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SYMMETRIC AND REVERSED MULTIPLE STATIONARY AUTOREGRESSIVE SERIES

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Let $\{X_t\}$ be a p-dimensional stationary autoregressive series. The main result is the determination of the autoregressive matrices of the series which is reversed in time with respect to $\{X_t\}$. The series which is reversed with respect to itself is called symmetric. The conditions for the symmetry of $\{X_t\}$ are given in the paper. The inverse of the covariance matrix is evaluated for the finite part of the symmetric autoregressive series.

0. Summary. Consider a p-dimensional stationary autoregressive series $\{X_t\}$ of order n. Denote its matrix of covariance functions by $(R_{jk}(t))$. A p-dimensional stationary autoregressive series $\{Z_t\}$ is reversed (in time) with respect to $\{X_t\}$, if $\{Z_t\}$ is of the same order as $\{X_t\}$ and its matrix of covariance functions equals $(R_{jk}(-t))$. The series $\{X_t\}$ which is reversed with respect to itself is called symmetric. Obviously, $\{X_t\}$ is symmetric if and only if $R_{jk}(t) = R_{jk}(-t)$ holds for $1 \leqq j,\ k \leqq p$ and $-\infty < t < \infty$. Let $\{X_t\}$ have autoregressive matrices $A_0, \cdots, A_n$. The autoregressive matrices $B_0, \cdots, B_n$ are found for $\{Z_t\}$ which is reversed with respect to $\{X_t\}$. Bartlett considered the same problem for $n = 1$ (see [3], Section 9.3, or [4]). In this special case our results coincide with his. Further, the paper contains necessary and sufficient conditions on $A_0, \cdots, A_n$ in order for $\{X_t\}$ to be symmetric. The explicit formula for the inverse of the covariance matrix Var $(X_1', \cdots, X_N')'$ is given when $\{X_t\}$ is symmetric and $N \geqq 2n$.

The results concerning the reversed series are applicable in the theory of tests of fit for multiple autoregressive series (see [4]). Another field of application is the "backward extrapolation", when $\{X_t\}_{t=0}^{N}$ is known and X_s is to be estimated for some $s < 0$.

1. Preliminaries. Let $\{Y_t\}_{t=-\infty}^{\infty}$ be a series of uncorrelated p-dimensional random vectors such that $EY_t = 0$ and Var $Y_t = I$, where Var Y_t denotes the covariance matrix of vector Y_t and I is the unit matrix. Let $A_0, \cdots, A_n$ be $p \times p$ matrices with real elements such that

(i) $\det A_0 \neq 0$

(ii) $A_n \neq 0$

(iii) all the roots of equation $\det\left(\sum_{j=0}^{n} A_j \lambda^{n-j}\right) = 0$

are smaller than 1 in absolute value.

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Define series $\{X_t\}_{t=-\infty}^{\infty}$ by the recurrent formula

$$\sum_{j=0}^{n} A_j X_{t-j} = Y_t , \qquad -\infty < t < \infty , \tag{1}$$

or equivalently

$$X_t = \sum_{j=1}^{n} U_j X_{t-j} + A_0^{-1} Y_t , \qquad -\infty < t < \infty , \tag{2}$$

where

$$U_j = -A_0^{-1} A_j , \qquad 1 \leqq j \leqq n . \tag{3}$$

Then $\{X_t\}$ is the p-dimensional autoregressive series of order n. It is stationary under above conditions, as is well known. $A_0, \cdots, A_n$ will be called the autoregressive matrices corresponding to $\{X_t\}$. Put $X_t = (X_t^1, \cdots, X_t^p)'$, where the prime denotes the transposition. Covariance function $R_{jk}(t)$ is defined by the formula $R_{jk}(t) = EX_t^j X_0^k (1 \leqq j, k \leqq p; -\infty < t < \infty)$, as $EX_t = 0$ obviously holds.

LEMMA 1. *The series $\{X_t\}$ has the matrix of spectral densities*

$$f(\lambda) = (f_{jk}(\lambda))_{j,k=1}^{p} = (2\pi)^{-1}[\bar{Q}'(\lambda)Q(\lambda)]^{-1} , \qquad -\pi \leqq \lambda \leqq \pi , \tag{4}$$

where

$$Q(\lambda) = \sum_{j=0}^{n} A_j e^{-ij\lambda} , \qquad \bar{Q}'(\lambda) = \sum_{j=0}^{n} A_j' e^{ij\lambda} .$$

