{k\to\infty} \frac{n_k}{N_k} < 1 , \tag{10.67}$$

then

$$\lim_{k\to\infty} \frac{\sigma^2(Z_{n_k}^{(k)})}{\sigma_k(n_k, n_k)} = 1 . \tag{10.68}$$

From this we can conclude that (3.26) holds for $m = n$ by an indirect proof as follows. Assume that (3.26) with $m = n$ does not hold. Then we can pick a subsequence $\{k'\}$ for which (10.67) holds and such that

$$\liminf_{k'\to\infty} |r_\sigma^{(k')}(n_{k'}, n_{k'})| > 0 . \tag{10.69}$$

As (10.67) is fulfilled for $n_{k'}$ so is, according to what is proved, also (10.68). But (10.68) and (10.69) then contradict each other, yielding that (3.26) holds for $m = n$.

Let m_k and n_k, $k = 1, 2, \cdots$, be such that

$$0 < \liminf_{k\to\infty} \frac{m_k}{N_k} \leqq \limsup_{k\to\infty} \frac{n_k}{N_k} < 1 \tag{10.70}$$

and

$$\liminf_{k\to\infty} \left(\frac{n_k}{N_k} - \frac{m_k}{N_k}\right) > 0 . \tag{10.71}$$

From (a) and (b) in Theorem 3.3 and from Lemma 10.13 we then conclude that

$$\begin{bmatrix} \sigma^2(Z_{m_k}^{(k)}) & \operatorname{Cov}(Z_{m_k}^{(k)}, Z_{n_k}^{(k)}) \\ \operatorname{Cov}(Z_{m_k}^{(k)}, Z_{n_k}^{(k)}) & \sigma^2(Z_{n_k}^{(k)}) \end{bmatrix} \begin{bmatrix} \sigma_k^2(m_k) & \sigma_k(m_k, n_k) \\ \sigma_k(m_k, n_k) & \sigma_k^2(n_k) \end{bmatrix}^{-1} \to I . \tag{10.72}$$

The off-diagonal element in the matrix product in (10.72) yields

$$\frac{\operatorname{Cov}(Z_{m_k}^{(k)}, Z_{n_k}^{(k)})\sigma_k^2(m_k) - \sigma^2(Z_{m_k}^{(k)})\sigma_k(m_k, n_k)}{\sigma_k^2(m_k)\sigma_k^2(n_k) - \sigma_k(m_k, n_k)^2} = \alpha(m_k, n_k) \to 0 , \tag{10.73}$$

which can be written

$$\begin{aligned} \operatorname{Cov}(Z_{m_k}^{(k)}, Z_{n_k}^{(k)}) = \sigma_k(m_k, n_k) + \sigma_k(m_k)\sigma(n_k)\Bigg[\left(\frac{\sigma^2(Z_{m_k}^{(k)})}{\sigma_k^2(m_k)} - 1\right)\frac{\sigma_k(m_k, n_k)}{\sigma_k(m_k)\sigma_k(n_k)} \\ + \alpha(m_k, n_k)\left(\frac{\sigma_k(m_k)}{\sigma_k(n_k)} - \frac{\sigma_k(m_k, n_k)^2}{\sigma_k^3(m_k)\sigma(n_k)}\right)\Bigg] . \end{aligned} \tag{10.74}$$

From Lemma 13.1 and (10.46) it follows that the function $\sigma_k(m, n)$ is nonnegative. Thus,

$$|\sigma_k(m, n)| \leqq \sigma_k(m)\cdot\sigma_k(n) . \tag{10.75}$$

Moreover, by using the results in Lemmas 10.9 and 10.10 one can in a fairly straightforward way deduce that if (3.21)—(3.23) are fulfilled, then

$$\limsup_{k\to\infty} \max_{\tau_1 N_k \leqq m, n \leqq \tau_2 N_k} \frac{\sigma_k(n)}{\sigma_k(m)} < \infty \tag{10.76}$$

for $0 < \tau_1 < \tau_2 < 1$. By an argument similar to that which was used to derive (3.26) for $m = n$ from (10.67) and (10.68), we now can conclude from (10.70)—(10.76) that the following relation holds, (10.77) $$\limsup_{k\to\infty} \max_{\tau_1 N_k \leqq m \leqq \lambda_1 N_k < \lambda_2 N_k \leqq n \leqq \tau_2 N_k} |r_\sigma^{(k)}(m, n)| = 0$$

for $0 < \tau_1 < \lambda_1 < \lambda_2 < \tau_2 < 1$.

Thereby we have almost proved Theorem 3.2. It now only remains to get rid of the separating λ's in (10.77). As a full proof will be lengthy but quite straightforward we omit it. We content ourselves with pointing out the following "continuity" property of Cov (Z_m, Z_n), and we leave the details to the reader.

(10.78) $$|\mathrm{Cov}\,(Z_{\alpha N}, Z_{\beta N}) - \sigma_2(Z_{\alpha N})| \leqq \alpha^{\frac{1}{2}}|\beta - \alpha|^{\frac{1}{2}} \cdot N \cdot M^2 C(\rho, \max(\alpha, \beta))$$

where M, ρ and $C(\cdot, \cdot)$ are as in (2.23), (2.20) and (1.4). By using Lemmas 7.1 and 11.1 we obtain (10.78) as follows. Assume that $\alpha N < \beta N$.

(10.79) $$\begin{aligned} &|\mathrm{Cov}\,(Z_{\alpha N}, Z_{\beta N}) - \sigma^2(Z_{\alpha N})| \\ &\quad = |EZ_{\alpha N}^c(Z_{\beta N}^c - Z_{\alpha N}^c)| \leqq (E(Z_{\alpha N}^c)^2)^{\frac{1}{2}}(E(Z_{\beta N}^c - Z_{\alpha N}^c)^2)^{\frac{1}{2}} \\ &\quad \leqq (\alpha N \cdot M^2 C(\rho, \alpha))^{\frac{1}{2}}((\beta N - \alpha N) M^2 \cdot C(\rho, \beta))^{\frac{1}{2}} . \end{aligned}$$

Now (10.78) follows from (10.79). This concludes the proof of Theorems 3.2 and 3.3.

11. Verification of the *C*-conditions. In this section we shall prove Lemma 10.6, i.e. we shall show that the conditions (C0)—(C4) are satisfied.

Condition (C0) is trivially satisfied, as we are dealing with a sum process. We prepare ourselves for the verification of (C1)′ with the following lemma, which is an extension of Lemma 7.1.

LEMMA 11.1. *With the same assumptions as in Lemma* 7.1 *we have for* $0 \leqq n < n + m < N$, $u \geqq 0$.

(11.1) $$E|\textstyle\sum_{\nu=n+1}^{n+m} D_\nu^c|^u \leqq m^{u/2} \cdot M^u \cdot C_u\left(\rho, \dfrac{n+m}{N}\right).$$

PROOF. We have

(11.2) $$\begin{aligned} E|\textstyle\sum_{\nu=n+1}^{n+m} D_\nu^c|^u \leqq\ & C_u E E^{\mathscr{B}_n}|\textstyle\sum_{\nu=n+1}^{n+m} (D_\nu - E^{\mathscr{B}_n} D_\nu)| \\ &+ C_u E|\textstyle\sum_{\nu=n+1}^{n+m} (E^{\mathscr{B}_n} D_\nu - ED_\nu)|^u . \end{aligned}$$

Let

(11.3) $$\pi(\mathscr{B}_n) = \{d_s : s \in G(\mathscr{B}_n)\}$$

where $G(\mathscr{B}_n)$ is defined in (6.2). According to Lemmas 6.1 and 7.1 we have

(11.4) $$\begin{aligned} E^{\mathscr{B}_n}|\textstyle\sum_{\nu=n+1}^{n+m} (D_\nu - E^{\mathscr{B}_n} D_\nu)|^u &\leqq m^{u/2} \cdot M(\pi(\mathscr{B}_n))^u \cdot C_u\left(\rho, \dfrac{m}{N-n}\right) \\ &= m^{u/2} \cdot M^u \cdot C_u\left(\rho, \dfrac{n+m}{N}\right). \end{aligned}$$

The following estimate is easily derived from Lemma 8.6,

(11.5) $$E|E^{\mathscr{B}_{\nu-1}} D_\nu - ED_\nu|^u \leqq \left(\frac{\nu^{\frac{1}{2}}}{N} M\right)^u \cdot C_u\left(\rho, \frac{\nu}{N}\right).$$

By Hölder's inequality and (11.5) we get for $u \geqq 1$,

$$
\begin{aligned}
(11.6)\quad & E\left|\sum_{\nu=n+1}^{n+m}(E^{\mathscr{B}_n}D_\nu - ED_\nu)\right|^u \\
& \leqq m^{u-1}\sum_{\nu=n+1}^{n+m} E|E^{\mathscr{B}_{\nu-1}}D_\nu - ED_\nu|^u \\
& \leqq m^u\left(\frac{(n+m)^{\frac{1}{2}}}{N}M\right)^u C_u\left(\rho, \frac{n+m}{N}\right) \leqq m^{u/2}C_u\left(\rho, \frac{n+m}{N}\right).
\end{aligned}
$$

Thereby the lemma is proved for $u \geqq 1$. The extension to $u \geqq 0$ is straightforward.

