

ALTERNATIVE AXIOMATIZATIONS OF ONE-WAY EXPECTED UTILITY

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An axiomatization for one-way expected utility, $P \succ Q \Rightarrow E(u, P) > E(u, Q)$, is given which places no restriction on the cardinality of the consequence set. In comparison with a previous axiomatization [*Ann. Math. Statist.* **42** (1971) 572-577] for finite consequence sets, the new axioms strengthen the Archimedean property and weaken the order-independence conditions. In particular, the new theory avoids the criticism of indifference due to threshold phenomena that affects the previous axiomatization.

1. Theoretical summary. This note supplements [2] by establishing an alternative axiomatization for one-way expected utility that compares quite favorably with the previous axiomatization. The notation is the same as in [2]: X is a nonempty set, \mathcal{P} is the set of simple probability measures on X (assigning probability 1 to finite subsets of X), \succ ("is preferred to") is a binary relation on \mathcal{P} , and $E(u, P) = \sum u(x)P(x)$. We shall be concerned with

PROPOSITION 1. *There is a real-valued function u on X such that, for all $P, Q \in \mathcal{P}$,*
(1)
$$P \succ Q \Rightarrow E(u, P) > E(u, Q).$$

The following theorem, proved in [2], is similar to a theorem of Aumann [1].

THEOREM A. *Suppose that X is finite. Then Proposition 1 is true if the following four conditions hold throughout \mathcal{P} :*

- A1. \succ is transitive.
- A2. $\alpha \in (0, 1)$ and $P \succ Q \Rightarrow \alpha P + (1 - \alpha)R \succ \alpha Q + (1 - \alpha)R$.
- A3. $\alpha \in (0, 1)$ and $\alpha P + (1 - \alpha)R \succ \alpha Q + (1 - \alpha)R \Rightarrow P \succ Q$.
- A4. $\alpha P + (1 - \alpha)R \succ \alpha Q + (1 - \alpha)S$ for all $\alpha \in (0, 1] \Rightarrow$ not $(S \succ R)$.

In the present paper we shall prove

THEOREM C. *Proposition 1 is true if the following three conditions hold throughout \mathcal{P} :*

- C1. \succ is irreflexive.
- C2. $\alpha \in (0, 1)$ and $P \succ Q$ and $R \succ S \Rightarrow \alpha P + (1 - \alpha)R \succ \alpha Q + (1 - \alpha)S$.
- C3. $P \succ Q$ and $R \succ S \Rightarrow$ there is an $\alpha \in (0, 1)$ such that $\alpha P + (1 - \alpha)S \succ \alpha Q + (1 - \alpha)R$.

The latter theorem strengthens the necessary Archimedean axiom A4 to a non-necessary axiom C3, and simultaneously weakens the order and independence axioms (C1 is implied by A4, and C2 is implied by A1 and A2). Although C2

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is not necessary for (1), it and C1 can be replaced by the following necessary axiom from [2]:

B1. $[m \in \{1, 2, \dots\}, \alpha_j > 0 \text{ and } P^j \succ Q^j \text{ for } j = 1, \dots, m, \text{ and } \sum \alpha_j = 1] \Rightarrow \sum \alpha_j P^j \neq \sum \alpha_j Q^j.$

This is admittedly less “elegant” than C1 and C2. It is easily verified that B1 is implied by C1 and C2. Hence we shall prove Theorem C by proving the more general

THEOREM B. *B1 and C3 imply Proposition 1.*

Theorem A couples a necessary Archimedean axiom with nonnecessary order-independence conditions; Theorem B couples a necessary order-independence axiom with a nonnecessary Archimedean axiom.

The approach of Theorems B and C is of interest for at least three reasons.

(i) Theorems B and C place no restriction on the cardinality of X . An example in Kannai [5], due to M. Perles, shows that Proposition 1 can be false when A1, A2, A3 and A4 hold and X is denumerable. Hence the finite restriction of Theorem A is essential.

(ii) There seems to be a widespread dissatisfaction with independence axioms such as A2, whose preservation of preference under dilution violates expected threshold phenomena. This is discussed in greater detail in [3]. Note that C2 requires preference in both antecedents (instead of $P \succ Q$ and $R = R$ as in A2) to carry the preference conclusion, and hence is not liable to the threshold criticism of A2. The avoidance of A2, even with the addition of the strong Archimedean axiom C3, may be viewed as an improvement by some readers.

(iii) In [2] I said that “there is no set of sufficient conditions for Proposition 1 that is more elegant than” {A1, A2, A3, A4}. Theorem C is offered as evidence against this contention.