PROOF. See [7] or [5]. Note that somewhat different notation was used in [7]. By $f(\lambda)$ we mean the matrix corresponding to $(R_{jk}(t))$ in the usual sense:

$$R_{jk}(t) = \int_{-\pi}^{\pi} e^{it\lambda} f_{jk}(\lambda)\, d\lambda , \qquad 1 \leqq j, k \leqq p; -\infty < t < \infty . \tag{5}$$

LEMMA 2. *Let $N > n$. Denote $B = \operatorname{Var}(X_1', \cdots, X_n')'$, $G = \operatorname{Var}(X_1', \cdots, X_N')'$. The matrix B is regular and it is the unique solution of the equation*

$$B = MBM' + \Lambda , \tag{6}$$

where

$$M = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I \\ U_n & U_{n-1} & U_{n-2} & \cdots & U_1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & (A_0' A_0)^{-1} \end{pmatrix} \tag{7}$$

are the matrices of the type $np \times np$ written in terms of the blocks of the type $p \times p$. Put $B^{-1} = E = (E_{st})_{s,t=1}^{n}$, where E_{st} are $p \times p$ blocks. The matrix G is regular and for its inverse $G^{-1} = H = (H_{st})_{s,t=1}^{N}$ written in terms of the $p \times p$ blocks the following formulas hold:

$$H_{st} = E_{st} + \sum_{k=n+1}^{\min(n+s,n+t,N)} A_{k-s}' A_{k-t} \qquad \textit{for} \quad 1 \leqq s, t \leqq n , \tag{8}$$

$$H_{st} = \sum_{k=\max(s,t)}^{\min(n+s,n+t,N)} A_{k-s}' A_{k-t} \qquad \textit{otherwise.} \tag{9}$$

PROOF. See [1].

Lemma 2 and Theorem 8 (see later) are generalizations of results for the one-dimensional case. The references concerning this case are mentioned in [1]. Still further reference is [2].

LEMMA 3. *The roots of the equation* $\det(\sum_{j=0}^{n} A_j \lambda^{n-j}) = 0$ *are the same as the roots of the matrix* M.

PROOF. See [1].

2. The reversed autoregressive series. Let $\{Z_t\}$ be the p-dimensional autoregressive series defined by

$$\sum_{j=0}^{n} B_j Z_{t-j} = Y_t\,, \qquad -\infty < t < \infty\,, \tag{10}$$

where

$$\det B_0 \neq 0 \tag{11}$$

and

(12) the equation $\det(\sum_{j=0}^{n} B_j \lambda^{n-j}) = 0$ has all the roots smaller than 1 in absolute value.

Then $\{Z_t\}$ is stationary. Let $\{X_t\}$ defined by (1) have the matrix of the covariance functions $(R_{jk}(t))_{j,k=1}^{p}$. If $\{Z_t\}$ has the matrix of the covariance functions $(R_{jk}(-t))_{j,k=1}^{p}$, we say that $\{Z_t\}$ is reversed (in time) with respect to $\{X_t\}$. On the other hand, $\{Z_t\}$ is reversed with respect to $\{X_t\}$ when $\mathrm{Cov}(Z_t^{\,j}, Z_0^{\,k}) = \mathrm{Cov}(X_0^{\,j}, X_t^{\,k})$ for $1 \leqq j, k \leqq p$ and $-\infty < t < \infty$, where $Z_t = (Z_t^1, \cdots, Z_t^p)'$. We shall prove that the reversed series exists and derive its autoregressive matrices. We shall see that the assumption $A_n \neq 0$ implies $B_n \neq 0$ so that the reversed series is of the same order as the given one.

THEOREM 4. *The series* $\{Z_t\}$ *is reversed with respect to* $\{X_t\}$ *if and only if its autoregressive matrices satisfy* (11), (12) *and*

$$\sum_{k=0}^{n-h} A'_{h+k} A_k = \sum_{k=h}^{n} B'_{k-h} B_k\,, \qquad 0 \leqq h \leqq n\,. \tag{13}$$

PROOF. The conditions (11) and (12) are self-evident. Denote the matrix of spectral densities of $\{Z_t\}$ by $g(\lambda)$. According to Lemma 1

$$g(\lambda) = (2\pi)^{-1}[\bar{S}'(\lambda)S(\lambda)]^{-1}\,, \qquad -\pi \leqq \lambda \leqq \pi\,,$$

holds, where

$$S(\lambda) = \sum_{j=0}^{n} B_j e^{-ij\lambda}\,, \qquad \bar{S}'(\lambda) = \sum_{j=0}^{n} B_j' e^{ij\lambda}\,.$$