Verification of (C1)$'$. Let f_ν and g_μ be according to (9.5) and (9.6). From Lemmas 8.1 and 11.1 we get for $0 \leqq \beta < \alpha < 1$

$$
\begin{aligned}
(11.7)\quad & \limsup_{k\to\infty} E\langle S_\alpha^{(k)} - S_\beta^{(k)}, f_h\rangle^2 \\
& = \frac{1}{\gamma^{2h}}\limsup_{k\to\infty}\frac{1}{N_k}E\left(\sum_{\nu=\beta N_k+1}^{\alpha N_k} Y_\nu^c(h,0)\right)^2 \\
& \leqq \frac{1}{\gamma^{2h}}\limsup_{k\to\infty}\frac{1}{N_k}\left((\alpha-\beta)N_k\cdot L^{2h}\cdot C\left(\rho_k, \frac{[\alpha N_k]}{N_k}\right)\right) \\
& \leqq (\alpha-\beta)\left(\frac{L}{\gamma}\right)^{2h}C(\alpha)
\end{aligned}
$$

for some function $C(\alpha)$ which is continuous on $0 \leqq \alpha < 1$. In a quite similar manner we get

$$
(11.8)\quad \limsup_{k\to\infty} E\langle S_\alpha^{(k)} - S_\beta^{(k)}, g_l\rangle^2 \leqq (\alpha-\beta)\left(\frac{L}{\gamma}\right)^{2l}M^2C(\alpha)\,.
$$

Now condition (C1)$'$ is easily verified by using (11.7) and (11.8) and the fact that $L/\gamma < 1$.

Verification of (C2). We shall show that condition (C2d) is fulfilled for every natural number d. By using Lemma 8.7 we get the following formula

$$
\begin{aligned}
(11.9)\quad E^{\mathscr{B}_n}Y_{n+m}^c(i,j) &= E^{\mathscr{B}_n}E^{\mathscr{B}_{n+m-1}}Y_{n+m}^c(i,j) \\
&= \frac{1}{N}\Bigg[\frac{v_{n+m-1}(i+1,j)}{v_{n+m-1}(1,0)^2}Z_n^c(1,0) \\
&\qquad - \frac{1}{v_{n+m-1}(1,0)}Z_n^c(i+1,j)\Bigg] + Q(n,m,i,j)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
(11.10)\quad Q(n,m,i,j) = {} & \frac{1}{N}\Bigg[\frac{v_{n+m-1}(i+1,j)}{v_{n+m-1}(1,0)^2}E^{\mathscr{B}_n}(Z_{n+m-1}^c(1,0) - Z_n^c(1,0)) \\
& - \frac{1}{v_{n+m-1}(1,0)}E^{\mathscr{B}_n}(Z_{n+m-1}^c(i+1,j) - Z_n^c(i+1,j))\Bigg] \\
& + E^{\mathscr{B}_n}R^{(2)}(n+m,i,j)\,,
\end{aligned}
$$

where $R^{(2)}$ is defined in (8.31). According to Lemma 11.1 we have

(11.11) $$E|Z^c_{n+m-1}(i, j) - Z_n^c(i, j)|^u = E\left|\frac{1}{N} \sum_{\nu=n+1}^{n+m-1} Y_\nu^c(i, j)\right|^u \leq \left(\frac{m^{\frac{1}{2}}}{N} L^i M^j\right)^u C_u\left(\rho, \frac{n+m}{N}\right).$$

By using the estimates (11.11) and (8.32) we get (after some computation) from (11.10),

(11.12) $$E|Q(n, m, i, j)|^u \leq \left(\frac{m^{\frac{1}{2}}}{N^2} + \frac{n+m}{N^2}\right)^u L^{iu} M^{ju} C_u\left(\rho, \frac{n+m}{N}\right)$$

where $C_u(\cdot, \cdot)$ is as in (1.4). Formula (11.9) can be written more compactly as follows.

(11.13) $$E^{\mathscr{B}_{\alpha N}} X_{\alpha N+m} = \frac{1}{N}[M(\alpha, m)S_\alpha + R(\alpha, m)]$$

where

(11.14) $$M(\alpha, m) = \left[\begin{array}{l|l} m^{(1)}_{\nu\mu}(\alpha, m) & m^{(2)}_{\nu\mu}(\alpha, m) \\ \nu, \mu = 1, 2, \cdots & \nu = 1, 2, \cdots, \mu = 0, 1, 2, \cdots \\ \hline m^{(3)}_{\nu\mu}(\alpha, m) & m^{(4)}_{\nu\mu}(\alpha, m) \\ \nu = 0, 1, 2, \cdots, \mu = 1, 2, \cdots & \nu, \mu = 0, 1, 2, \cdots \end{array}\right]$$

where $m_{\nu\mu}(\alpha, m) \equiv 0$ in all but the following cases:

(11.15) $$m^{(1)}_{\nu 1}(\alpha, m) = \frac{v_{\alpha N+m-1}(\nu+1, 0)}{v_{\alpha N+m-1}(1, 0)^2 \cdot \gamma^{\nu-1}}, \qquad \nu = 1, 2, \cdots$$

(11.16) $$m^{(3)}_{\nu 1}(\alpha, m) = \frac{v_{\alpha N+m-1}(\nu+1, 1)}{v_{\alpha N+m-1}(1, 0)^2 \cdot \gamma^{\nu-1}}, \qquad \nu = 0, 1, 2, \cdots$$

(11.17) $$m^{(p)}_{\nu,\nu+1}(\alpha, m) = -\frac{\gamma}{v_{\alpha N+m-1}(1, 0)},$$
$$\nu = 1, 2, \cdots, p = 1, \nu = 0, 1, 2, \cdots, p = 4.$$

Furthermore, $R(\alpha, m)$ is a random element with values in H whose components are as follows:

(11.18) $$\langle R(\alpha, m), f_\nu\rangle = \frac{1}{N^{\frac{1}{2}}} Q(\alpha N, m, \nu, 0) \cdot \frac{1}{\gamma^\nu}, \qquad \nu = 1, 2, \cdots$$

(11.19) $$\langle R(\alpha, m), g_\mu\rangle = \frac{1}{N^{\frac{1}{2}}} Q(\alpha N, m, \mu, 1) \cdot \frac{1}{\gamma^\mu}, \qquad \mu = 0, 1, 2, \cdots.$$

By applying Lemma 12.1 in [4] the assertion (c) in Lemma 10.6 now follows from the above formulas and from Lemma 10.5. We omit the computations which are straightforward but somewhat lengthy.

Verification of (C3). We shall prove assertion (d) in Lemma 10.6 by applying Lemma 12.2 in [4]. From (8.37) we get

$$
\begin{aligned}
(11.20)\quad & E^{\mathscr{B}_n} Y^c_{n+m}(i_1, j_1) Y^c_{n+m}(i_2, j_2) \\
&= E^{\mathscr{B}_n} E^{\mathscr{B}_{n+m-1}} Y^c_{n+m}(i_1, j_1) Y^c_{n+m}(i_2, j_2) \\
&= \frac{v_{n+m-1}(i_1 + i_2 + 1, j_1 + j_2)}{v_{n+m-1}(1, 0)} - \frac{v_{n+m-1}(i_1 + 1, j_1) \cdot v_{n+m-1}(i_2 + 1, j_2)}{v_{n+m-1}(1, 0)^2} \\
&\quad + E^{\mathscr{B}_n} R(n + m - 1, i_1, i_2, j_1, j_2) ,
\end{aligned}
$$

where the last term can be estimated by using (8.38). From (11.20), (8.38) and Lemma 10.4 it is readily verified that condition 2′ in Lemma 12.2 in [4] is fulfilled for $D(\alpha)$ according to (10.24). We omit the details. We are through if we show that also condition 3′ in Lemma 12.2 in [4] is fulfilled. In the following lemma we give the essential estimate and we leave the rest of the verification to the reader.

