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BAYESIAN PRIOR DISTRIBUTIONS FOR SYSTEMS WITH EXPONENTIAL FAILURE-TIME DATA

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In this paper, confidence bounds on the reliability of a serial system composed of exponential subsystems are considered. Both the classical and the Bayesian analyses are discussed. The main result is that for the case in which there are no previous data, then there are no prior distributions on the subsystem reliabilities that are independent of current data and that yield the uniformly most accurate unbiased confidence bounds available through classical techniques.

1. Introduction. In this paper, confidence bounds for system reliability as determined from subsystem information are considered. We restrict attention to those systems composed of independent subsystems whose lives are each exponentially distributed. For this case, if we let θ_i be the mean time to failure for the ith subsystem, the reliability, defined as the probability that that subsystem will operate successfully at least until a specified time t_i, is given by

$$R_i = e^{-t_i/\theta_i} .$$

The reliability for a system of k subsystems is then given by $R = g(R_1, \cdots, R_k)$ where g is determined from the system configuration. In what follows, we assume that life-test data for the ith subsystem is obtained by placing n_i units on life-test and then terminating the test at the time of the r_ith failure. The data then take the form

$$x_{i1}, x_{i2}, \cdots, x_{ir_i}; \quad n_i, \quad r_i$$

where x_{ij} represents the time of the jth failure of the ith subsystem. It is known [4] that $z_i = x_{i1} + \cdots + x_{ir_i} + (n_i - r_i)x_{ir_i}$ is a sufficient statistic to estimate the θ_i and hence the R_i and that $2z_i/\theta_i$ is distributed as chi-square with $2r_i$ degrees of freedom.

In the case of serial systems for which $R = \prod_{i=1}^k R_i$, uniformly most accurate (UMA) unbiased confidence bounds, developed by classical techniques, are available as a function of the z_i when $t_i = t_0$ for $i = 1, \cdots, k$. The procedure for obtaining these is discussed in Section 2. For more complex systems, confidence bounds in the Bayesian sense can be computed. This procedure is discussed in Section 3. In particular, the R_i are treated as random variables and it is necessary to specify their "prior" distributions. A usual technique is to assume a convenient distributional form and then choose the parameters of that form so as to fit a preconceived idea of the distribution of the R_i. Even if we disregard the objection that the R_i are not true random variables, we still run

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into difficulty since the incorporation of previous data into prior distributions invariably assumes a form for that prior distribution when no data is avilable. Several prior distributions to handle this situation have been presented in the literature. For the systems to which we are restricting our attention, the two most mentioned priors appearing in the literature are the uniform distribution over (0, 1) (see Springer and Thompson [10]) and the improper prior, $R^{-1}(\ln 1/R)^{-1}$. The latter is discussed by Burnett and Wales [2] and corresponds to what is sometimes called the fiducial approach, yielding the optimum classical confidence bound on R, for a system of one subsystem. Recent research at North American Rockwell Corporation has demonstrated that these two priors lead to confidence bounds which are not exact in general [8]. In particular, the improper prior yielded numbers which, though exact in the Bayesian sense, were conservative in the classical sense, and the uniform prior distribution usually led to confidence bounds lower than those predicted by the improper prior distribution. Thus, both forms predicted confidence bounds which were in general not equal to the classical optimum bounds.

The question then arises as to whether or not there exist prior distributions on the R_i that do yield the classical optimum bounds for a serial system of more than one subsystem. It was a desire to find these priors that motivated this work. Such priors would go a long way towards reducing the conflict between classical and Bayesian statisticians. A more practical result, however, would be that these priors would provide insight as to how to proceed for more complex systems for which classical techniques are not available. In particular, it was found that there do not exist such prior distributions that are independent of the current data and that yield the optimum bounds for a serial system. A proof of this statement is given in Section 4. Also offered are two prior distributions, or more properly weighting functions in one case, which are valid for special cases. These have been of aid in furthering research in this area, e.g., see Mann and Grubbs [9].

