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# UNIFORM STRONG CONSISTENCY OF RAO-BLACKWELL DISTRIBUTION FUNCTION ESTIMATORS

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In the independent sampling model, Rao-Blackwell distribution function estimators $\tilde{F}_n(x)$ obtained by conditioning on sufficient statistics $T_n(X_1, \cdots, X_n)$ are considered. If for each $n \geqq 1$, $T_n$ is symmetric in $X_1, \cdots, X_n$ and $T_{n+1}$ is $\mathscr{B}(T_n, X_{n+1})$ measurable, it is shown that $\tilde{F}_n(x)$ converges strongly to the corresponding $F(x)$ and uniformly in $x$. This is a direct generalization of the Glivenko-Cantelli theorem.

**1. Introduction and summary.** Berk [1] has shown that for Rao–Blackwell estimators based on a sequence of symmetric and transitive statistics, strong convergence to the corresponding expectation holds. Here, under the same conditions on the statistics, the uniform strong convergence of Rao–Blackwell distribution function estimators $\tilde{F}_n(x)$ to the corresponding $F(x)$ is shown. See [6] for further results and additonal references for such estimators.

**2. Definitions and notation.** Let $(R^\infty, \mathscr{D}^\infty, P^\infty)$ be the probability space corresponding to a sequence of independent identically distributed random variables on the Borel line $(R, \mathscr{D})$, with common probability measure $P \in \mathscr{P}$.

For each $n \geqq 1$, denote by $X_n : R^\infty \to R$, the $n$th projection of $R^\infty$ by $U_n : R^\infty \to R^n$, the vector of order statistics, a function of $X_1, \cdots, X_n$, only; and by $T_n : R^\infty \to R^{k_n}$ ($k_n$ some integer $\leqq n$) any sufficient statistic, also a function of $X_1, \cdots, X_n$, only. Let $\mathscr{C}_n \subset \mathscr{D}^\infty$ be the $\sigma$-algebra induced by $U_n$, and $\mathscr{B}_n \subset \mathscr{D}^\infty$ be that induced by $T_n$.

Following Berk's Condition A:

DEFINITION 2.1. $(T_n)_{n \geqq 1}$ is said to be a sequence of *symmetric* and *transitive* statistics if for each $n \geqq 1$,

(i) $\mathscr{B}_n \subset \mathscr{C}_n$ and,
(ii) $\mathscr{B}_{n+1} \subset \mathscr{B}(T_n, X_{n+1})$,

the compound $\sigma$-algebra induced by $T_n$ and $X_{n+1}$,

Let $F(x)$ be the distribution function corresponding to $P$, and denote by $\tilde{F}_n(x)$ the $\mathscr{B}_n$ measurable random variable $E\{1_{[X_1 \leqq x]} \mid \mathscr{B}_n\}$. Note that $E\{1_{[X_1 \leqq x]} \mid \mathscr{C}_n\}$ is the usual empirical distribution function.

**3. Uniform strong consistency of $\tilde{F}_n(x)$.**

THEOREM 3.1. *If $(T_n)_{n \geqq 1}$ is a sequence of symmetric and transitive statistics, then*

$$\sup_{x \in R} |\tilde{F}_n(x) - F(x)| \to 0 \text{ a.s.}$$

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PROOF. The sequence of rv's $(Z_n)_{n\geqq 1}$ defined by:

$$Z_n = \sup_{x\in R} |E\{1_{[X_1\leqq x]} \mid \mathscr{C}_n\} - F(x)| ,$$

converges a.s. to 0 by the Glivenko–Cantelli Lemma; moreover, from the symmetry property,

$$E\{E\{1_{[X_1\leqq x} \mid \mathscr{C}_k\} \mid \mathscr{B}_n\} = \tilde{F}_n(x) \text{ a.s.} \qquad \text{for } k = 1, 2, \cdots, n .$$

Using Jensen's inequality, it follows that

$$E\{Z_n \mid \mathscr{B}_n\} \geqq \sup_{x\in R} |\tilde{F}_n(x) - F(x)| \geqq 0 ;$$

so it will suffice to show that:

$$E\{Z_n \mid \mathscr{B}_n\} \to 0 \text{ a.s.} \qquad \text{as } n \to \infty .$$

Now

$$E\{Z_n \mid \mathscr{B}_n\} = E\{Z_n \mid \mathscr{B}(T_n, X_{n+1}, \cdots)\} \text{ a.s.} ,$$

by the independence of $Z_n$ and $(X_{n+1}, \cdots)$, and the sequence of $\sigma$-algebras $\mathscr{B}(T_n, X_{n+1}, \cdots)$ is monotonically contracting (by transitivity (ii) with a.s. trivial tail $\sigma$-algebra (by the Hewitt–Savage 0–1 law); therefore,

$$E\{Z_n \mid \mathscr{B}(T_n, X_{n+1}, \cdots)\} \to 0 \text{ a.s.}$$

(see Loève [4] page 409).

*Note.* For the multivariate generalizations of the Glivenko–Cantelli Lemma given by Ranga Rao [5], Wolfowitz [7], [8] and Blum [3], the corresponding analogue to the above theorem can be obtained whenever the sequence of statistics is symmetric and transitive.

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