LEMMA 11.2. *For* $1 \leqq m_1 \neq m_2 < N - n$ *we have*

$$
(11.21)\quad E|E^{\mathscr{B}_n} Y^c_{n+m_1}(i_1, j_1) Y^c_{n+m_2}(i_2, j_2)| \leqq \frac{1}{N} L^{i_1+i_2} M^{j_1+j_2} C\Big(\rho, \frac{n + \max(m_1, m_2)}{N}\Big)
$$

where $C(\cdot, \cdot)$ *is as in* (1.4).

PROOF. We shall use the following inequality, where notation and assumptions are as in Lemma 8.9. For $1 \leqq m_1 \neq m_2 < N - n$ we have,

$$
(11.22)\quad E|E^{\mathscr{B}_n} D^c_{n+m_1} H^c_{n+m_2}| \leqq \frac{1}{N} M(\pi_1) \cdot M(\pi_2) \cdot C\Big(\rho, \frac{n + \max(m_1, m_2)}{N}\Big) .
$$

The inequality (11.22) can be derived from (8.43) by a conditioning argument which is quite analogous to that which was used in the proof of Lemma 8.9 in the extension from $\nu_1 = 1$ to a general value of ν_1. Therefore we omit the proof.

Now choose $\pi_1 = \pi(i_1, j_1)$ and $\pi_2 = \pi(i_2, j_2)$ (see (8.6)). In view of Lemma 8.1 and (8.13), (11.22) then yields (11.21) and the lemma is proved.

Verification of (C4). We shall verify (C4) by showing that the condition in the remark to Lemma 12.3 in [4] is fulfilled. From Lemmas 8.1 and 11.1 we conclude, with f as in (9.5),

$$
\begin{aligned}
(11.23)\quad & \limsup_{k\to\infty} E\langle S^{(k)}_{\alpha+\Delta} - S_\alpha^{(k)}, f_i\rangle^4 \\
&= \limsup_{k\to\infty} E \frac{1}{N_k^2}\Big(\textstyle\sum_{\nu=\alpha N_k+1}^{(\alpha+\Delta)N_k} Y_\nu^{(k)}(i, 0)^c \cdot \frac{1}{\gamma^i}\Big)^4 \\
&\leqq \frac{1}{\gamma^{4i}} \limsup_{k\to\infty} \frac{1}{N_k^2} (\Delta N_k)^2 \cdot L^4 \cdot C\Big(\rho_k, \frac{[(\alpha + \Delta)N_k]}{N_k}\Big) \\
&= \Delta^2 \Big(\frac{L}{\gamma}\Big)^{4i} \limsup_{k\to\infty} C(\rho_k, \alpha + \Delta) , \qquad i = 1, 2, \cdots .
\end{aligned}
$$

In the same manner we get, with g as in (9.6),

$$\limsup_{k\to\infty} E\langle S^{(k)}_{\alpha+\Delta} - S_{\alpha}^{(k)}, g_i\rangle \leqq \Delta^2\left(\frac{L}{\gamma}\right)^{4i} \cdot M^4 \limsup_{k\to\infty} C(\rho_k, \alpha+\Delta)\,, \qquad i = 0, 1, 2, \cdots . \tag{11.24}$$

Now only straightforward computations remain to show that the condition in the remark to Lemma 12.3 in [4] is fulfilled. Thereby (C4) is verified and the proof of Lemma 10.6 is complete.

12. Solution of the differential equations. We shall here prove Lemma 10.7. According to Theorem B in [4], (10.26)—(10.30) are uniquely solvable. One way of proving Lemma 10.7 would be to check that (10.31) with components as in (10.32)—(10.35) actually satisfies (10.26)—(10.30). However, we shall produce the solution in a more constructive way.

It is easily checked that (10.27)—(10.30) are homogeneous in the parameter γ, which enters in $M(\alpha)$ and $D(\alpha)$ (see (10.23) and (10.24)) in the following sense. If $\lambda^{(i)}_{\nu\mu}(\alpha, \beta)$ solve (10.27)—(10.30) for the original $M(\alpha)$ and $D(\alpha)$, then $\gamma^{(\nu+\mu)}\lambda^{(i)}_{\nu\mu}(\alpha, \beta)$ solve (10.27)—(10.30) with $\gamma = 1$ in $M(\alpha)$ and $D(\alpha)$. It is therefore no loss of generality to assume that $\gamma = 1$, and we do so throughout this section.

We point out that the mass $H(x, y)$ lies on a rectangle $0 < l \leqq x \leqq L < \infty$, $-M < y < M$. From this it follows that we do not have any problems of convergence of the integrals which will be considered in the sequel. Concerning integrals we adopt the convention that when no domain of integration is specified in double integrals, the integration is over R^2.

We state without proof an elementary result about partial differential equations, which will be used repeatedly in the course of the solution.

LEMMA 12.1. *Consider the differential equation*

$$\frac{\partial}{\partial x}u(x, y_1, y_2, \cdots, y_m) + a'(x) \sum_{\nu=1}^{m} \frac{\partial}{\partial y_\nu}u(x, y_1, y_2, \cdots, y_m) = R(x, y_1, y_2, \cdots, y_m)\,, \qquad x_0 < x < x_1,\ -\infty < y_\nu < \infty \tag{12.1}$$

with given, sufficiently smooth boundary data along the hyperplane $x = x_0$.

This equation is uniquely solvable and the solution is

$$\begin{aligned} u(x, y_1, y_2, \cdots, y_m) &= u(x_0, y_1 - a(x) + a(x_0), \cdots, y_m - a(x) + a(x_0)) \\ &\quad + \int_{x_0}^{x} R(u, y_1 - a(x) + a(u), \cdots, y_m - a(x) + a(u))\, du\,. \end{aligned} \tag{12.2}$$

The following formula (12.3) will be used repeatedly without explicit reference every time.

LEMMA 12.2. *For $\rho(\alpha)$ and $\kappa_{10}(\alpha)$ according to Definitions* 10.1 *and* 10.2 *we have*

$$\rho'(\alpha) = \frac{1}{\kappa_{10}(\alpha)}\,, \qquad 0 < \alpha < 1\,. \tag{12.3}$$

PROOF. By differentiating with respect to α in (10.7) we get

$$-1 = -\rho'(\alpha) \textstyle\int_0^\infty xe^{-\rho(\alpha)x}\, dH(x) = -\rho'(\alpha)\kappa_{10}(\alpha)\,, \qquad 0 < \alpha < 1\,,$$

and (12.3) follows.

We now begin the solution of (10.26)—(10.30) and we start with the equation (10.27). To simplify writing, we write $\Lambda(\alpha)$ and $\lambda_{\nu\mu}(\alpha)$ instead of $\Lambda(\alpha, \alpha)$ and $\lambda_{\nu\mu}(\alpha, \alpha)$.

By writing (10.27) in terms of matrices we get the following systems of differential equations.

From the upper left-hand block:

$$\begin{aligned}(12.4)\qquad \frac{d}{d\alpha}\lambda_{\nu\mu}^{(1)}(\alpha) &= \frac{-1}{\kappa_{10}(\alpha)}(\lambda_{\nu+1,\mu}^{(1)}(\alpha) + \lambda_{\nu,\mu+1}^{(1)}(\alpha)) + \frac{\kappa_{\nu+1,0}(\alpha)}{\kappa_{10}(\alpha)^2}\lambda_{1\mu}^{(1)}(\alpha) \\ &\quad + \frac{\kappa_{\mu+1,0}(\alpha)}{\kappa_{10}(\alpha)^2}\lambda_{1\nu}^{(1)}(\alpha) + \frac{\kappa_{\nu+\mu+1,0}(\alpha)}{\kappa_{10}(\alpha)} - \frac{\kappa_{\nu+1,0}(\alpha)\kappa_{\mu+1,0}(\alpha)}{\kappa_{10}(\alpha)^2}\,, \\ &\qquad 0 < \alpha < 1\,, \quad \nu, \mu = 1, 2, \cdots.\end{aligned}$$

From the upper right-hand block:

$$\begin{aligned}(12.5)\qquad \frac{d}{d\alpha}\lambda_{\nu\mu}^{(2)}(\alpha) &= \frac{-1}{\kappa_{10}(\alpha)}(\lambda_{\nu+1,\mu}^{(2)}(\alpha) + \lambda_{\nu,\mu+1}^{(2)}(\alpha)) + \frac{\kappa_{\nu+1,0}(\alpha)}{\kappa_{10}(\alpha)^2}\lambda_{1\mu}^{(2)}(\alpha) \\ &\quad + \frac{\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)^2}\lambda_{1\nu}^{(1)}(\alpha) + \frac{\kappa_{\nu+\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)} - \frac{\kappa_{\nu+1,0}(\alpha)\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)^2}\,, \\ &\qquad 0 < \alpha < 1\,, \quad \nu = 1, 2, \cdots, \mu = 0, 1, 2, \cdots.\end{aligned}$$

Because of symmetry the lower left-hand block will add nothing new.