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2. Uniformly most accurate unbiased confidence intervals for the reliabilty of a serial system. For the case in which each subsystem is required to operate the entire mission time, t_0, the reliability of a serial system is given by

$$R = \prod_{i=1}^{k} R_i = \prod_{i=1}^{k} e^{-t_0/\theta_i} = e^{-t_0\phi} ,$$

where

$$\phi = \sum_{i=1}^{k} 1/\theta_i .$$

Thus, making confidence statements about R for a specified t_0 is equivalent to making confidence statements about ϕ.

Performing the transformation $w = z_1$ and $u_i = z_1 - z_i$, $i = 2, \cdots, k$, Lentner and Buehler [7] derived for the case $k = 2$ the conditional distribution of w given

$\mathbf{u}$, $H(w \mid \mathbf{u}; \phi)$. Ahmed El Mawaziny [3] extended this work for arbitrary k to yield:

$$(1) \qquad H(w \mid \mathbf{u}; \phi) = A^{-1}(\mathbf{u}; \phi) \sum_{i_1=0}^{a_1} \sum_{i_2=0, i_j \neq i_m}^{a_2} \cdots \sum_{i_k=0}^{a_k} \prod_{j=1, j \neq m}^{k} \binom{a_j}{i_j} \times [\phi(u_m - u_j)]^{a_j - i_j} \Gamma_{\phi(w-u_m)}(a_m + \sum_{j=1, j \neq m}^{k} i_j + 1) \quad \text{for } w > 0,$$

where, for convenience, we have let

$$a_i = r_i - 1$$
$$u_1 = 0$$
$$u_m = \max\{u_1, u_2, \cdots, u_k\}$$
$$A(\mathbf{u}, \phi) = \sum_{i_1=0, i_j \neq i_m}^{a_1} \cdots \sum_{i_k=0}^{a_k} (a_m + \sum_{j=1, j \neq m}^{k} i_j)! \prod_{j=1, j \neq m}^{k} \binom{a_j}{i_j} [\phi(u_m - u_j)]^{a_j - i_j}$$

and $\Gamma_x(y + 1)$ is the incomplete gamma function defined by

$$\Gamma_x(y + 1) = \int_0^x e^{-t} t^y \, dt .$$

If more than one u_i attains the value of $\max\{u_1, \cdots, u_k\}$, we may define m as the minimum of the i for which this is the case. The value of $H(w \mid \mathbf{u}; \phi)$ is not affected by this choice. (In any case, this event occurs with probability zero.) It is seen that this distribution depends on the θ_i solely through ϕ. Utilizing the one-to-one correspondence between hypothesis testing and confidence intervals and noting that this distribution is the tool used to yield a uniformly most powerful (UMP) unbiased test of level α to test H_1: $\phi \geqq \phi_0$ against K: $\phi < \phi_0$ (Theorem 3 page 136, Lehmann [6]), one may obtain a UMA unbiased confidence interval for ϕ. In particular, if w and $\mathbf{u}$ are observed, then by solving $H(w \mid \mathbf{u}; \phi) = 1 - \alpha$ for ϕ (call this ϕ_m) one obtains an upper confidence bound for ϕ with a $1 - \alpha$ level of confidence. The uniqueness of the solution, ϕ_m, follows from Corollary 3 page 80 of Lehmann [6]. Since R is a strictly decreasing function of ϕ, if we let $R_m = e^{-\phi_m t_0}$, then $S = \{R \mid R_m \leqq R\}$ becomes a UMA unbiased confidence set for the system reliability R. Further, we can demonstrate that the UMA unbiased confidence sets are essentially unique. This follows from the one-to-one correspondence between UMA unbiased confidence sets and UMP unbiased tests (Theorem 2 page 261 of Ferguson) and the fact that the latter are themselves essentially unique in terms of the sufficient and complete statistics $(w, \mathbf{u})$ (remarks page 229 of Lehmann [6]).