2. The strong Archimedean axiom. A word about the strong Archimedean axiom C3 is in order since it differs from the following traditional axiom of von Neumann and Morgenstern [6]:

C3'. $P \succ Q \text{ and } Q \succ R \Rightarrow \text{there are } \alpha, \beta \in (0, 1) \text{ such that } \alpha P + (1 - \alpha)R \succ Q \text{ and } Q \succ \beta P + (1 - \beta)R.$

Clearly, $C3 \Rightarrow C3'$, and both are necessary for two-way expected utility, which has \Rightarrow in place of \Rightarrow in (1). Despite the fact that C3 is stronger than C3', the discussion of C3' that is scattered throughout the expected-utility literature applies almost equally well to C3. In particular, I would expect that those who are disposed to accept the von Neumann–Morgenstern axiom C3' would also favor C3.

In contrast to Theorem C, C1, C2 and C3' do not imply Proposition 1. This is shown by $X = \{x, y, z, w\}$ with \succ equal to (z, w) , plus $(\alpha x + (1 - \alpha)w, \alpha y + (1 - \alpha)z)$ for all $\alpha \in (0, 1]$, plus all preference statements generated from these by the use of C2. (In these statements, x, y, z, w stand for the probability

distributions that assign probability 1 to the displayed element.) This example satisfies C1, C2 and C3' but violates A4.

It is possible to obtain sufficient conditions for Proposition 1 by adding other seemingly inoffensive assumptions to C1, C2 and C3' (or to B1 and C3'), but the resultant axiomatization is bulkier than that of Theorem C (or Theorem B) and I shall not present it here.

3. Proof of Theorem B. We shall base the proof of Theorem B on an existence theorem for an order preserving linear functional for a real vector space V . A cone $C \subseteq V$ has the defining property $p, q \in C$ and $\lambda, \mu > 0 \Rightarrow \lambda p + \mu q \in C$. We shall say that a cone C is *Archimedean* iff $p, q \in C \Rightarrow \lambda p - q \in C$ and $q - \mu p \in C$ for some positive real λ and μ . θ denotes the origin of V .

LEMMA 1. *Let C be a cone in a real vector space V . If $\theta \notin C$ and C is Archimedean then there exists a linear functional f on V such that $f(p) > 0$ for all $p \in C$.*

A proof of Lemma 1 can be based on Theorem 3.1 in Hausner and Wendel [4]. Given C as in the lemma, let \mathcal{H} be the set of all cones C' with $\theta \notin C'$ and $C \subseteq C'$. By Zorn's lemma, \mathcal{H} has a maximal element C^* . Defining $p >^* q \Leftrightarrow p - q \in C^*$, $(V, >^*)$ is easily seen to be a simply ordered vector space. Because C is Archimedean, it is included in one of the equivalence classes of C^* under the equivalence relation \sim , where $p \sim q$ iff $\lambda p - q \in C^*$ and $q - \mu p \in C^*$ (i.e., $\lambda p >^* q$ and $q >^* \mu p$) for some positive real λ and μ . It then follows from the Hausner-Wendel lexicographic embedding theorem that there exists a linear functional f on V such that $f(p) > 0$ for all $p \in C$.

We now proceed with the proof of Theorem B, assuming that B1 and C3 hold. Let V be the vector space of all real-valued functions on X that vanish on all but a finite number of elements in X , with the usual operations of pointwise addition and scalar multiplication for the functions in V . θ is the function that is uniformly zero on X .

Using point probabilities to represent the simple measures in \mathcal{P} , $\mathcal{P} \subseteq V$. Let $D = \{P - Q : P, Q \in \mathcal{P} \text{ and } P \succ Q\}$, and let C be the cone in V generated by D :
 $p \in C \Leftrightarrow$ there exist $\lambda_k > 0$ and $p_k \in D$ such that $p = \sum \lambda_k p_k$, where all sums are nonempty and finite. Clearly, B1 $\Rightarrow \theta \notin C$, and it is routine to show that axiom C3 implies that C is Archimedean.

Therefore, by Lemma 1, there exists a linear functional f on V such that $f(p) > 0$ for all $p \in C$. Define u on X by $u(x) = f(P)$ when $P \in \mathcal{P}$ and $P(x) = 1$. Then, by linearity, $f(P) = \sum u(x)P(x)$. Therefore $P \succ Q \Rightarrow P - Q \in C \Rightarrow f(P - Q) > 0 \Rightarrow f(P) > f(Q) \Rightarrow \sum u(x)P(x) > \sum u(x)Q(x)$, which is (1).

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