From the lower right-hand block:

$$\begin{aligned}(12.6)\qquad \frac{d}{d\alpha}\lambda_{\nu\mu}^{(4)}(\alpha) &= \frac{-1}{\lambda_{10}(\alpha)}(\lambda_{\nu+1,\mu}^{(4)}(\alpha) + \lambda_{\mu+1,\nu}^{(4)}(\alpha)) + \frac{\kappa_{\nu+1,1}(\alpha)}{\kappa_{10}(\alpha)^2}\lambda_{1\mu}^{(2)}(\alpha) \\ &\quad + \frac{\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)^2}\lambda_{1\nu}^{(2)}(\alpha) + \frac{\kappa_{\nu+\mu+1,2}(\alpha)}{\kappa_{10}(\alpha)} - \frac{\kappa_{\nu+1,1}(\alpha)\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)^2}\,, \\ &\qquad 0 < \alpha < 1\,, \quad \nu, \mu = 0, 1, 2, \cdots.\end{aligned}$$

Furthermore, from (10.28) we get the following initial values

$$(12.7)\qquad \lambda_{\nu\mu}(0) = 0 \qquad \text{for all } \lambda_{\nu\mu}(\alpha) \text{ in } \Lambda(\alpha)\,.$$

Next we introduce some functions which will turn up during the solving process.

$$(12.8)\qquad K_\mu(\alpha, t) = \textstyle\sum_{\nu=0}^\infty \frac{t^\nu}{\nu!}\kappa_{\nu+1,\mu}(\alpha) = \iint xy^\mu e^{tx}e^{-\rho(\alpha)x}\, dH\,, \qquad \mu = 0, 1, 2, \cdots$$

$$(12.9)\qquad V_i(\alpha, t, s) = \textstyle\sum_{\nu=0}^\infty \sum_{\mu=0}^\infty \frac{t^\nu}{\nu!}\cdot\frac{s^\mu}{\mu!}\kappa_{\nu+\mu+1,i}(\alpha) = \iint xy^i e^{tx}e^{sx}e^{-\rho(\alpha)x}\, dH\,, \qquad i = 0, 1, 2, \cdots$$

$$(12.10)\qquad \tau_{\nu\mu}(\alpha) = \iint x^\nu y^\mu(1 - e^{-\rho(\alpha)x})e^{-\rho(\alpha)x}\, dH\,, \qquad \nu, \mu = 0, 1, 2, \cdots$$

(12.11) $$T_\mu(\alpha, t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \tau_{\nu\mu}(\alpha) = \iint y^\mu e^{tx}(1 - e^{-\rho(\alpha)x}) e^{-\rho(\alpha)x}\, dH\,, \qquad \mu = 0, 1, 2, \cdots$$

(12.12) $$Q_i(\alpha, t, s) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{t^\nu}{\nu!} \cdot \frac{s^\mu}{\mu!} \tau_{\nu+\mu,i}(\alpha) = \iint y^i e^{tx} e^{sx}(1 - e^{-\rho(\alpha)x}) e^{-\rho(\alpha)x}\, dH\,, \qquad i = 0, 1, 2, \cdots .$$

The following relations, which are easily checked, will be needed in the sequel. When checking (12.14) remember (12.3)

(12.13) $$K_\mu(u, t - \rho(\alpha) + \rho(u)) = K_\mu(\alpha, t)\,,$$

(12.14) $$\int_0^\alpha \frac{1}{\kappa_{10}(u)} V_i(u, t - \rho(\alpha) + \rho(u), s - \rho(\alpha) + \rho(u))\, du = Q_i(\alpha, t, s)\,, \qquad i = 0, 1, 2, \cdots .$$

Next we make the following observation. Let $\{\lambda_{\nu\mu}^{(1)}(\alpha);\ \nu, \mu = 1, 2, \cdots\}$ solve (12.4). Then, the enlarged system $\{\lambda_{\nu\mu}^{(1)}(\alpha);\ \nu, \mu = 0, 1, 2, \cdots\}$, where

(12.15) $$\lambda_{\nu\mu}^{(1)}(\alpha) \equiv 0\,, \qquad 0 \leqq \alpha < 1\,, \quad \text{when} \quad \nu \cdot \mu = 0$$

solves the enlarged system (12.4) which is obtained by letting ν and μ run over $0, 1, 2, \cdots$. In fact, as is easily realized, this enlargement of the solution is unique under the assumption that $\lambda_{\nu\mu}^{(1)}(0) = 0$ for $\nu \cdot \mu = 0$. In the same manner a solution $\{\lambda_{\nu\mu}^{(2)}(\alpha);\ \nu = 1, 2, \cdots, \mu = 0, 1, 2, \cdots\}$ of (12.5) can be enlarged to a solution $\{\lambda_{\nu\mu}^{(2)}(\alpha);\ \nu, \mu = 0, 1, 2, \cdots\}$ of the enlarged system (12.5) which is obtained by letting also ν run over 0. Such an enlargement is obtained by putting

(12.16) $$\lambda_{0\mu}^{(2)}(\alpha) \equiv 0\,, \qquad 0 \leqq \alpha < 1\,, \quad \mu = 0, 1, 2, \cdots,$$

and it is unique under the assumption that $\lambda_{0\mu}^{(2)}(0) = 0$, $\mu = 0, 1, 2, \cdots$. In the sequel we consider the systems (12.4), (12.5) and (12.6) for $\nu, \mu = 0, 1, 2, \cdots$.

We introduce the following functions.

(12.17) $$\Phi^{(p)}(\alpha, t, s) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{t^\nu}{\nu!} \cdot \frac{s^\mu}{\mu!} \lambda_{\lambda\mu}^{(p)}(\alpha)\,, \qquad p = 1, 2, 4$$

(12.18) $$\chi^{(p)}(\alpha, t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \lambda_{1\nu}^{(p)}(\alpha)\,, \qquad p = 1, 2, 4\,.$$

We note the following relations, which are easily checked.

(12.19) $$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{t^\nu}{\nu!} \cdot \frac{s^\mu}{\mu!} \lambda_{\nu+1,\mu}^{(p)}(\alpha) = \frac{\partial}{\partial t} \Phi^{(p)}(\alpha, t, s)\,, \qquad p = 1, 2, 4$$

(12.20) $$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{t^\nu}{\nu!} \cdot \frac{s^\mu}{\mu!} \lambda_{\nu,\mu+1}^{(p)}(\alpha) = \frac{\partial}{\partial s} \Phi^{(p)}(\alpha, t, s)\,, \qquad p = 1, 2, 4\,.$$

Next we observe that the system (12.4) only contains $\lambda^{(1)}$-functions. Thus this system can be solved separately and we start by doing this. We multiply (12.4) by $t^\nu s^\mu/\nu!\mu!$ and sum over $\nu, \mu = 0, 1, 2, \cdots$. Observing the formulas (12.8), (12.9), (12.17)—(12.20) and (12.3) we get the following equation.

$$(12.21)\qquad \frac{\partial}{\partial\alpha}\Phi^{(1)}(\alpha, t, s) + \rho'(\alpha)\frac{\partial}{\partial t}\Phi^{(1)}(\alpha, t, s) + \rho'(\alpha)\frac{\partial}{\partial s}\Phi^{(1)}(\alpha, t, s) = R^{(1)}(\alpha, t, s)\,, \qquad 0 < \alpha < 1\,, \quad -\infty < t, \quad s < \infty\,,$$

where

$$(12.22)\qquad R^{(1)}(\alpha, t, s) = \frac{1}{\kappa_{10}(\alpha)^2}(K_0(\alpha, t)\chi^{(1)}(\alpha, s) + K_0(\alpha, s)\chi^{(1)}(\alpha, t)) + \frac{V_0(\alpha, t, s)}{\kappa_{10}(\alpha)^2} - \frac{K_0(\alpha, t)\cdot K_0(\alpha, s)}{\kappa_{10}(\alpha)^2}\,.$$

Furthermore, from (12.7) and (12.15) we get the following boundary values.

$$(12.23)\qquad \Phi^{(1)}(0, t, s) \equiv 0\,, \qquad -\infty < t, \quad s < \infty\,.$$

According to Lemma 12.1 the solution of (12.21) and (12.23) is

$$(12.24)\qquad \Phi^{(1)}(\alpha, t, s) = \int_0^\alpha R^{(1)}(u, t - \rho(\alpha) + \rho(u), s - \rho(\alpha) + \rho(u))\, du\,.$$