3. Bayesian confidence interval for R. In a Bayesian procedure, the R_i are treated as random variables. In order to alleviate later complications, we will consider the distribution of ϕ_i rather than R_i. Using Bayes' theorem, the posterior density function of $\phi_i = 1/\theta_i$ is

$$(2) \qquad f(\phi_i \mid z_i; r_i) = c_i h_i(z_i \mid \phi_i; r_i) p_i(\phi_i) \qquad \phi_i > 0,$$

where $p_i(\phi_i)$ is the "prior" density function of ϕ_i and c_i is the constant of normalization. The density of z_i given ϕ_i can be found using the fact that $2z_i\phi_i$ is distributed as chi-square with $2r_i$ degrees of freedom. It is

(3) $$h_i(z_i \mid \phi_i; r_i) = \phi_i^{r_i} z_i^{r_i - 1} e^{-z_i \phi_i} / \Gamma(r_i) .$$

Assuming that the ϕ_i are independent from each other, one may find the distribution of $\phi = \phi_1 + \cdots + \phi_k$ given $\mathbf{z} = (z_1, \cdots, z_k)$ by taking a convolution integral of $\prod_{i=1}^k f_i(\phi_i \mid z_i; r_i)$. This is easily done using the well-known convolution property of the Laplace transform. In particular, we may write

$$\mathscr{L}\{f(\phi \mid \mathbf{z}; \mathbf{r})\} = \prod_{i=1}^k \mathscr{L}\{f_i(\phi_i \mid z_i; r_i)\} ,$$

or

$$f(\phi \mid \mathbf{z}; \mathbf{r}) = \mathscr{L}^{-1}\{\prod_{i=1}^k \mathscr{L}\{f_i(\phi_i \mid z_i; r_i)\}\} ,$$

where $\mathbf{r} = (r_1, \cdots, r_k)$. Once $f(\phi \mid \mathbf{z}; \mathbf{r})$ is determined, a $(1 - \alpha)$-level upper confidence bound for ϕ can be obtained by solving

$$\int_0^{\phi_B} f(\phi' \mid \mathbf{z}; \mathbf{r})\, d\phi' = 1 - \alpha$$

for ϕ_B.

In practice, $f(\phi \mid \mathbf{z}; \mathbf{r})$ may be difficult to obtain analytically, and in such cases Monte Carlo techniques are helpful. Such techniques can also be useful when working with non-serial systems. In these cases, the distribution of the system reliability may be simulated by sampling from the f_i and combining these samples according to $R = g(R_1, \cdots, R_k)$ where g is determined by the system's logical configuration.

4. Choosing the prior in the absence of prior information. The aforementioned analysis of complex systems using Bayesian procedures in conjunction with Monte Carlo techniques cannot proceed until the priors are specified. An optimum property of these priors would be that in the absence of previous information they yield UMA unbiased confidence intervals for serial systems. The following will demonstrate that no such sets of optimum priors exist that are independent of the current data $\mathbf{z}$. The proof will be by contradiction.

It was shown in Section 2 that the UMA unbiased confidence intervals attained using El Mawaziny's derived distribution, $H(w \mid \mathbf{u}; \phi)$, are unique with probability one. Thus, if the confidence intervals predicted by the Bayesian technique are to be UMA unbiased, they must agree with El Mawaziny's intervals point for point. That is, if $\mathbf{z} = (z_1, \cdots, z_i)$ is observed, and if ϕ_B and ϕ_m are such that

$$\int_0^{\phi_B} f(\phi' \mid \mathbf{z}; \mathbf{r})\, d\phi' = 1 - \alpha , \qquad H(w \mid \mathbf{u}; \phi_m) = 1 - \alpha ,$$

then $P\{\phi_B = \phi_m\} = 1$. Then, dropping the subscripts, we may write

(4) $$\int_0^{\phi} f(\phi' \mid \mathbf{z}; \mathbf{r})\, d\phi' = H(w \mid \mathbf{u}; \phi)$$