Clearly, $\{Z_t\}$ is reversed with respect to $\{X_t\}$ if and only if $f(\lambda) = \overline{g(\lambda)}$, i.e., if and only if $\bar{Q}'(\lambda)Q(\lambda) = S'(\lambda)\bar{S}(\lambda)$. This leads to the condition

$$\sum_{h=-n}^{n} e^{ih\lambda} \sum_{k=\max(0,-h)}^{\min(n,n-h)} A'_{h+k} A_k = \sum_{h=-n}^{n} e^{ih\lambda} \sum_{k=\max(0,h)}^{\min(n,n+h)} B'_{k-h} B_k\,, \qquad -\pi \leqq \lambda \leqq \pi\,.$$

Thus the coefficients of $e^{ih\lambda}$ must be the same on both sides, $-n \leqq h \leqq n$, and we obtain (13). The proof is finished.

Put

$$V_j = -B_0^{-1} B_j\,, \qquad 1 \leqq j \leqq n\,. \tag{14}$$

The relation (10) may be written equivalently in the form

$$Z_t = \sum_{j=1}^{n} V_j Z_{t-j} + B_0^{-1} Y_t\,, \qquad -\infty < t < \infty\,. \tag{15}$$

LEMMA 5. *Let* $B = \operatorname{Var}(X_1', \cdots, X_n')'$, $B^{-1} = E = (E_{st})_{s,t=1}^n$. *Put* $E_{st} = 0$ *if* $\max(s, t) > n$. *Then*

$$\begin{aligned} A'_{n-s} A_{n-t} = E_{st} E_{s+1,t+1} + (E_{s+1,1} + A'_{n-s} A_n) \\ \times (E_{11} + A_n' A_n)^{-1}(E_{1,t+1} + A_n' A_{n-t}) \end{aligned} \tag{16}$$

holds for $1 \leqq s, t \leqq n$.

PROOF. Denote $G = \operatorname{Var}(X_1', \cdots, X'_{n+1})'$. According to Lemma 2

$$G^{-1} = H = \begin{pmatrix} (E_{st} + A'_{n+1-s} + A_{n+1-t})_{s,t=1}^n & (A'_{n+1-s} A_0)_{s=1}^n \\ (A_0' A_{n+1-t})_{t=1}^n & A_0' A_0 \end{pmatrix},$$

where H is divided into the four submatrices so that H is of the type

$$\begin{pmatrix} np \times np & np \times p \\ p \times np & p \times p \end{pmatrix}.$$

Now, introduce matrices $Q = (Q_{11}, Q_{12}, \cdots, Q_{1n})$, $R = (R_{st})_{s,t=1}^n$, where Q_{1t} and R_{st} are the blocks of the type $p \times p$ defined by the following formulas:

$$Q_{1t} = E_{1,t+1} + A_n' A_{n-t}\,, \qquad R_{st} = E_{s+1,t+1} + A'_{n-s} A_{n-t}\,, \qquad 1 \leqq s, t \leqq n\,.$$

Then

$$H = \begin{pmatrix} E_{11} + A_n' A_n & Q \\ Q' & R \end{pmatrix}.$$

Consider this division and evaluate H^{-1} as the inverse of the matrix divided into the four blocks. The well-known formula gives (see [6], Chapter 1b, Example 2.7, for example)

$$H^{-1} = \begin{pmatrix} * & * \\ * & [R - Q'(E_{11} + A_n' A_n)^{-1} Q]^{-1} \end{pmatrix},$$

where the symbol * denotes the blocks which are not important for our purpose. Since $H^{-1} = G$, we have

$$\operatorname{Var}(X_2', \cdots, X'_{n+1})' = [R - Q'(E_{11} + A_n' A_n)^{-1} Q]^{-1}\,.$$

Because $\{X_t\}$ is stationary, we have $\operatorname{Var}(X_2', \cdots, X'_{n+1})' = \operatorname{Var}(X_1', \cdots, X_n')' = B$ and $E = R - Q'(E_{11} + A_n' A_n)^{-1} Q$ holds. Writing this equality in the blocks we obtain (16).

THEOREM 6. *The autoregressive matrices* $B_0, \cdots, B_n$ *belong to the series* $\{Z_t\}$ *reversed with respect to* $\{X_t\}$ *if and only if*

$$B_0' B_j = E_{1,j+1} + A_n' A_{n-j}\,, \qquad 0 \leqq j \leqq n\,, \tag{17}$$

holds, where $E_{1,n+1} = 0$.

Any solution $B_0, \cdots, B_n$ *of* (17) *has the following properties.*

(i) B_0 *is regular.*

(ii) *The roots of the equation* $\det(\sum_{j=0}^{n} B_j \lambda^{n-j}) = 0$ *are the same as those of* $\det(\sum_{j=0}^{n} A_j \lambda^{n-j}) = 0$.