From (12.24), (12.22), (12.13) and (12.14) we get,

$$(12.25)\qquad \Phi^{(1)}(\alpha, t, s) = K_0(\alpha, t)A^{(1)}(\alpha, s) + K_0(\alpha, s)A^{(1)}(\alpha, t) + Q_0(\alpha, t, s) - K_0(\alpha, t)K_0(\alpha, s)\cdot b(\alpha)\,,$$

where

$$(12.26)\qquad A^{(1)}(\alpha, t) = \int_0^\alpha \frac{\chi^{(1)}(u, t - \rho(\alpha) + \rho(u))}{\kappa_{10}(u)^2}\, du$$

and

$$(12.27)\qquad b(\alpha) = \int_0^\alpha \frac{du}{\kappa_{10}(u)^2}\,.$$

The function $A^{(1)}$ in (12.25) is still unknown, and we proceed to determine it. The following relation is a consequence of (12.15)

$$(12.28)\qquad \Phi^{(1)}(\alpha, t, 0) \equiv \Phi^{(1)}(\alpha, 0, s) \equiv 0\,.$$

By putting $t = s = 0$ in (12.25) and by observing (12.28) we obtain,

$$(12.29)\qquad 2K_0(\alpha, 0)A^{(1)}(\alpha, 0) + Q_0(\alpha, 0, 0) - K_0(\alpha, 0)^2 b(\alpha) = 0\,,$$

which yields

$$(12.30)\qquad A^{(1)}(\alpha, 0) = -\frac{Q_0(\alpha, 0, 0)}{2K_0(\alpha, 0)} + \tfrac{1}{2}K_0(\alpha, 0)b(\alpha)\,.$$

By putting $s = 0$ in (12.25) and by considering (12.28) we obtain,

$$(12.31)\qquad K_0(\alpha, t)A^{(1)}(\alpha, 0) + K_0(\alpha, 0)A^{(1)}(\alpha, t) + Q_0(\alpha, t, 0) - K_0(\alpha, t)K_0(\alpha, 0)b(\alpha) = 0\,.$$

By inserting (12.30) into (12.31) and solving for $A^{(1)}(t)$ we get, by observing (12.10)—(12.12),

$$(12.32)\qquad A^{(1)}(\alpha, t) = -\frac{Q_0(\alpha, t, 0)}{K_0(\alpha, 0)} + \frac{1}{2}\frac{Q_0(\alpha, 0, 0)\cdot K_0(\alpha, t)}{K_0(\alpha, 0)^2} + \tfrac{1}{2}K_0(\alpha, t)b(\alpha)$$
$$= -\frac{T_0(\alpha, t)}{\kappa_{10}(\alpha)} + \frac{1}{2}\frac{\tau_{00}(\alpha)\cdot K_0(\alpha, t)}{\kappa_{10}(\alpha)^2} + \tfrac{1}{2}K_0(\alpha, t)b(\alpha) .$$

By inserting (12.32) into (12.25) we get

$$(12.33)\qquad \Phi^{(1)}(\alpha, t, s) = Q_0(\alpha, t, s) - \frac{K_0(\alpha, t)T_0(\alpha, s) + K_0(\alpha, s)T_0(\alpha, t)}{\kappa_{10}(\alpha)}$$
$$+ \tau_{00}(\alpha)\frac{K_0(\alpha, t)\cdot K_0(\alpha, s)}{\kappa_{10}(\alpha)^2}$$
$$= \iint \left(\frac{K_0(\alpha, t)}{\kappa_{10}(\alpha)} - e^{tx}\right)\left(\frac{K_0(\alpha, s)}{\kappa_{10}(\alpha)} - e^{sx}\right)$$
$$\times (1 - e^{-\rho(\alpha)x})e^{-\rho(\alpha)x}\, dH .$$

according to (12.10)—(12.12).

Thereby we have solved (12.4). Explicit formulas for $\lambda^{(1)}_{\nu\mu}(\alpha)$ can be obtained from (12.33) by expanding the right-hand side in powers of t and s.

We now proceed with equation (12.5) which we shall treat in a quite analogous way. We multiply (12.5) by $t^\nu s^\mu/\nu!\mu!$ and sum over $\nu, \mu = 0, 1, 2, \cdots$. By considering the formulas (12.8), (12.9), (12.17)—(12.20) we get the following equation.

$$(12.34)\qquad \frac{\partial}{\partial\alpha}\Phi^{(2)}(\alpha, t, s) + \rho'(\alpha)\frac{\partial}{\partial t}\Phi^{(2)}(\alpha, t, s)$$
$$+ \rho'(\alpha)\frac{\partial}{\partial s}\Phi^{(2)}(\alpha, t, s) = R^{(2)}(\alpha, t, s)$$

where

$$(12.35)\qquad R^{(2)}(\alpha, t, s) = \frac{1}{\kappa_{10}(\alpha)^2}(K_0(\alpha, t)\chi^{(2)}(\alpha, s) + K_1(\alpha, s)\chi^{(1)}(\alpha, t))$$
$$+ \frac{V_1(\alpha, t, s)}{\kappa_{10}(\alpha)} - \frac{K_0(\alpha, t)K_1(\alpha, s)}{\kappa_{10}(\alpha)^2} .$$

Furthermore, according to (12.7) and (12.16) we have

$$(12.36)\qquad \Phi^{(2)}(0, t, s) \equiv 0 .$$

From Lemma 12.1 and the formulas (12.13) and (12.14) we get the following solution of (12.34) and (12.36)

$$(12.37)\qquad \Phi^{(2)}(\alpha, t, s) = K_0(\alpha, t)A^{(2)}(\alpha, s) + K_1(\alpha, s)A^{(1)}(\alpha, t)$$
$$+ Q_1(\alpha, t, s) - K_0(\alpha, t)K_1(\alpha, s)b(\alpha)$$

where

$$A^{(2)}(\alpha, s) = \int_0^\alpha \frac{\chi^{(2)}(u, s - \rho(\alpha) + \rho(u))}{\kappa_{10}(u)^2} du \tag{12.38}$$

and $A^{(1)}(\alpha, t)$ and $b(\alpha)$ are defined in (12.26) and (12.27). From (12.16) we conclude that

$$\Phi^{(2)}(\alpha, 0, s) \equiv 0 . \tag{12.39}$$

By putting $t = 0$ in (12.37) we get from (12.39), (12.30) and (12.10)—(12.12),

$$\begin{aligned} A^{(2)}(\alpha, s) &= -\frac{Q_1(\alpha, 0, s)}{K_0(\alpha, 0)} + K_1(\alpha, s)b(\alpha) - \frac{K_1(\alpha, s)}{K_0(\alpha, 0)} A^{(1)}(\alpha, 0) \\ &= -\frac{T_1(\alpha, s)}{\kappa_{10}(\alpha)} + \frac{\tau_{00}(\alpha)K_1(\alpha, s)}{2\kappa_{10}(\alpha)^2} + \tfrac{1}{2}K_1(\alpha, s)b(\alpha) . \end{aligned} \tag{12.40}$$

By inserting (12.40) and (12.32) into (12.37) and by paying regard to (12.8)—(12.12) we get

$$\begin{aligned} \Phi^{(2)}(\alpha, t, s) &= Q_1(\alpha, t, s) - \frac{K_0(\alpha, t)T_1(\alpha, s) + K_1(\alpha, s)T_0(\alpha, t)}{\kappa_{10}(\alpha)} \\ &\quad + \frac{\tau_{00}(\alpha)K_0(\alpha, t)K_1(\alpha, s)}{\kappa_{10}(\alpha)^2} \\ &= \iint \left(\frac{K_0(\alpha, t)}{\kappa_{10}(\alpha)} - e^{tx}\right)\left(\frac{K_1(\alpha, s)}{\kappa_{10}(\alpha)} - ye^{sx}\right) \\ &\quad \times (1 - e^{-\rho(\alpha)x})e^{-\rho(\alpha)x}\, dH . \end{aligned} \tag{12.41}$$

Thereby (12.5) is also solved and we continue with (12.6). As before we multiply (12.6) by $t^\nu s^\mu/\nu!\mu!$ and sum over $\nu, \mu = 0, 1, 2, \cdots$. By observing (12.3), (12.8), (12.9) and (12.17)—(12.20) we get

$$\begin{aligned} &\frac{\partial}{\partial \alpha}\Phi^{(4)}(\alpha, t, s) + \rho'(\alpha)\frac{\partial}{\partial t}\Phi^{(4)}(\alpha, t, s) \\ &\qquad + \rho'(\alpha)\frac{\partial}{\partial s}\Phi^{(4)}(\alpha, t, s) = R^{(4)}(\alpha, t, s) \end{aligned} \tag{12.42}$$

where

$$\begin{aligned} R^{(4)}(\alpha, t, s) &= \frac{1}{\kappa_{10}(\alpha)^2}(K_1(\alpha, t)\chi^{(2)}(\alpha, s) + K_1(\alpha, s)\chi^{(2)}(\alpha, t)) \\ &\quad + \frac{V_2(\alpha, t, s)}{\kappa_{10}(\alpha)} - \frac{K_1(\alpha, t)\cdot K_1(\alpha, s)}{\kappa_{10}(\alpha)^2} . \end{aligned} \tag{12.43}$$