and this relation should hold for all $\phi > 0$ and $\mathbf{z}$ such that $z_i > 0, i = 1, \cdots, k$. That this is so follows from the fact that if two continuous functions are equal almost everywhere in some region, then the equality holds everywhere in that region. Letting $H_\phi(w \mid \mathbf{u}; \phi)$ denote the derivative of $H(w \mid \mathbf{u}; \phi)$ with respect to ϕ, we obtain from (4) that $f(\phi \mid \mathbf{z}; \mathbf{r}) = H_\phi(w \mid \mathbf{u}; \phi)$, or that

$$\begin{aligned} \mathscr{L}\{H_\phi(w \mid \mathbf{u}; \phi)\} &= \mathscr{L}\{f(\phi \mid \mathbf{z}; \mathbf{r})\} \\ &= \prod_{i=1}^k \mathscr{L}\{f_i(\phi_i \mid \mathbf{z}_i; \mathbf{r}_i)\} \\ &= \prod_{i=1}^k \mathscr{L}\{c_i h_i(z_i \mid \phi_i; r_i) p_i(\phi_i)\} . \end{aligned}$$

Now since the $p_i(\phi_i)$ are assumed to be independent of the z_j $i, j = 1, \cdots, k$, any set of solutions found by assuming particular values of the z_i should be true for all z_i. To this end, assume $z_1 = z_2 = \cdots = z_k$. Then, letting $q = \sum_{i=1}^k (r_i - 1)$, we have from (1)

$$H(w \mid \mathbf{0}, \phi) = (q!)^{-1}\Gamma_{\phi w}(q+1) = \int_0^{\phi w} \frac{e^{-t} t^q}{q!}\, dt ,$$

and hence by the Fundamental Theorem of Calculus,

$$H_\phi(w \mid \mathbf{0}; \phi) = w^{q+1} e^{-\phi w} \phi^q / q! .$$

Then, by Formula 4.5.3 [1]

$$\begin{aligned} \mathscr{L}\{H_\phi(w \mid \mathbf{0}; \phi)\} &= w^{q+1}(s+w)^{-(q+1)} \\ &= w^{1-k}(s+w)^{k-1} \prod_{i=1}^k w^{r_i}(s+w)^{-r_i}. \end{aligned}$$

Also, when $z_1 = \cdots = z_k = w$, we have

$$\prod_{i=1}^k \mathscr{L}\{f_i(\phi_i \mid z_i; r_i)\} = \prod_{i=1}^k \mathscr{L}\{f_i(\phi_i \mid w; r_i)\} ,$$

Using the fact that for nonzero functions m_i, n_i, if $\prod_{i=1}^k m_i(x_i) = c \prod_{i=1}^k n_i(x_i)$, then $m_i(x_i) = d_i n_i(x_i)$ where the d_i are independent of x and $\prod_{i=1}^k d_i = c$, we have

$$\mathscr{L}\{f_i(\phi_i \mid w; r_i)\} = q_i(s)(s+w)^{-r_i} w^{r_i} ,$$

or that

$$f_i(\phi_i \mid w; r_i) = \mathscr{L}^{-1}\{q_i(s)(s+w)^{-r_i} w^{r_i}\} ,$$

where the $q_i(s)$ do not depend upon $\mathbf{r}$ and $\prod_{i=1}^k q_i(s) = w^{1-k}(s+w)^{k-1}$. Since the confidence interval is invariant under permutations of the subsystems and since ϕ_i is assumed independent of z_j, $j \neq i$, then $f_i(\phi_i \mid z_i; r_i) = f_j(\phi_i \mid z_i; r_i)$ and thus $q_i(s) = w^{1/k-1}(s+w)^{1-1/k}$. Therefore, using Formula 5.4.1 [1], we may write

$$f_i(\phi_i \mid w; r_i) = w^{r_i - 1 + 1/k} e^{-w\phi_i} \phi_i^{\,r_i - 2 + 1/k} / \Gamma(r_i - 1 + 1/k) \tag{5}$$

and then, using (2) and (3),

$$p_i(\phi_i) \propto (1/\phi_i)^{2-1/k} \qquad \phi_i > 0 . \tag{6}$$

In order for the above Laplace transforms to exist, it is required that $r_i + 1/k - 1 > 0$. By definition, r_i is greater than or equal to one and k is finite. Thus, the strict inequality is always satisfied.