(iii) *If* $A_n \neq 0$, *then* $B_n \neq 0$.

The matrices V_j *introduced in* (14) *are defined by* (17) *uniquely. It holds that*

$$V_j = -(E_{11} + A_n' A_n)^{-1}(E_{1,j+1} + A_n' A_{n-j}), \qquad 1 \leqq j \leqq n, \tag{18}$$

where $E_{st} = 0$ *for* $\max(s, t) > n$.

PROOF. First we prove the necessity of the condition (17). Let $N \geqq 2n$ and put $G = \operatorname{Var}(X_1', \cdots, X_N')' = (G_{st})_{s,t=1}^{N}$. Let the series $\{Z_t\}$ defined by (10) be reversed with respect to $\{X_t\}$. Denote $K = \operatorname{Var}(Z_1', \cdots, Z_N')'$ and introduce matrix L of the type $Np \times Np$ written in terms of the $p \times p$ blocks

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & I & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ I & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{19}$$

Then $G = LKL$ holds and it implies $G^{-1} = LK^{-1}L$ because of $L^{-1} = L$. Write $G^{-1} = H = (H_{st})_{s,t=1}^{N}$, $K^{-1} = C = (C_{st})_{s,t=1}^{N}$ in terms of $p \times p$ blocks. The relation $H = LCL$ implies $H_{st} = C_{N+1-s,N+1-t}$ for $1 \leqq s, t \leqq N$. Using (8) and (9) for $s = 1$; $t = 1, 2, \cdots, n$ we get (17) for $0 \leqq j < n$. The condition (17) for $j = n$ follows from (13) as the special case for $h = n$.

And now the sufficiency of (17). We use Theorem 4. (17) gives for $j = 0$ that $B_0' B_0 = E_{11} + A_n' A_n$. The matrix B is regular according to Lemma 2 and thus it is positive definite. The same is true for $E = B^{-1}$ and for E_{11}. Then $E_{11} + A_n' A_n$ is positive definite and B_0 must be regular. The relation (11) as well as assertion (i) of our theorem is proved. As for (13), we get from (17)

$$B_k' B_j = (E_{k+1,1} + A_{n-k}' A_n)(E_{11} + A_n' A_n)^{-1}(E_{1,j+1} + A_n' A_{n-j}), \qquad 1 \leqq j, k \leqq n,$$

because $E_{1,k+1}' = E_{k+1,1}$ with regard to the symmetry of the matrix E. (Remember, that $E_{st} = 0$ for $\max(s, t) > n$.) Making use of formula (16) we have

$$\begin{aligned} \textstyle\sum_{k=0}^{n-h} A_{h+k}' A_k &= A_n' A_{n-h} + \textstyle\sum_{k=0}^{n-h-1} (E_{n-h-k,n-k} - E_{n-h-k+1,n-k+1}) \\ &\quad + \textstyle\sum_{k=0}^{n-h-1} (E_{n-h-k+1,1} + A_{h+k}' A_n)(E_{11} + A_n' A_n)^{-1}(E_{1,n-k+1} + A_n' A_k) \\ &= A_n' A_{n-h} + E_{1,h+1} + \textstyle\sum_{k=0}^{n-h-1} B_{n-h-k}' B_{n-k} = \textstyle\sum_{k=h}^{n} B_{k-h}' B_k . \end{aligned}$$

The condition (12) follows from assertion (ii) and, therefore, let us prove (ii). Suppose first that the matrix A_n is regular. Then U_n is regular, too. Let us start with equation (6). By evaluating the inverse matrices on both sides, we obtain

$$E = (M' + EM^{-1}\Lambda)^{-1}EM^{-1}. \tag{20}$$

It may easily be proved that

$$M^{-1} = \begin{pmatrix} A & U_n^{-1} \\ D_I & 0 \end{pmatrix},$$

where $A = -U_n^{-1}(U_{n-1}, \cdots, U_1)$, $D_I = \operatorname{diag}(I, \cdots, I)$. Let L be defined by (19) for $N = n$. We have from (20)

$$L(M + \Lambda M'^{-1}E)^{-1}L = LE^{-1}M'EL. \tag{21}$$

Put $D_j = U_j + (A_0'A_0)^{-1}U_n'^{-1}E_{1,n-j+1}$ for $1 \leqq j \leqq n$ and let $D = (D_{n-1}, \cdots, D_1)$. After some computation we get