From (12.7) we get the boundary condition

$$\Phi^{(4)}(0, t, s) \equiv 0 . \tag{12.44}$$

Now solve (12.42) and (12.44) with the aid of Lemma 12.1 and remember (12.13) and (12.14). Then we get

$$(12.45)\qquad \Phi^{(4)}(\alpha, t, s) = K_1(\alpha, t)A^{(2)}(\alpha, s) + K_1(\alpha, s)A^{(2)}(\alpha, t) + Q_2(\alpha, t, s) - K_1(\alpha, t)K_1(\alpha, s)\cdot b(\alpha)\,,$$

where $A^{(2)}$ and $b(\alpha)$ are defined in (12.38) and (12.27). By inserting (12.40) into (12.45) we get

$$(12.46)\qquad \Phi^{(4)}(\alpha, t, s) = Q_2(\alpha, t, s) - \frac{K_1(\alpha, t)T_1(\alpha, s) + K_1(\alpha, s)T_1(\alpha, t)}{\kappa_{10}(\alpha)} + \frac{\tau_{00}(\alpha)K_1(\alpha, t)K_1(\alpha, s)}{\kappa_{10}(\alpha)^2}$$
$$= \iint \left(\frac{K_1(\alpha, t)}{\kappa_{10}(\alpha)} - ye^{tx}\right)\left(\frac{K_1(\alpha, s)}{\kappa_{10}(\alpha)} - ye^{sx}\right) \times (1 - e^{-\rho(\alpha)x})e^{-\rho(\alpha)x}\, dH\,.$$

Thereby we have solved (12.6). Thus, the equation (10.27) is solved and we continue with (10.29). By writing (10.29) in its matrix representation we get the following systems of differential equations.

From the upper left-hand block:

$$(12.47)\qquad \frac{d}{d\beta}\lambda^{(1)}_{\nu\mu}(\alpha, \beta) = -\frac{\lambda^{(1)}_{\nu+1,\mu}(\alpha, \beta)}{\kappa_{10}(\beta)} + \frac{\kappa_{\nu+1,0}(\beta)}{\kappa_{10}(\beta)^2}\lambda^{(1)}_{1\mu}(\alpha, \beta)\,,$$
$$0 \leqq \alpha < \beta < 1\,, \quad \nu, \mu = 1, 2, 3, \cdots\,.$$

From the upper right-hand block:

$$(12.48)\qquad \frac{d}{d\beta}\lambda^{(2)}_{\nu\mu}(\alpha, \beta) = -\frac{\lambda^{(2)}_{\nu+1,\mu}(\alpha, \beta)}{\kappa_{10}(\beta)} + \frac{\kappa_{\nu+1,0}(\beta)}{\kappa_{10}(\beta)^2}\lambda^{(2)}_{1\mu}(\alpha, \beta)\,,$$
$$0 \leqq \alpha < \beta < 1\,, \quad \nu = 1, 2, 3, \cdots\,, \quad \mu = 0, 1, 2, \cdots\,.$$

From the lower right-hand block:

$$(12.49)\qquad \frac{d}{d\beta}\lambda^{(4)}_{\nu\mu}(\alpha, \beta) = -\frac{\lambda^{(4)}_{\nu+1,\mu}(\alpha, \beta)}{\kappa_{10}(\beta)} + \frac{\kappa_{\nu+1,1}(\beta)}{\kappa_{10}(\beta)^2}\lambda^{(2)}_{1\mu}(\alpha, \beta)\,,$$
$$0 \leqq \alpha < \beta < 1\,, \quad \nu, \mu = 0, 1, 2, \cdots\,.$$

As before we make the observation that the solutions of (12.47) and (12.48) can be enlarged to solutions of the enlarged system which are obtained by letting ν and μ both run over $0, 1, 2, \cdots$. Such enlarged solutions are obtained by

$$(12.50)\qquad \lambda^{(1)}_{\nu\mu}(\alpha, \beta) \equiv 0\,, \qquad 0 \leqq \alpha < \beta < 1 \quad \text{for} \quad \nu\cdot\mu = 0$$

$$(12.51)\qquad \lambda^{(2)}_{0\mu}(\alpha, \beta) \equiv 0\,, \qquad 0 \leqq \alpha < \beta < 1 \quad \text{for} \quad \mu = 0, 1, 2, \cdots\,.$$

These enlargements are unique under the assumptions that their initial values for $\beta = \alpha$ are 0. In the sequel we consider the systems (12.47)—(12.49) for $\nu, \mu = 0, 1, 2, \cdots$.

We introduce the following functions,

$$(12.52)\qquad \Phi^{(p)}(\alpha, \beta, t, s) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{t^\nu}{\nu!}\cdot\frac{s^\mu}{\mu!}\lambda^{(p)}_{\nu\mu}(\alpha, \beta)\,, \qquad p = 1, 2, 4$$

(12.53) $$\chi^{(p)}(\alpha, \beta, s) = \sum_{\mu=0}^{\infty} \frac{s^\mu}{\mu!} \lambda_{1\mu}^{(p)}(\alpha, \beta) , \qquad p = 1, 2, 4 .$$

By multiplying by $t^\nu s^\mu/\nu!\mu!$ in (12.47) and summing over $\nu, \mu = 0, 1, 2, \cdots$ we get

(12.54) $$\frac{\partial}{\partial\beta}\Phi^{(1)}(\alpha, \beta, t, s) + \rho'(\beta)\frac{\partial}{\partial t}\Phi^{(1)}(\alpha, \beta, t, s) = \frac{K_0(\beta, t)\chi^{(1)}(\alpha, \beta, s)}{\kappa_{10}(\beta)^2} .$$

Furthermore, according to (10.26) we have the following boundary values

(12.55) $$\Phi^{(1)}(\alpha, \alpha, t, s) = \Phi^{(1)}(\alpha, t, s) ,$$

where $\Phi^{(1)}(\alpha, t, s)$ is defined in (12.17) and computed in (12.33). From Lemma 12.1 and (12.13) we obtain that the solution of (12.54) and (12.55) is

(12.56) $$\Phi^{(1)}(\alpha, \beta, t, s) = \Phi^{(1)}(\alpha, t - \rho(\beta) + \rho(\alpha), s) + K_0(\beta, t)A^{(1)}(\alpha, \beta, s) , \qquad 0 \leqq \alpha \leqq \beta < 1 ,$$

where

(12.57) $$A^{(1)}(\alpha, \beta, s) = \int_\alpha^\beta \frac{\chi^{(1)}(\alpha, u, s)}{\kappa_{10}(u)^2}\, du .$$

From (12.50) we get

(12.58) $$\Phi^{(1)}(\alpha, \beta, 0, s) \equiv 0 .$$

By putting $t = 0$ in (12.56) and by observing (12.58) we get

(12.59) $$A^{(1)}(\alpha, \beta, s) = -\frac{1}{\kappa_{10}(\beta)}\Phi^{(1)}(\alpha, -\rho(\beta) + \rho(\alpha), s) .$$

By inserting (12.59) into (12.56) we obtain

(12.60) $$\Phi^{(1)}(\alpha, \beta, t, s) = \Phi^{(1)}(\alpha, t - \rho(\beta) + \rho(\alpha), s) - \frac{K_0(\beta, t)}{\kappa_{10}(\beta)}\Phi^{(1)}(\alpha, -\rho(\beta) + \rho(\alpha), s) .$$

From (12.60), (12.33) and (12.13) we get after some computation

(12.61) $$\Phi^{(1)}(\alpha, \beta, t, s) = \iint \left(\frac{K_0(\beta, t)}{\kappa_{10}(\beta)} - e^{tx}\right)\left(\frac{K_0(\alpha, s)}{\kappa_{10}(\alpha)} - e^{sx}\right) \times (1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH .$$

By expanding in powers of t and s we get from (12.61), (12.52) and (12.8)

(12.62) $$\lambda_{\nu\mu}^{(1)}(\alpha, \beta) = \iint \left(\frac{\kappa_{\nu+1,0}(\beta)}{\kappa_{10}(\beta)} - x^\nu\right)\left(\frac{\kappa_{\mu+1,0}(\alpha)}{\kappa_{10}(\alpha)} - x^\mu\right)(1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH .$$