It may be observed that if $k = 1$, then this prior reduces to $p_1(\phi_1) = 1/\phi_1$ or, in terms of R_1,

$$p_1(R_1) = R_1^{-1}(\ln 1/R_1)^{-1} ,$$

which is the "improper" prior density already mentioned in the introduction that corresponds to the fiducial approach and which yields the UMA confidence bounds for a system of one subsystem [4]. Therefore, the fact that (6) is improper does not as yet imply that there is no solution.

Since we have assumed $w = z_1 = \cdots = z_k$, (5) becomes

$$f_i(\phi_i \mid z_i; r_i) = z_i^{r_i - 1 + 1/k} e^{-z_i \phi_i} \phi_i^{r_i - 2 + 1/k} / \Gamma(r_i - 1 + 1/k) . \tag{7}$$

Since we are assuming the existence of priors on ϕ_i which are independent of the jth subsystem, we conclude that the posterior distribution on ϕ_i cannot be affected by the fact that $z_j = z_i$. Hence, under these assumptions, (7) should hold for arbitrary $\mathbf{z}$. In order to see if these posterior densities do indeed yield the UMA unbiased bounds, we can simplify the task by assuming $r_i = 1$ for $i = 1, \cdots, k$ and $z_1 < z_i$ for $i = 2, \cdots, k$. Then (1) becomes

$$H = 1 - e^{-z_1 \phi} .$$

Differentiating and taking the Laplace transform, we have

$$\mathscr{L}\{H_\phi(w \mid \mathbf{u}; \mathbf{r} = \mathbf{1})\} = z_1(s + z_1)^{-1} . \tag{8}$$

However, for $r_i = 1$, we have

$$\prod_{i=1}^k \mathscr{L}\{f_i(\phi_i \mid z_i; r_i)\} = \prod_{i=1}^k z_i^{1/k}(s + z_i)^{-1/k} ,$$

and since $0 < z_1 < z_i$, then

$$\prod_{i=1}^k z_i^{1/k}(s + z_i)^{-1/k} \neq z_1(s + z)^{-1} . \tag{9}$$

Since the only assumption used in deriving (6) was that there existed priors independent of $\mathbf{z}$ that yield UMA unbiased bounds and since (9) indicates that (4) does not hold for $z_1 \neq z_j$ when $i \neq j$, then the assumption that such priors existed must be false.

Even though the priors defined by (6) do not lead to UMA unbiased bounds, it may be of interest to see if they provide a useful approximation. A computer program had been written by the Mathematics and Statistics Group at North American Rockwell Corporation for the IBM/360 that computes the Bayesian confidence bounds for system reliability using Monte-Carlo techniques when the prior densities are of the form

$$P_i(R_i) \propto R_i^{\beta_{0i}}(\ln 1/R_i)^{r_{0i}} ,$$

where β_{0_i} and r_{0_i} are arbitrary real numbers with the restriction that the corresponding posterior density be normalizable. This program also computes the UMA unbiased bounds for a serial system using the conditional distribution that El Mawaziny derived. Some of the results for a system of five subsystems are listed below. Except for run 1, the z_i were randomly generated using $2z_i/\theta_i \sim X^2_{2r_i}$ where $r_1 = 3$, $r_2 = 4$, $r_3 = 4$, $r_4 = 2$, $r_5 = 2$, and $\theta_1 = 12$, $\theta_2 = 13$, $\theta_3 = 15$, $\theta_4 = 10$, $\theta_5 = 11$. The mission time was taken as 1.0. The true reliability is then $R = \exp[-\sum_{i=1}^5 1/\theta_i] = 0.66$. The level of confidence is 0.90.