$$M + \Lambda M'^{-1}E = \begin{pmatrix} 0 & D_I \\ D_n & D \end{pmatrix}.$$

Further

$$L(M + \Lambda M'^{-1}E)^{-1}L = \begin{pmatrix} 0 & D_I \\ D_n^{-1} & R \end{pmatrix},$$

where $R = -D_n^{-1}(D_1, \cdots, D_{n-1})$. In view of (3) and (18), which is an easy consequence of (17) and (i), we come to

$$D_n^{-1} = [U_n + (A_0'A_0)^{-1}U_n'^{-1}E_{11}]^{-1} = -(E_{11} + A_n'A_n)^{-1}A_n'A_0 = V_n,$$

$$-D_n^{-1}D_j = (E_{11} + A_n'A_n)^{-1}A_n'A_0[U_j + (A_0'A_0)^{-1}U_n'^{-1}E_{1,n-j+1}] = V_{n-j} \quad \text{for} \quad 1 \leqq j < n.$$

We know that $L = L^{-1}$. Relation (21) implies that the matrix $L(M + \Lambda M'^{-1}E)^{-1}L$ has the same roots as the matrix M', which has the same roots as M. Denote $S = (V_{n-1}, \cdots, V_1)$. Obviously, M has the same roots as the matrix

$$\begin{pmatrix} 0 & D_I \\ V_n & S \end{pmatrix},$$

which has the same roots as the equation $\det(\sum_{j=0}^n B_j \lambda^{n-j}) = 0$ according to Lemma 3. But the roots of M are the same as those of $\det(\sum_{j=0}^n A_j \lambda^{n-j}) = 0$.

If A_n is not regular, we obtain the proof of (ii) by the well-known limit procedure, when a sequence of regular matrices tending to A_n is chosen. We omit the details.

The assertion (iii) follows from (17) immediately, when we put $j = n$. Theorem 6 is proved.

The matrix B_0 may be an arbitrary solution of $B_0B_0' = E_{11} + A_n'A_n$, e. g. $B_0 = (E_{11} + A_n'A_n)^{\frac{1}{2}}$. If B_0 is chosen, then $B_1, \cdots, B_n$ are determined by (17) uniquely.

In the special case $n = 1$ put $U_1 = U$, $V_1 = V$. After some computations we obtain $V = BU'E$, $(B_0'B_0)^{-1} = B - BU'EUB$, which is the same as the results derived by Bartlett in [3], Section 9.3, and in [4].

3. Symmetric autoregressive series. The p-dimensional stationary autoregressive series $\{X_t\}$ is called symmetric, if it is reversed with respect to itself. We give a simple condition for such a symmetry.

THEOREM 7. *The stationary autoregressive series* $\{X_t\}$ *is symmetric if and only if*

$$\sum_{k=0}^{n-h} A'_{h+k} A_k = \sum_{k=0}^{n-h} A_k' A_{k+h}\,, \qquad 1 \leqq h \leqq n\,. \tag{22}$$

PROOF. Theorem 7 follows immediately from Theorem 4.

For example, the autoregressive series of the first order is symmetric if and only if the product $A_0'A_1$ is the symmetric matrix. For $p = 1$ the problem is trivial because every one-dimensional stationary series is symmetric, as is well known.

If $\{X_t\}$ is symmetric, then the blocks H_{st} and E_{st} mentioned in Lemma 2 may be evaluated explicitly.

THEOREM 8. *Let* $\{X_t\}$ *be symmetric. Then*

$$H_{st} = H_{N+1-s,N+1-t} \qquad \text{for} \quad 1 \leqq s, t \leqq N\,, \tag{23}$$

$$E_{st} = \sum_{k=1}^{\min(s,t)} (A'_{s-k} A_{t-k} - A'_{n+k-s} A_{n+k-t}) \qquad \text{for} \quad 1 \leqq s, t \leqq n\,. \tag{24}$$

If $N \geqq 2n$, *then*

$$H_{st} = \sum_{k=1}^{\min(s,t)} A'_{s-k} A_{t-k} \qquad \text{for} \quad 1 \leqq s, t \leqq n\,. \tag{25}$$

PROOF. Write G in terms of the $p \times p$ blocks G_{st}, $G = (G_{st})_{s,t=1}^{N}$. If $\{X_t\}$ is symmetric, then $G_{st} = G_{ts}$ for $1 \leqq s, t \leqq N$. It is easy to see that $G = LGL$, where L is defined in (19). Since $L^{-1} = L$, we get $G^{-1} = LG^{-1}L$. This implies formula (23). If $N \geqq 2n$, then (25) is the consequence of (23) and (9). Finally, (24) follows from (8) and (25). It is clear that E_{st} does not depend on N.

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