Thus we have obtained the solution which was claimed in (10.32). Next we consider the system (12.48) and we proceed as usual. By multiplying by $t^\nu s^\mu/\nu!\mu!$ and summing over $\nu, \mu = 0, 1, 2, \cdots$ we obtain

(12.63) $$\frac{\partial}{\partial\beta}\Phi^{(2)}(\alpha, \beta, t, s) + \rho'(\beta)\frac{\partial}{\partial t}\Phi^{(2)}(\alpha, \beta, t, s) = \frac{K_0(\beta, t)\chi^{(2)}(\alpha, \beta, s)}{\kappa_{10}(\beta)^2} .$$

According to (10.28) and (12.51) we have the following boundary values,

$$\Phi^{(2)}(\alpha, \alpha, t, s) \equiv \Phi^{(2)}(\alpha, t, s) \tag{12.64}$$

$$\Phi^{(2)}(\alpha, \beta, 0, s) \equiv 0 . \tag{12.65}$$

By proceeding as before we obtain the following solution of (12.63) and (12.64),

$$\Phi^{(2)}(\alpha, \beta, t, s) = \Phi^{(2)}(\alpha, t - \rho(\beta) + \rho(\alpha), s) + K_0(\beta, t)A^{(2)}(\alpha, \beta, s) \tag{12.66}$$

where

$$A^{(2)}(\alpha, \beta, s) = \int_\alpha^\beta \frac{\chi^{(2)}(\alpha, u, s)}{\kappa_{10}(u)^2}\, du . \tag{12.67}$$

From (12.66) and (12.65) we obtain,

$$A^{(2)}(\alpha, \beta, s) = -\frac{1}{\kappa_{10}(\beta)}\Phi^{(2)}(\alpha, -\rho(\beta) + \rho(\alpha), s) . \tag{12.68}$$

By inserting (12.68) into (12.66) we get

$$\begin{aligned}\Phi^{(2)}(\alpha, \beta, t, s) = {} & \Phi^{(2)}(\alpha, t - \rho(\beta) + \rho(\alpha), s) \\ & -\frac{K_0(\beta, t)}{\kappa_{10}(\beta)}\Phi^{(2)}(\alpha, -\rho(\beta) + \rho(\alpha), s) .\end{aligned} \tag{12.69}$$

From (12.69) and (12.41) we get after some computation

$$\begin{aligned}\Phi^{(2)}(\alpha, \beta, t, s) = {} & \iint \left(\frac{K_0(\beta, t)}{\kappa_{10}(\beta)} - e^{tx}\right)\left(\frac{K_1(\alpha, s)}{\kappa_{10}(\alpha)} - ye^{sx}\right) \\ & \times (1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH .\end{aligned} \tag{12.70}$$

By expanding in powers of t and s in (12.70) and observing (12.52) and (12.8) we get

$$\lambda^{(2)}_{\nu\mu}(\alpha, \beta) = \iint \left(\frac{\kappa_{\nu+1,0}(\beta)}{\kappa_{10}(\beta)} - x^\nu\right)\left(\frac{\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)} - yx^\mu\right)(1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH . \tag{12.71}$$

This is the solution which was claimed in (10.33). We now conclude by solving (12.49). In the usual way we get the following equation

$$\frac{\partial}{\partial\beta}\Phi^{(4)}(\alpha, \beta, t, s) + \rho'(\beta)\frac{\partial}{\partial t}\Phi^{(4)}(\alpha, \beta, t, s) = \frac{K_1(\beta, t)\chi^{(2)}(\alpha, \beta, s)}{\kappa_{10}(\beta)^2} . \tag{12.72}$$

Furthermore, from (10.26) we get the boundary values

$$\Phi^{(4)}(\alpha, \alpha, t, s) = \Phi^{(4)}(\alpha, t, s) . \tag{12.73}$$

By using Lemma 12.1 and (12.68) we get after some computation the following solution of (12.72) and (12.73)

$$\begin{aligned}\Phi^{(4)}(\alpha, \beta, t, s) = {} & \Phi^{(4)}(\alpha, t - \rho(\beta) + \rho(\alpha), s) \\ & -\frac{K_1(\beta, t)}{\kappa_{10}(\beta)}\Phi^{(2)}(\alpha, -\rho(\beta) + \rho(\alpha), s) .\end{aligned} \tag{12.74}$$

From (12.74), (12.41), (12.46) and (12.13) we get, again after some computation,

$$\Phi^{(4)}(\alpha, \beta, t, s) = \iint \left(\frac{K_1(\beta, t)}{\kappa_{10}(\beta)} - ye^{tx}\right)\left(\frac{K_1(\alpha, t)}{\kappa_{10}(\alpha)} - ye^{sx}\right) \times (1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH . \tag{12.75}$$

By expanding in powers of t and s in (12.73) we get

$$\lambda^{(4)}_{\nu\mu}(\alpha, \beta) = \iint \left(\frac{\kappa_{\nu+1,1}(\beta)}{\kappa_{10}(\beta)} - yx^{\nu}\right)\left(\frac{\kappa_{\mu+1,1}(\alpha)}{\kappa_{10}(\alpha)} - yx^{\mu}\right) \times (1 - e^{-\rho(\alpha)x})e^{-\rho(\beta)x}\, dH . \tag{12.76}$$

This concludes the proof of Lemma 10.7.

13. Positivity of $B(\alpha, \beta)$. In this section we shall prove Lemma 10.9. Put

$$\theta(\alpha; x, y) = \left(\frac{\kappa_{11}(\alpha)}{\kappa_{10}(\alpha)} - y\right)(1 - e^{-\rho(\alpha)x}) , \qquad 0 \leqq \alpha < 1 \tag{13.1}$$

$$\psi(\alpha; x, y) = \left(\frac{\kappa_{11}(\alpha)}{\kappa_{10}(\alpha)} - y\right)e^{-\rho(\alpha)x} , \qquad 0 \leqq \alpha < 1 . \tag{13.2}$$

We assume throughout this section that

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_d < 1 . \tag{13.3}$$

It is obvious that it suffices to prove Lemma 10.9 under the assumption (13.3). Put

$$\begin{aligned} d_{\nu\mu}(x, y) &= \theta(\alpha_\nu; x, y)\cdot\psi(\alpha_\mu; x, y) , && \nu \leqq \mu \\ &= \theta(\alpha_\mu; x, y)\cdot\psi(\alpha_\nu; x, y) , && \mu \leqq \nu ; \quad \nu, \mu = 1, 2, \cdots, d \end{aligned} \tag{13.4}$$

and

$$D(\alpha_1, \alpha_2, \cdots, \alpha_d; x, y) = [d_{\nu\mu}(x, y); \nu, \mu = 1, 2, \cdots, d] . \tag{13.5}$$

Then we have

$$[B(\alpha_\nu, \alpha_\mu); \nu, \mu = 1, 2, \cdots, d] = \iint_{R^2} D(\alpha_1, \cdots, \alpha_d; x, y)\, dH(x, y) . \tag{13.6}$$

LEMMA 13.1. *Assume that* (13.3) *holds. Then the matrix* $D(\alpha_1, \alpha_2, \cdots, \alpha_d; x, y)$ *is*

(a) *nonnegative as soon as* $x \geqq 0$,
(b) *positive if and only if* $x > 0$ *and*

$$\frac{\kappa_{11}(\alpha_\nu)}{\kappa_{10}(\alpha_\nu)} - y \neq 0 , \qquad \nu = 1, 2, \cdots, d . \tag{13.7}$$

PROOF. We shall apply the results in Chapter 3, Section 3, pages 110–112, in [3]. We have

$$\theta(\alpha; x, y)\cdot\psi(\alpha; x, y) = \left(\frac{\kappa_{11}(\alpha)}{\kappa_{10}(\alpha)} - y\right)^2(1 - e^{-\rho(\alpha)x})e^{-\rho(\alpha)x} , \tag{13.8}$$

and if $\kappa_{11}(\alpha)/\kappa_{10}(\alpha) - y \neq 0$,

$$\theta(\alpha; x, y)/\psi(\alpha; x, y) = e^{\rho(\alpha)x} - 1 . \tag{13.9}$$

The function $e^{\rho(\alpha)x} - 1$ is, as a function of α, strictly increasing when $x > 0$ (cf. Lemma 10.1). The claim in (b) now follows from Theorem 3.1 in [3], (13.8) and (13.9). The assertion in (a) follows from (b) and the fact that $D(\alpha_1, \cdots, \alpha_d; x, y)$ is continuous in x and y. Thus the lemma is proved.

The positivity of $[B(\alpha_\nu, \alpha_\mu); \nu, \mu = 1, 2, \cdots, d]$ will now follow from (13.6) and Lemma 13.1 if we show that $D(\alpha_1, \alpha_2, \cdots, \alpha_d; x, y)$ is positive on a (x, y)-set of positive $H(x, y)$-measure. To do this we shall need the following lemma.