TABLE 1

Run	z_1	z_2	z_3	z_4	z_5	Classical Bound	Bayesian Bound
1	30.0	30.0	30.0	30.0	30.0	0.60	0.60
2	72.7	42.9	142.3	9.6	28.0	0.57	0.63
3	29.1	61.9	36.3	27.9	8.0	0.48	0.53
4	55.3	80.7	44.6	22.3	9.0	0.54	0.59
5	42.0	65.5	26.1	7.8	14.1	0.45	0.50
6	43.7	58.3	84.6	24.0	19.5	0.67	0.69
7	55.4	20.4	73.0	22.6	51.3	0.63	0.64
8	43.6	63.0	25.1	25.4	31.1	0.63	0.64
9	8.1	37.4	48.4	7.0	14.3	0.33	0.36
10	41.3	17.0	110.3	18.3	17.2	0.54	0.55
11	60.6	59.4	113.5	18.9	7.8	0.51	0.58

As Table 1 indicates, (7) leads to liberal estimates of $R_{0.90}$. This was found to be typically true for other serial systems tested.

5. The priors as weighting functions. The question of whether or not there exist priors independent of the data that yield the UMA unbiased confidence bounds may be generalized to whether or not there exist functions $v_i(\phi_i, \mathbf{z}, \mathbf{r})$ dependent upon the data that satisfy the following:

$$\int_0^\phi \mathscr{L}^{-1}\{\textstyle\prod_{i=1}^k \mathscr{L}\{f_i(\phi_i \mid \mathbf{z}; \mathbf{r})\}\} \, d\phi = H(w \mid \mathbf{u}, \phi) \tag{10}$$

where

$$f_i(\phi_i \mid \mathbf{z}; \mathbf{r}) = h_i(z_i \mid \phi_i, r_i) v_i(\phi_i, \mathbf{z}; \mathbf{r}) \tag{11}$$

and

$$f_i \text{ is such that } \int_0^\infty f_i(\phi_i \mid z; r) \, d\phi_i = 1 \, .$$

In other words, the v_i are such that the functions produced in (11) may be treated as the posterior densities of ϕ_i and that these densities lead to the UMA unbiased bounds predicted using the classical technique.

Differentiating (10), taking the Laplace transform of both sides, and using (3), we may write

$$\textstyle\prod_{i=1}^k \mathscr{L}\{\phi_i^{r_i} z_i^{r_i-1} e^{-z_i\phi_i} v_i(\phi_i, \mathbf{z}, \mathbf{r})/\Gamma(r_i)\} = \mathscr{L}\{H_\phi(w \mid \mathbf{u}, \phi)\} \, . \tag{12}$$

For arbitrary w and $\mathbf{u}$, one may find the Laplace transform of $H_\phi(w \mid u; \phi)$. For the special case $\mathbf{r} = \mathbf{1}$, we have

$$\mathscr{L}\{H_\phi(w \mid \mathbf{u}; \phi)\} = z_m(s + z_m)^{-1} \qquad z_m = \min\{z_i\} \, . \tag{13}$$

If we again require the posterior densities to be invariant under permutation, then taking the kth root of (12) yields

$$\begin{aligned} f_i(\phi_i \mid \mathbf{z}, \mathbf{1}) &= \mathscr{L}^{-1}\{z_m^{1/k}(s + z_m)^{-1/k}\} \\ &= z_m^{1/k} \phi^{1/k-1} e^{-z_m\phi_i}/\Gamma(1/k) \, . \end{aligned} \tag{14}$$

Solving for v_i, we get

$$v_i(\phi_i, \mathbf{z}, \mathbf{1}) = (z_m^{1/k}/\Gamma(1/k))\phi_i^{-2+1/k}e^{-(z_m - z_i)\phi_i} .$$

The kernel of this, of course, reduces to (6) when all the z_i are equal.

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