LEMMA 13.2. *Let $\mu_1, \mu_2, \cdots, \mu_d$ be probability measures on R(=the real line) which all have means. Let the measure μ be equivalent with each of $\mu_1, \mu_2, \cdots, \mu_d$ (i.e. μ and μ_ν are absolutely continuous with respect to each other). Furthermore we assume μ to be non-degenerate. Then,*

$$\mu(\{y: \textstyle\int_{-\infty}^{\infty} t\, d\mu_\nu(t) < y, \nu = 1, 2, \cdots, d\}) > 0 . \tag{13.10}$$

PROOF. As μ is non-degenerate, so is also μ_ν because of the equivalence. Then, as is easily seen, we have

$$\mu_\nu(\{y: \textstyle\int_{-\infty}^{\infty} t\, d\mu_\nu(t) < y\}) > 0 . \tag{13.11}$$

In view of the equivalence between μ and μ_ν we get from (13.11),

$$\mu(\{y: \textstyle\int_{-\infty}^{\infty} t\, d\mu_\nu(t) < y\}) > 0 , \qquad \nu = 1, 2, \cdots, d . \tag{13.12}$$

Upon some thought it is realized that (13.10) and (13.12) are equivalent. Thus the lemma is proved.

We are now prepared to finish the proof of Lemma 10.9. According to (10.11) and (12.3) we have,

$$\frac{\kappa_{11}(\alpha)}{\kappa_{10}(\alpha)} = \int_0^\infty \int_{-\infty}^{\infty} ut\rho'(\alpha)e^{-\rho(\alpha)u}\, dH(u, t) = \int_{-\infty}^{\infty} t\, d\mu(t; \alpha) , \tag{13.13}$$

where the measure $\mu(\cdot; \alpha)$, $0 < \alpha < 1$, is given by

$$\mu((-\infty, y]; \alpha) = \int_0^\infty \int_{-\infty}^{y} u\rho'(\alpha)e^{-\rho(\alpha)u}\, dH(u, t) . \tag{13.14}$$

According to (12.3) we have

$$1 = \int_0^\infty \int_{-\infty}^{\infty} u\rho'(\alpha)e^{-\rho(\alpha)u}\, dH(u, t) , \qquad 0 < \alpha < 1 . \tag{13.15}$$

Furthermore,

$$u\rho'(\alpha)e^{-\rho(\alpha)u} > 0 \qquad \text{for} \quad u > 0 , \quad 0 < \alpha < 1 . \tag{13.16}$$

Let $\mu^*(\cdot)$ be the marginal mass, along the y-axis, corresponding to $H(x, y)$. We have

(i) $\mu(\cdot; \alpha)$ (see (13.14)) is a probability measure for every $\alpha \in (0, 1)$.
(ii) $\mu(\cdot; \alpha)$ is equivalent to $\mu^*(\cdot)$ for every $\alpha \in (0, 1)$.
(iii) $\mu^*(\cdot)$ is non-degenerate.

(i) follows from (13.15). (ii) is a consequence of (13.14), (13.16) and the fact that $H(x, y)$ does not have positive mass along the y-axis. (iii) is another way of expressing (10.50). Now, from Lemma 13.2, (i), (ii) and (iii) above and from (13.13) we conclude that for each (fixed) $\alpha_1, \alpha_2, \cdots, \alpha_d$ satisfying (13.3) we have

$$\mu^*\left(\left\{y: \frac{\kappa_{11}(\alpha_\nu)}{\kappa_{10}(\alpha_\nu)} < y, \nu = 1, 2, \cdots, d\right\}\right) > 0 . \tag{13.17}$$

From (13.17) and (b) in Lemma 13.1 it follows that, under the assumption (10.50), the matrix $D(\alpha_1, \alpha_2, \cdots, \alpha_d; x, y)$ is positive on a set of positive $H(x, y)$-measure. This concludes the proof of Lemma 10.9.

14. On the Horvitz-Thompson estimator. The *population total* τ_π corresponding to the population $\pi = (a_1, a_2, \cdots, a_N)$ is

$$\tau_\pi = a_1 + a_2 + \cdots + a_N . \tag{14.1}$$

As usual $I_1, I_2, \cdots, I_N$ denotes a **p**-permutation of $1, 2, \cdots, N$ and $Y_1, Y_2, \cdots, Y_N$ the corresponding **p**-permutation of the elements in $\pi = (a_1, a_2, \cdots, a_N)$.

The Horvitz-Thompson estimator (for the population total) based on a sample of size n, $(HT)_n$, was defined in (1.2) and (1.3). The interest in this estimator depends on the fact that it is an unbiased estimator for the population total. This follows immediately from (1.2) and (3.7). It is easily seen that $(HT)_n$ can also be expressed in the following form

$$(HT)_n = \sum_{\nu=1}^n Y_\nu/\Delta(I_\nu, n) . \tag{14.2}$$

From (14.2) we see that $(HT)_n$ can be viewed as the sample sum in a **p**-sample of size n from the population $(a_1/\Delta(1, n), a_2/\Delta(2, n), \cdots, a_N/\Delta(N, n))$. The estimator $(HT)_n$ is not however usable in general in practical situations, as we do not have exact knowledge of $\Delta(s, n)$. The natural attempt to circumvent this obstacle is to replace the $\Delta(s, n) : s$ in $(HT)_n$ by their approximations (see (3.4)). Then we obtain the following "quasi" Horvitz-Thompson estimator,

$$(QHT)_n = \sum_{s=1}^N W_s \frac{a_s}{1 - e^{-p_s t(n)}} = \sum_{\nu=1}^n a_{I_\nu}/(1 - \exp(-p_{I_\nu} t(n))) . \tag{14.3}$$

As is seen from (14.3), the estimator $(QHT)_n$ can be viewed as the sample sum in a **p**-sample of size n from the following population,

$$\pi^*(\mathbf{p}, \pi, n) = (a_1/(1 - e^{-p_1 t(n)}), a_2/(1 - e^{-p_2 t(n)}), \cdots, a_N/(1 - e^{-p_N t(n)})) . \tag{14.4}$$

We shall not go into the matter of the asymptotic behaviour of $(HT)_n$ and $(QHT)_n$ in detail. We shall be satisfied with the following result which is a straightforward application on Theorem 3.3.

THEOREM 14.1. *Let* $(\mathbf{p}_k, \pi_k)$ *and* n_k, $k = 1, 2, \cdots$, *satisfy* (3.21), (3.22), (3.34) *and the following condition. The population sequence* $\pi^*(\mathbf{p}_k, \pi_k, n_k)$, $k = 1, 2, \cdots$, (*according to* (14.4)) *satisfies* (3.23). *Then* $(QHT)_{n_k}^{(k)}$ (*according to* (14.3)) *is asymptotically* $N(\tau_{\pi_k}, \delta_k^2(n_k))$*-distributed as* $k \to \infty$, *where* τ_π *is as in* (14.1) *and where*

$$(14.5) \qquad \delta^2(n) = \sum_{s=1}^{N} \left(\frac{a_s}{1 - e^{-p_s t(n)}} - \frac{\sum_{r=1}^{N} p_r a_r e^{-p_r t(n)}/(1 - e^{-p_r t(n)})}{\sum_{r=1}^{N} p_r e^{-p_r t(n)}} \right)^2 \times (1 - e^{-p_s t(n)}) e^{-p_s t(n)} .$$

REFERENCES

[1] HÁJEK, J. (1960). Limiting distributions in simple random sampling from a finite population. *Publ. Math. Inst. Hungary Acad. Sci.* **5** 361–374.

[2] HÁJEK, J. (1964). Asymptotic theory of rejective sampling with varying probabilities from a finite population. *Ann. Math. Statist.* **35** 1491–1523.

[3] KARLIN, S. (1968). *Total Positivity*. Stanford Univ. Press.

[4] ROSÉN, B. (1969a). Asymptotic normality of sums of random elements with values in a real separable Hilbert space. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.* **13** 221–255.

[5] ROSÉN, B. (1969b). Asymptotic normality in a coupon collector's problem. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.* **13** 256–279.

[6] ROSÉN, B. (1970). On the coupon collector's waiting time. *Ann. Math. Statist.* **41** 1952–1969.

[7] ROSÉN, B. (1971). Asymptotic theory for successive sampling with varying probabilities without replacement, I. *Ann. Math. Statist.* **43** 401–425.

[8] SUKHATME, P. V. and SUKHATME, B. V. (1970). *Sampling Theory of Surveys with Applications*, 2nd revised ed. Asia Publishing House, London.